MINIMUM COST SPANNING TREE PROBLEMS WITH
RESTRICTIONS ON COALITIONS

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Abstract. We focus on minimum cost spanning tree problems (mcstps) where agents need to be connected with a source directly or indirectly through other agents with the minimum cost. Mcstps have been studied as cooperative games, where all of agents cooperate to construct a minimum cost spanning tree (mcst), and share the cost of the mcst. There actually exist situations where some coalitions of agents are not admissible. In this paper, we consider restrictions on coalitions of agents for mcstps and define feasible networks with respect to set systems. We study the existence of spanning feasible networks and feasible arborescences. In addition, we study minimum cost feasible networks and cost allocation in a special case.

1. Introduction

We are concerned with minimum cost spanning tree problems, for short mcstps. The mcstp is a situation where a set of agents located at different geographical points wants to obtain the specific resources only from a common supplier called a source. Agents want to directly or indirectly connect to the source with as cheap cost as possible, namely, a minimum cost spanning tree (mcst) is considered. In the economy and social life there are many situations that can be modeled in this way. Dutta and Kar [9] gave an example that several towns may draw power from a common power plant, and hence have to share the cost of the distribution network.

Conventional studies about mcstps do not consider any restriction on networks. However, there exist mcstps with such restrictions. For example, Two towns A and B need to be connected to a power plant, but A and B cannot be connected directly due to geographical restrictions. Such a situation can

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be treated by modifying connection costs of pairs of agent and the source [13],
namely, the connection cost between A and B is the infinity. On the other
hand, we can consider restrictions which cannot be treated only by modifying
connection costs. For example, Bergantiños et al. [3] have studied $k$-hop span-
ning trees where paths from the source to agents should be through at most $k$
agents. Fernández and Puerto [8] have considered networks whose paths should
be shortest paths from the source to agents. Moreover, we can consider the
situation where town A cannot connect with a power plant unless town B con-
nects with the power plant and the situation where A and B cannot connect
with a power plant using the common path. Some of those situations can be
considered as restrictions on cooperative relations among agents.

In this paper, we study mcstps with restrictions on coalitions of agents. In
several studies, restricted coalitions are defined as set systems, called feasible
coaition systems [4, 13]. So, we introduce a set system to an mcstp. From the
set system, we define feasible networks using feasible paths from the source 0
(0-paths). The agents in a feasible 0-path should form a coalition, because, if
not, the agents cannot share the supply from the source. We consider directed
networks instead of undirected networks as opposed to conventional mcstps to
emphasize directions of paths from the source.

In an mcstp with a feasible coalition system, a minimum cost spanning
feasible (directed) network or arborescence with root source 0 is sought. How-
ever, there exist situations where even spanning feasible networks do not exist.
Therefore, we study conditions on the feasible coalition system for the existence
of spanning feasible networks or feasible arborescences.

Even if there exist feasible arborescences, it is difficult to propose reason-
able cost allocation rules, because we may not have useful characteristics of
minimum cost feasible arborescences. Moreover, the minimum cost of feasible
spanning networks is less than that of feasible arborescences. Therefore, we
study a sufficient condition under which feasibility of networks can be han-
dled by using an asymmetric cost matrix. In that case, we can apply the cost
allocation rule proposed by Dutta and Mishra [6].

Our paper is organized as follows. In section 2 we introduce mcstps and
we define directed networks for mcstps. In addition, we introduce minimum
cost arborescence and Dutta and Mishra’s allocation rule via Edmonds’ algo-

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rithm. In section 3 we define feasibility of networks. We express restrictions of
cooation relations by set systems [4]. Then we introduce feasible networks,
strongly feasible networks and induced networks. In section 4 we introduce
some properties of $\mathcal{F}$ and show existence of spanning feasible networks as well
as feasible arborescences. In section 5 we study cost allocation for mcstps with
the feasible coalition system $\mathcal{F}$. We show an example to which Dutta and
Mishra’s approach is applicable. In section 6 we summarize this paper and mention future researches.

