A METHOD FOR CHECKING THE BASIC CONSTRAINT QUALIFICATION AND ITS APPLICATION TO CONVEX QUADRATIC FUNCTIONS

SHUNSUKE YAMAMOTO AND DAISHI KUROIWA

Abstract. We study a method for checking the basic constraint qualification (BCQ) in convex optimization and its application to convex quadratic functions.

1. Introduction

In this paper, we consider the convex optimization model problem:

\[
\begin{align*}
\text{(P)} \quad & \text{minimize} & f(x) \\
& \text{subject to} & g_i(x) \leq 0, \quad \text{for each } i \in I,
\end{align*}
\]

where \( I \) is an arbitrary index set, \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g_i : \mathbb{R}^n \to \mathbb{R}, \ i \in I, \) are convex functions. In convex optimization, research on constraint qualification is very important, for example it ensures the existence of Lagrange multipliers. Results on constraint qualification have played very important role in the development of convex optimization.

In 2008, it was shown that the basic constraint qualification (BCQ) is a necessary and sufficient constraint qualification for the optimality condition, see [2]. To check the BCQ at a feasible point, however, we must calculate the subdifferential of all \( g_i \) and the normal cone of the feasible set at the point. So, it is difficult to check whether BCQ holds or not at every feasible point.

On the other hand, a simple method for checking whether the BCQ holds or not at every feasible point, is given, see [3]. To check the BCQ at all feasible points by the method, we need to calculate only \( \text{cone} \text{co} \bigcup_{i \in I} \text{epi} g_i^* \).

The purpose of this paper is to study the method, in [3], by a specific class of functions which are presented by quadratic. We show results which related to check the BCQ without calculating \( g^* \). The layout of the paper is as

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follows. In section 2, we describe our notation and present preliminary results. In section 3, we study a characterization in the quadratic case.

2. Preliminaries

In this section, we describe our notation and present preliminary results. The space of all \((n \times n)\) symmetric matrices is denoted by \(S^n\). Let \(f : \mathbb{R}^n \to \mathbb{R}\) be a function. \(f\) is said to be convex if for any \(x, y \in \mathbb{R}^n\) and for any \(\lambda \in (0, 1)\),
\[
f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y),
\]
and \(f\) is said to be strictly convex if for any \(x, y \in \mathbb{R}^n\) with \(x \neq y\) and for any \(\lambda \in (0, 1)\),
\[
f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y).
\]
Also \(f\) is said to be quadratic if \(f\) is written by the following form:
\[
f(x) = \langle x, Ax \rangle + \langle a, x \rangle + \alpha, \quad \forall x \in \mathbb{R}^n,
\]
where \(A \in S^n, a \in \mathbb{R}^n\) and \(\alpha \in \mathbb{R}\). Let \(C\) be a set in \(\mathbb{R}^n\). We denote the closure, the interior, the boundary, the conical hull and the convex hull of \(C\) by \(\text{cl} C\), \(\text{int} C\), \(\text{bd} C\), \(\text{cone} C\) and \(\text{co} C\), respectively. The epigraph of \(f\), denoted by \(\text{epi} f\), is defined by
\[
\text{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \text{dom} f, f(x) \leq r\}.
\]
The conjugate function of \(f\), \(f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\), is defined by
\[
f^*(u) = \sup \{\langle u, x \rangle - f(x) \mid x \in \mathbb{R}^n\},
\]
where \(\langle u, x \rangle\) denotes the inner product of two vectors \(u\) and \(x\). The subdifferential of \(f\) at \(x \in \mathbb{R}^n\), denoted by \(\partial f(x)\), is defined by
\[
\partial f(x) = \{\xi \in \mathbb{R}^n \mid f(x) + \langle \xi, y - x \rangle \leq f(y), \forall y \in \mathbb{R}^n\}.
\]
For \(f : \mathbb{R}^n \to \mathbb{R}\), the gradient of \(f\) at \(\bar{x}\) is denoted by \(\nabla f(\bar{x})\).