2. Preliminaries

We consider minimum cost spanning tree problems (for short mcstps). The mcstp is a situation where some agents are willing to be connected to a source with saving their cost as cheap as possible. The set of agents and the source are denoted by \( N = \{1, \ldots, n\} \) and 0, respectively. In the sequel we use \( N_0 = N \cup \{0\} \). Connection costs are provided by \( C \) which is an \( n + 1 \) order real valued symmetric square matrix. Each \( ij \) element of \( C \), denoted by \( c_{ij} \), is the cost for connecting the pair \( i, j \) of agents/source. We assume that the costs are nonnegative, namely, \( c_{ij} \geq 0 \). Notice that for every pair \( i, j \), we have \( c_{ij} = c_{ji} \). The mcstp is denoted by \((N_0, C)\). The set of all cost matrices for \( N \) is denote by \( \mathcal{C}_N \). Throughout this paper, the set of agents is fixed to \( N \). So, we can identify all of mcstps by \( \mathcal{C}_N \).

We introduce directed networks for mcstps \((N_0, C)\). A directed network is given by a subset of \( E_{N_0} = \{(i, j) \mid i \in N_0, j \in N \text{ and } i \neq j\} \). Each pair \((i, j) \in E_{N_0} \) can be seen the directed edge from \( i \) to \( j \). Notice that we ignore directed edges of the form \((i, i)\) as well as of directed edges from any \( i \in N \) to the source 0. We call “directed network” and “directed edge” as “network” and “edge” simply. For a network \( E \subseteq E_{N_0} \), a subnetwork of \( E \) is a subset of \( E \). The cost of network \( E \) is defined by \( c(E) = \sum_{(i, j) \in E} c_{ij} \).

For \( i, i' \in N_0 \) such that \( i \neq i' \), a (directed) path from \( i \) to \( i' \) is a sequence \( i_1, i_2, \ldots, i_k \) of \( N_0 \) where \( k \geq 2 \), \( i_1 = i \), \( i_k = i' \) and elements \( i_1, i_2, \ldots, i_k \) are different from each other. A (directed) cycle is a sequence \( i_1, i_2, \ldots, i_k \) of \( N_0 \) where \( k \geq 3 \), \( i_1 = i_k \) and element \( i_1, i_2, \ldots, i_{k-1} \) are different from each other. Additionally, when the first element of a path is 0, the path is called a 0-path.

For a path \( \pi = i_1, i_2, \ldots, i_k \), a subpath of \( \pi \) is a path which is described by \( i_{l_1}, i_{l_1+1}, \ldots, i_{l'} \) for some \( l, l' \) such that \( 1 \leq l < l' \leq k \). For a subpath of a 0-path \( \pi \), when the first element of the subpath is also 0, it is called a 0-subpath of \( \pi \).

For a network \( E \) and a path \( \pi \), we say that \( \pi \) is included in \( E \) or \( \pi \) exists in \( E \) when \((i_l, i_{l+1}) \in E \) for all \( l = 1, 2, \ldots, k - 1 \). Two different elements \( i, j \in N_0 \) are called connected in \( E \) if there exists a path from \( i \) to \( j \) in \( E \).

We introduce spanning networks and arborescences (spanning trees) with rooted at 0.

**Definition 2.1.** A network \( E \subseteq E_{N_0} \) is called a spanning network rooted at 0 if for each \( i \in N \) there exists a path from 0 to \( i \) in \( E \). Moreover, a network \( E \) is called an arborescences rooted at 0 if for each \( i \in N \) there exists exactly one path from 0 to \( i \) in \( E \).
Clearly an arborescence is a special case of spanning networks. It is known that a network \( E \) is an arborescence rooted at 0 if and only if \( E \) contains no cycle, 0 has no incoming edge, and every node \( i \in N \) has only one incoming edge. The sets of all spanning networks and arborescences rooted at 0 for \( N \) are denoted by \( S_N \) and \( A_N \), respectively. For simplicity we omit “rooted at 0” of spanning networks (as well as arborescences), because we only consider those rooted at 0.

To reduce the total cost of a network that supplies of service from 0 to all agents, a spanning network with the minimum cost is required. Such a network is necessarily an arborescence since all costs are nonnegative. A minimum cost arborescence (for short MCA) \( E \in A_N \) corresponding to a cost matrix \( C \) is satisfies that \( c(E) \leq c(E') \) for all \( E' \in A_N \). Let \( MA(C) \) denote the set of MCA corresponding to \( C \), and \( T(C) \) the total cost associated with any element \( E \in MA(C) \).

In cooperative game theory, one of major interests for mcstps is how to allocate the cost of MCA to the agents. In the following sections, we consider restrictions to spanning networks. Without any restriction, MCA can be successfully obtained by Prim’s or Kruskal’s algorithm [12, 11] for symmetric cost matrices and by Edmonds’ algorithm [7] for asymmetric ones (Chu and Liu [5] also proposed an algorithm for asymmetric matrices). Moreover, several cost allocation rules have been proposed [3]. Unfortunately, in restriction cases even obtaining MCAs becomes difficult, and furthermore minimum cost spanning networks may not be arborescences. However, in the special case where all spanning feasible networks correspond to spanning subnetworks of some network \( E \), we can apply Dutta and Mishra’s allocation rule [6], which was originally proposed for the network \( E_{N_0} \) and an asymmetric cost matrix. To apply Dutta and Mishra’s approach to that special case, we modify all costs of edges which does not appear in \( E \) to some large value or the infinity.

Now, we introduce Dutta and Mishra’s cost allocation rule for mcstp \( (N_0, C) \). Notice that \( C \) can be asymmetric. First we introduce irreducible forms of cost matrix \( C \). An irreducible form \( CR \) is a minimal cost matrix satisfying \( T(CR) = T(C) \). The special irreducible form which is defined via Edmonds’ algorithm [6] is used to define the cost allocation rule:

**Edmonds’ Algorithm.**

**Stage (1)-0:** For each \( j \in \{1, \ldots, N\} \),

\[
\Delta_j^0 = \min_{j \in N_0 \setminus \{j\}} c_{ij}^0
\]
Set $N^{0} = \{ \{i\} \mid i \in N_{0}\}$. For each $i \in \{0, \ldots, n\}$ and $j \in \{1, \ldots, n\}$, \[c_{i j}^{0} = c_{i j} - \Delta_{j}^{0}\]

**Stage (1)-t:** Select one cycle $\sigma^{t-1} = N_{i_{1}}^{t-1}, N_{i_{2}}^{t-1}, \ldots, N_{i_{k}}^{t-1}$ such that $N_{i_{1}}^{t-1} = N_{i_{k}}^{t-1}$ and $\tilde{c}_{N_{i_{1}}^{t-1}N_{i_{2}}^{t-1}} = 0$ for $l = 1, 2, \ldots, k - 1$. If there is no cycle satisfies above then the algorithm is terminated. Construct a sub-partition $N^{t} = \{N_{0}^{t}, \ldots, N_{K}^{t}\}$ of $N^{t-1}$ such that $N^{t} = \{N_{j}^{t-1} \mid N_{j}^{t-1} \neq N_{i_{l}}^{t-1}, l = 1, 2, \ldots, k - 1\} \cup \{\bigcup_{l=1}^{k} N_{i_{l}}^{l-1}\}$. Note that since 0 cannot be part of any cycle, 0 is one of the singleton in this partition. Denote $N_{0}^{t} = \{0\}$ and $N_{K}^{t} = \bigcup_{l=1}^{k-1} N_{i_{l}}^{l-1}$. For each $k \in \{0, \ldots, K\}$ and $l \in \{1, \ldots, K\}$, define the cost of $N_{k}^{t}$ and $N_{l}^{t}$ \[\tilde{c}_{N_{k}^{t}N_{l}^{t}}^{t} = \min_{N_{i}^{t-1} \subseteq N_{k}^{t}, N_{j}^{t-1} \subseteq N_{l}^{t}} \tilde{c}_{N_{i}^{t-1}N_{j}^{t-1}}^{t}\]