For nonempty convex set \(C \subseteq \mathbb{R}^n\), the indicator function \(\delta_C : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is defined by
\[
\delta_C(x) = \begin{cases} 
0 & x \in C, \\
+\infty & x \notin C.
\end{cases}
\]
For any \(x \in C\), the normal cone of \(C\) at \(x\), denoted by \(N_C(x)\), is defined by
\[
N_C(x) = \{\xi \in \mathbb{R}^n \mid \langle \xi, y - x \rangle \leq 0, \forall y \in C\}.
\]
In this paper, we consider the following convex programming problem:
\[
(P) \left\{ \begin{array}{ll}
\text{minimize} & f(x) \\
\text{subject to} & g_i(x) \leq 0, \quad \text{for each } i \in I,
\end{array} \right.
\]
where $I$ is an arbitrary finite set, and $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I$, are convex functions. Let $S$ denote the feasible set of (P), that is,

$$S = \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, \forall i \in I \}.$$

The family $\{g_i \mid i \in I\}$ is said to satisfy the basic constraint qualification (BCQ) at $\bar{x} \in S$ if

$$N_S(\bar{x}) = \text{cone} \bigcup_{i \in I(\bar{x})} \partial g_i(\bar{x}),$$

where $I(\bar{x}) = \{ i \in I \mid g_i(\bar{x}) = 0 \}$. In particular, if $\bar{x} \in \text{int}S$, then the BCQ holds at $\bar{x}$.

At first, we introduce a previous result of the BCQ.

**Theorem 2.1** ([2]). Let $\bar{x}$ be a feasible point of (P). Then the following statements are equivalent:

(i) The family $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.

(ii) For each convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\bar{x}$ is a minimizer of (P) if and only if there exist a finite subset $J \subseteq I(\bar{x})$ and $\lambda \in \mathbb{R}^J_+$, such that

$$0 \in \partial f(\bar{x}) + \sum_{i \in J} \lambda_i \partial g_i(\bar{x}).$$

By Theorem 2.1, the BCQ is a necessary and sufficient constraint qualification for the convex optimization. However, to check the BCQ at a feasible point, we must calculate the subdifferential of all $g_i$ and the normal cone of the feasible set at the point. So, it is difficult to check whether BCQ holds or not at every feasible point. Recently, a simple method for checking whether the BCQ holds or not at every feasible point, is given, see [3].

**Theorem 2.2** ([3]). Let $\bar{x}$ be a feasible point of (P). Then the following statements are equivalent:

(i) The family $\{g_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.

(ii) $\left\{ y \in \mathbb{R}^n \mid (y, \langle y, \bar{x} \rangle) \in \text{cl conec} \bigcup_{i \in I} \text{epi} g_i^* \right\} \subseteq \left\{ y \in \mathbb{R}^n \mid (y, \langle y, \bar{x} \rangle) \in \text{cone} \bigcup_{i \in I} \text{epi} g_i^* \right\}$.

Based on the Theorem 2.2, we can check whether the BCQ holds or not at all feasible points by checking the intersection of cone $\bigcup_{i \in I} \text{epi} g_i^*$ and the graph of linear function $\langle \cdot, \bar{x} \rangle$. In particular, when $n \leq 2$, it is easy to check the BCQ by illustrating $\bigcup_{i \in I} \text{epi} g_i^*$. However, when $n \geq 3$, it is difficult to illustrate $\bigcup_{i \in I} \text{epi} g_i^*$.

On the other hand, the result of related with quadratic has the following.
Theorem 2.3 ([1]). Let \( g : \mathbb{R}^n \to \mathbb{R} \) be quadratic function that is not identically zero. Suppose that \( S = \{ x \in \mathbb{R}^n \mid g(x) \leq 0 \} \) is nonempty set. Then the following statements are equivalent:

(i) There exists \( x_0 \in \mathbb{R}^n \) such that \( g(x_0) < 0 \).

(ii) For each quadratic function \( f : \mathbb{R}^n \to \mathbb{R} \), there exists \( \lambda \geq 0 \) such that
\[
\inf_{x \in S} f(x) = \inf_{x \in \mathbb{R}^n} \{ f(x) + \lambda g(x) \}.
\]

Remark 2.4. Condition (ii) of Theorem 2.3 implies \( \{g\} \) satisfies the BCQ at every point in \( S \). Indeed, for each \( \bar{x} \in S \), it is clear that \( N_S(\bar{x}) \supseteq \text{cone} \partial g(\bar{x}) \).

Conversely, for each \( a \in N_S(\bar{x}) \),
\[
\langle -a, \bar{x} \rangle = \inf_{x \in S} \langle -a, x \rangle.
\]

Then \( f(\bar{x}) = \inf_{x \in S} f(x) \). From (ii) of Theorem 2.3, there exists \( \lambda \geq 0 \) such that
\[
\inf_{x \in S} f(x) = \inf_{x \in \mathbb{R}^n} \{ f(x) + \lambda g(x) \}.
\]

For each \( x \in \mathbb{R}^n \),
\[
f(\bar{x}) \leq f(x) + \lambda g(x),
\]
\[
f(\bar{x}) - f(x) \leq \lambda g(x),
\]
\[
\langle a, x - \bar{x} \rangle \leq \lambda g(x) - \lambda g(\bar{x}).
\]

So, \( a \in \partial(\lambda g)(\bar{x}) = \lambda \partial g(\bar{x}) \subseteq \text{cone} \partial g(\bar{x}) \). Thus \( \{g\} \) satisfies BCQ at \( \bar{x} \).