Note that $\tilde{C}^{t}$ is a cost matrix on nodes $N^{t}$. For each $j \in \{1, \ldots, K\}$, define \[\Delta_{j}^{t} = \min_{N_{i}^{t} \neq N_{j}^{t}} \tilde{c}_{N_{i}^{t-1}N_{j}^{t-1}}^{t} \]

For each $i \in \{0, \ldots, K\}$ and $j \in \{1, \ldots, K\}$, update $\tilde{C}^{t}$ by \[\tilde{c}_{N_{i}^{t}N_{j}^{t}}^{t} = \tilde{c}_{N_{i}^{t-1}N_{j}^{t-1}}^{t} - \Delta_{j}^{t}\]

Since the source cannot be part of any cycle and since $N$ is finite, the algorithm must be terminated. The terminated stage is denoted by $T$. At the terminated stage $T$, for each supernode $N_{k}^{T-1}$, select an incoming edge $N_{k}^{T-1}N_{l}^{T-1}$ with $\tilde{c}_{N_{k}^{T-1}N_{l}^{T-1}}^{T-1} = 0$, then the resulting network $E^{T-1}$ over $N^{T-1}$ is an MCA for $\tilde{C}^{T-1}$. From the network $E^{T-1}$, we construct an MCA for the cost matrix $C$ by expanding supernodes one by one.

**Stage (2)-t:** Set $E^{T-1}$ as above. Set $t = T - 2$.

**Stage (2)-t:** If $t = -1$ then this procedure is terminated.

Let $\sigma^{t} = N_{i_{1}}^{t}, N_{i_{2}}^{t}, \ldots, N_{i_{k}}^{t}$ be the cycle selected in stage $t+1$. Moreover, let

\[N_{j}^{t+1}N_{i_{a}}^{t} = \arg\min_{N_{i_{a}}^{t} \subseteq \sigma^{t}} \tilde{c}_{N_{i_{a}}^{t+1}N_{j}^{t}} \quad \text{for } j \text{ such that } N_{j}^{t+1}N_{K_{k+1}}^{t+1} \in E^{t+1}\]

\[N_{l}(l)^{t+1}N_{i_{a}}^{t+1} = \arg\min_{N_{i_{a}}^{t+1} \subseteq \sigma^{t}} \tilde{c}_{N_{i_{a}}^{t+1}N_{l}^{t+1}} \quad \text{for } l \text{ such that } N_{K_{k+1}}^{t+1}N_{l}^{t+1} \in E^{t+1}\]
Note that there exists the unique \( j \) such as \( K_{t+1} \cap N_{t+1}^j \in E_{t+1} \). Define the network,

\[
E_t = \{ N_{t+1}^p N_{t+1}^q \in E \mid p, q \neq K_{t+1} \} \cup \{ N_{t+1}^j N_{t+1}^{i_s} \} \\
\cup \{ N_{i_1}^t N_{i_2}^t, \ldots, N_{i_{s-1}}^t N_{i_{s-1}}^t, \ldots, N_{i_k}^t N_{i_1}^t \}.
\]

Set \( t = t - 1 \).

This procedure outputs \( E_0 \) which is an MCA over \( N_0 \) for the cost matrix \( C \).

For every \( t = 0, \ldots, T - 1 \) and every \( i \in N \), input

\[
\delta_i^t = \Delta_k \quad \text{where} \quad i \in N_k^t.
\]

For every cost matrix \( C \), the irreducible form of \( C \), denoted by \( C_R \), is defined as follows [6]. For every \( i \in N_0 \) and \( j \in N \setminus i \),

\[
c_{ij}^R := \sum_{t'=0}^{t-1} \delta_{ij}^{t'}
\]

where \( t \) is the first stage when \( i \) and \( j \) belong in the same supernode, namely, \( \{i, j\} \subseteq N_k^t \) for some \( k \) and \( \{i, j\} \not\subseteq N_k^{t-1} \) for any \( k \). In spite of arbitrary selection of cycles and edges in the algorithm, the irreducible cost matrix \( C_R \) is well-defined. Dutta and Mishra states that \( T(C) = T(C_R) \).

Using \( C_R \), we define a cooperative game with a characteristic function. The set of players is the set of agents \( N \). For \( C_R \) and \( S \subseteq N \), the characteristic function \( v_{C_R} \) of \( S \) is defined by the cost of an MCA for the induced subnetwork \( E_S = \{(i, j) \mid i \in S_0, j \in S, i \neq j \} \) and \( C_R \). Dutta and Mishra [6] proposed the cost allocation rule \( f^* \) of a cost matrix \( C \) in terms of the Shapley value of the cost game \((N, v_{C_R})\):

\[
f^*(C) := Sh(N, v_{C_R})
\]

3. Feasible networks

In this section we consider restriction of cooperative relations among agents for an mcts problem and define feasibility of networks. We express restrictions of cooperative relations by a feasible coalition system \( F \subseteq 2^N \setminus \emptyset \) as in the game theory [4]. Every elements in \( F \) is a feasible coalition. We assume that agents in the same feasible coalition can share the supply of a service from the source 0. Hence, for an agent \( i \in N \), to supply the service from 0 to \( i \), all of agents in the path from 0 to \( i \) need to cooperate with \( i \). It leads the idea of feasible \( 0 \)-path.
Definition 3.1. Let $\mathcal{F}$ be a feasible coalition system. A 0-path $0, i_1, \ldots, i_k$ is called feasible for $\mathcal{F}$ or $\mathcal{F}$-feasible if $\{i_1, \ldots, i_k\}, \{i_1, \ldots, i_{k-1}\}, \ldots, \{i_1\} \in \mathcal{F}$.

When a given feasible coalition system $\mathcal{F}$ is obvious, we simply say a 0-path is feasible. Let $\Pi_\mathcal{F}$ be the set of all feasible 0-paths for $\mathcal{F}$.

Remark 3.2. We have defined that a 0-path $0 = 0; i_1; \ldots; i_k$ is feasible if all 0-subpaths of $\pi$ are also feasible. If some 0-subpath $0, i_1, \ldots, i_l$ is not feasible, the service do not reach the agents after $i_l$ along $\pi$.

For a feasible coalition system $\mathcal{F}$, a feasible coalition $S \in \mathcal{F}$ is called reachable in $\mathcal{F}$ if there exists a feasible 0-path $0 = 0; i_1; i_2; \ldots; i_k$ such that $S = \{i_1, i_2, \ldots, i_k\}$. Such a 0-path $\pi$ is called a feasible 0-path of $S$ for $\mathcal{F}$. The set of all reachable sets in $\mathcal{F}$ is denoted by $R(\mathcal{F})$. Additionally, the set of all maximal reachable sets with respect to inclusion is denoted by $Q(\mathcal{F})$.

Example 3.3. We give an example of feasible 0-paths. Let $N = \{1, 2, 3\}$ and $\mathcal{F} = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$.

- We consider feasible 0-paths of agent 1. Since $\{1\} \in \mathcal{F}$, a feasible 0-path is $0, 1$. Since $\{1\}, \{1, 2\} \in \mathcal{F}$, a feasible 0-path is $0, 2, 1$. Therefore feasible 0-paths of 1 are $0, 1$ and $0, 2, 1$.
- We consider feasible 0-paths of agent 2. Since $\{2\} \in \mathcal{F}$, a feasible 0-path is $0, 2$. Since $\{1\}, \{1, 2\} \in \mathcal{F}$, a feasible 0-path is $0, 1, 2$. Therefore feasible 0-paths of 2 are $0, 2$ and $0, 1, 2$.
- We consider feasible 0-paths of agent 3. Since $\{1\}, \{1, 3\} \in \mathcal{F}$, a feasible 0-path is $0, 1, 3$. Therefore a feasible 0-path of 1 is $0, 1, 3$.

Feasible 0-paths can be easily understood by using the set system in Figure 1.

Remark 3.4. For convenience of description we abbreviate a set of some sets such as “$\{\{1\}, \{1, 2\}\}$” to “$\{1, 12\}$”.