In this paper, Theorem 2.2 is investigated by a specific class of functions which are presented by convex quadratic.

3. Checking the BCQ for quadratic function

Lemma 3.1. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be a convex quadratic function, and \( S = \{ x \in \mathbb{R}^n \mid g(x) \leq 0 \} \) be nonempty set. Assume that \( g(x) \geq 0 \) for each \( x \in \mathbb{R}^n \). Then \( \nabla g^*(0) \in S \).

Proof. From \( g \) is a non-negative convex quadratic function, we have \( g^*(0) = 0 \) and \( \nabla g^*(0) \in \partial g^*(0) \). So, for each \( y \in \mathbb{R}^n \),
\[
\langle \nabla g^*(0), y - 0 \rangle + g^*(0) \leq g^*(y),
\]
\[
\langle \nabla g^*(0), y \rangle - g^*(y) \leq 0,
\]
\[
g(\nabla g^*(0)) \leq 0.
\]
Thus, \( \nabla g^*(0) \in S \). \( \Box \)

Theorem 3.2. Let \( g : \mathbb{R}^n \to \mathbb{R} \) be convex quadratic function that is not identically zero. Suppose that \( S = \{ x \in \mathbb{R}^n \mid g(x) \leq 0 \} \) is nonempty set. Then the following statements are equivalent:

(i) For each \( x \in \mathbb{R}^n \), \( g(x) \geq 0 \).

(ii) \( \{g\} \) doesn’t satisfy the BCQ at \( \nabla g^*(0) \).

(iii) There exists \( x_0 \in S \) such that \( \{g\} \) doesn’t satisfy the BCQ at \( x_0 \).
Proof. The implication (ii)⇒(iii) is clear.

Next, we prove (iii)⇒(i). Assume that (iii). By Remark 2.4, for each \( x \in \mathbb{R}^n \),
\( g(x) \geq 0 \).

Finally, we turn to the proof of (i)⇒(ii). Assume that (i). By Lemma 3.1,
\( g^*(0) = 0, \nabla g^*(0) \in \mathcal{S} \) and \( \nabla g^*(0) \in \partial g^*(0) \). Put
\[ B = \{(y, \beta) \in \mathbb{R}^n \times \mathbb{R} | \langle (\nabla g^*(0), -1), (y, \beta) \rangle = 0 \}. \]
Then,
\[ B \cap \text{epi} g^* = \{(0, 0)\}. \]
In fact, it can be checked easily that
\[ B \cap \text{epi} g^* \supseteq \{(0, 0)\}. \]
Conversely, assume that there exists \((x, \alpha) \in B \cap \text{epi} g^*\) such that \((x, \alpha) \neq (0, 0)\). By \( \nabla g^*(0) \in \partial g^*(0) \), for each \((y, \beta) \in \text{epi} g^*\),
\[ \langle \nabla g^*(0), y \rangle + g^*(0) \leq g^*(y) \leq \beta. \]
From \((x, \alpha) \in \text{epi} g^*, g^*(x) \leq \alpha\). Since \((x, \alpha) \in B, \langle \nabla g^*(0), x \rangle = \alpha. \]
So,
\[ \alpha = \langle \nabla g^*(0), x \rangle \leq g^*(x) \leq \alpha. \]
Thus, \( \alpha = g^*(x) \). From, \( g^*(0) = 0 \) and \((x, g^*(x)) \neq (0, 0), x = 0 \). For each \( \lambda \in (0, 1), (\lambda x, g^*(\lambda x)) \in \text{epi} g^* \) and \( \lambda x \neq 0 \). By the strict convexity of \( g^* \), we have
\[ g^*(- \lambda g^*(0) + \lambda x) < g^*(\lambda x), \]
\[ g^*( (1 - \lambda) g^*(0) + \lambda g^*(x) ) < g^*(x), \]
\[ g^*(\lambda x) < (1 - \lambda) g^*(0) + \lambda g^*(x), \]
\[ 0 < \lambda (\nabla g^*(0), x) - g^*(\lambda x), \]
\[ g^*(\lambda x) < \langle \nabla g^*(0), \lambda x \rangle + g^*(0). \]
This contradicts (3.2).