We define feasibility of networks from the feasibility of 0-paths. When a family of paths is given, the union of those paths implies the set of edges which are included in one of them.
Definition 3.5. Let $\mathcal{F}$ be a feasible coalition system. A network $E$ is said to be \textit{feasible} for $\mathcal{F}$ or \textit{$\mathcal{F}$-feasible} if $E$ is the union of some feasible 0-paths for $\mathcal{F}$. Moreover, a feasible network $E$ is said to be \textit{strongly $\mathcal{F}$-feasible} if all of 0-paths in $E$ are feasible.

Hence, for each edge $(p, q)$ in a feasible network $E$, there exists a feasible 0-path $0, i_1, \ldots, i_l, i_{l+1}, \ldots, i_k$ such that $i_l = p$ and $i_{l+1} = q$.

Definition 3.6. Let $\mathcal{F}$ be a feasible coalition system. The union of all feasible 0-paths for $\mathcal{F}$ is called the \textit{induced network} of $\mathcal{F}$ or \textit{$\mathcal{F}$-induced network}, and denoted by $E_\mathcal{F}$.

Obviously, $E_\mathcal{F}$ is also feasible. Each feasible network $E$ is included in $E_\mathcal{F}$.

We show the induced network for Example 3.3 in Figure 2. In the rest of this paper, we consider a spanning feasible network $E$ such that $E$ is feasible and for each $i \in N$ there exists a 0-path of $i$ in $E$. The next proposition says that all agents in a spanning feasible network can be supplied the service from the source 0.

Proposition 3.7. Let $\mathcal{F}$ be a feasible coalition system and $E$ be a spanning $\mathcal{F}$-feasible network. For each $i \in N$ there exists an $\mathcal{F}$-feasible 0-path of $i$ in $E$.

Proof. Since $E$ is spanning, for each $i \in N$ there exists a 0-path of $i$. Hence for each $i$ there exists at least one incoming edge, and the edge is a part of some feasible 0-path $\pi$. $\pi$ includes $i$. Hence, there exists a feasible 0-path of $i$ which is a 0-subpath of $\pi$.

When a feasible network is an arborescence, the feasibility and the strong feasibility for networks are reduced to the same.

Proposition 3.8. Let $\mathcal{F}$ be a feasible coalition system and $E$ be an arborescence. $E$ is $\mathcal{F}$-feasible if and only if $E$ is strongly $\mathcal{F}$-feasible.

Proof. (if part) This is obvious. (only if part) Suppose that $E$ is a feasible arborescence. Then for each $i \in N$ there exists exactly one 0-path of $i$, and it is feasible. Hence from Proposition 3.7, all of 0-path in $E$ are feasible.

The following proposition shows a characteristic of feasible arborescences.

Proposition 3.9. Let $\mathcal{F}$ be a feasible coalition system and let $E$ be a network. The following assertions are equivalent:

(a): $E$ is an $\mathcal{F}$-feasible arborescence,

(b): $E$ is a spanning strongly $\mathcal{F}$-feasible network and minimal with respect to inclusion,

(c): $E$ is feasible and for each $i \in N$ there exists exactly one $\mathcal{F}$-feasible 0-path of $i$ in $E$. 
Proof. (a ⇒ b) From Proposition 3.8, if \( E \) is a feasible arborescence then \( E \) is a spanning strongly feasible network. Since \( E \) is an arborescence, if some edges \( e \) of \( E \) is removed then \( E \setminus \{e\} \) is not spanning. It implies that \( E \) is minimal.

(b ⇒ c) Suppose that the condition of (b) holds. Then for each \( i \in N \) there exists exactly one feasible 0-path of \( i \) in \( E \). In fact, if there are more than one feasible 0-path of \( i \), we can remove one incoming edge corresponding to some feasible 0-paths of \( i \) preserving that \( E \) is spanning and strongly feasible.

(c ⇒ a) Suppose that the condition of (c) holds. If there are more than one incoming edge for some \( i \in N \) then there are more than one feasible 0-path of \( i \) in \( E \) because of \( E \)'s feasibility. That contradicts the condition of (c). Therefore \( E \) is an arborescence.