By using (3.1), we have
\[ \{y \in \mathbb{R}^n | \langle y, \nabla g^*(0) \rangle \in \text{cone} \text{epi} g^* \} = \{0\}. \]
Actually, we prove the left hand side is included in \{0\}. Let \( y \in \mathbb{R}^n \) be satisfying \( \langle y, \nabla g^*(0) \rangle \in \text{cone} \text{epi} g^* \). There exist \( \lambda \geq 0 \) and \((x, \alpha) \in \text{epi} g^*\) such that \( \langle y, \nabla g^*(0) \rangle = \lambda(x, \alpha) \). If \( \lambda = 0 \), then \( y = 0 \). If \( \lambda > 0 \), we have
\[ \langle \frac{1}{\lambda} y, \langle \frac{1}{\lambda} y, \nabla g^*(0) \rangle \rangle \in \text{epi} g^*, \]
and
\[ \langle \langle \frac{1}{\lambda} y, \langle \frac{1}{\lambda} y, \nabla g^*(0) \rangle \rangle, \nabla g^*(0), -1 \rangle \rangle = 0. \]
Thus, \((\frac{1}{r}y, \langle \frac{1}{r}y, \nabla g^*(0) \rangle) \in B \cap \text{epi} g^*\). Since \(B \cap \text{epi} g^* = \{(0, 0)\}\), we have \(y = 0\).

On the other hand, \(\{y \in \mathbb{R}^n \mid (y, \langle y, \nabla g^*(0) \rangle) \in \text{cl cone epi} g^*\} = \mathbb{R}^n\).

Actually, we prove \(\mathbb{R}^n\) is included in the right hand side. Let \(y \in \mathbb{R}^n\). Since \(g^*\) is a quadratic convex function and \(g^*(0) = 0\), there exist \(B \in S^n\) and \(a \in \mathbb{R}^n\) such that
\[
\begin{align*}
g^*(x) &= \langle x, Bx \rangle + \langle b, x \rangle \\
\nabla g^*(0) &= b.
\end{align*}
\]

For each \(\epsilon > 0\), put \(r = \frac{1}{\epsilon} \langle y, By \rangle + 1\).

Thus, \(\frac{1}{r} \langle y, By \rangle < r\).

\[
\frac{1}{r} \langle y, By \rangle + \langle b, y \rangle < \frac{1}{r} (\langle \nabla g^*(0), y \rangle + \epsilon)
\]

So, \((y, \langle \nabla g^*(0), y \rangle + \epsilon) \in \text{epi} g^*\).

Therefore, \((y, \langle \nabla g^*(0), y \rangle + \epsilon) \in \text{cl cone epi} g^*\).

By Theorem 2.2, \(\{g\}\) doesn’t satisfy the BCQ at \(\nabla g^*(0)\), and thus (ii) holds. This completes the proof. \(\square\)

By Theorem 2.2 and Theorem 3.2, we have the following corollary.

**Corollary 3.3.** Let \(g : \mathbb{R}^n \to \mathbb{R}\) be convex quadratic function that is not identically zero. Suppose that \(S = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}\) is nonempty set.

Then the following statements are equivalent:

(i) There exists \(x_0 \in \mathbb{R}^n\) such that \(f(x_0) < 0\).

(ii) \(\{g\}\) satisfies the BCQ at \(\nabla g^*(0)\).

(iii) For each \(x \in S\), \(\{g\}\) satisfies the BCQ at \(x\).

(iv) For each \(\bar{x}\) and \(f : \mathbb{R}^n \to \mathbb{R}\), convex, the following statements are equivalent:

(a) \(\bar{x}\) is a minimizer of the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \leq 0,
\end{align*}
\]

(b) There exists \(\lambda \geq 0\) such that \(f(x) + \lambda g(x) \geq f(\bar{x})\), and \(\lambda g(\bar{x}) = 0\).
Example 3.4. Let \( g(x) = x^2 + 2x \). Then \( S = \{ x \in \mathbb{R} \mid g(x) \leq 0 \} = [-2, 0] \), and \( g(-1) = -1 < 0 \). So, the BCQ holds at every point of \( S \).

Example 3.5. Let \( g(x) = \frac{a}{2} + \langle b, x \rangle + \frac{\|b\|^2}{2a} \), where \( a > 0 \) and \( b \in \mathbb{R}^n \). Then, \( S = \{ -\frac{b}{a} \} \), and for each \( x \in \mathbb{R}^n \), \( g(x) \geq 0 \). So, the BCQ doesn’t hold at \( -\frac{b}{a} \).

References


S. Yamamoto
Interdisciplinary Graduate School of Science and Engineering, Shimane University, Matsue, Shimane, 690-8504, Japan
E-mail address: yamamoto@math.shimane-u.ac.jp

D. Kuroiwa
Major in Interdisciplinary Science and Engineering, Shimane University, Matsue, Shimane, 690-8504, Japan
E-mail address: kuroiwa@math.shimane-u.ac.jp