The sets of all spanning feasible networks and feasible arborescences are denoted by \( \mathcal{S}_F \) and \( \mathcal{A}_F \), respectively.

4. Existence of spanning feasible networks and arborescences

In this section we study the existence of spanning feasible networks. For an mcstp without restrictions on coalitions, spanning networks and arborescences always exist and minimum cost networks should be arborescences. On the contrary, for an mcstp with restrictions on coalitions, feasible arborescences do not necessarily exist, and even if there exist some feasible arborescences, minimum cost spanning feasible networks are not arborescences. Furthermore, even spanning feasible networks may not exist. Therefore, we consider the existence of spanning feasible networks as well as feasible arborescences.

First of all, we show direct conditions for the existence of spanning feasible networks.

Proposition 4.1. For a feasible coalition system \( F \), the following statements hold.

(a): \( \mathcal{S}_F \neq \emptyset \) if and only if \( E_F \) is spanning.

(b): \( \mathcal{A}_F \neq \emptyset \) if and only if there exists a spanning strongly \( F \)-feasible network.

Proof. (a) (if part) This is obvious. (only if part) Let \( E \in \mathcal{S}_F \). Since \( E \) is spanning, \( E_F \supseteq E \) is also spanning.

(b) (only if part) This is obvious. (if part) Consider the set of spanning strongly \( F \)-feasible networks, and select a minimal one with respect to inclusion. It is an \( F \)-feasible arborescence by Proposition 3.9.

In this paper, we investigate conditions of feasible coalition systems \( F \) which guarantee the existence of spanning feasible networks. We introduce several
properties of $\mathcal{F}$ some of which have been proposed in cooperative game theory \cite{1,4,10}.

**Definition 4.2.** \cite{1} A feasible coalition system $\mathcal{F}$ is said to satisfy normality (NO) if it holds that $\bigcup_{S \in \mathcal{F}} S = N$.

This property implies each $i \in N$ is necessarily in at least one member of $\mathcal{F}$.

**Definition 4.3.** \cite{1} A feasible coalition system $\mathcal{F}$ is said to satisfy accessibility (AC) if for each $S \in \mathcal{F}$ which is not singleton, there exists $i \in S$ such that $S \setminus i \in \mathcal{F}$.

This property implies the existence of a sequence $S_1, S_2, \ldots, S_k$ for each $S \in \mathcal{F}$ where $S_1$ is a singleton, $S_k = S$ and $|S_{l+1} \setminus S_l| = 1$ for each $l = 1, 2, \ldots, k - 1$.

**Theorem 4.4.** A feasible coalition system $\mathcal{F}$ satisfies (NO) and (AC) then $\mathcal{A}_\mathcal{F} \neq \emptyset$.

**Proof.** Consider any feasible coalition $S \in \mathcal{F}$. Suppose $|S| = k$. Since $\mathcal{F}$ satisfies (AC), there exists $i_k \in S$ such that $S \setminus i_k = S_{k-1}$ then $S_{k-1} \in \mathcal{F}$. Similarly there also exists $i_{k-1} \in S_{k-1}$ such that $S_{k-1} \setminus i_{k-1} = S_{k-2} \in \mathcal{F}$. Also there also exists $i_2 \in S_2$ and $S_2 \setminus i_2 = S_1$. Repeating this operation, we have the feasible 0-path $0, i_1, \ldots, i_k$. If $\{i_1, i_2, \ldots, i_k\} \neq N$, there exists $S' \in \mathcal{F}$ such that $S' \cap (N \setminus S) \neq \emptyset$ by (NO). Suppose $|S'| = k'$. By the above operation, we have the feasible 0-path $0, i'_1, i'_2, \ldots, i'_{k'}$, where $S' = \{i'_1, i'_2, \ldots, i'_{k'}\}$. Repeating this operation, for each $i \in N$, there exists at least one obtained feasible 0-path. Take the union of those obtained feasible 0-paths, the network is spanning.

However, only (NO) and (AC) do not ensure $\mathcal{A}_\mathcal{F} \neq \emptyset$ as shown in the following example.

**Example 4.5.** Let $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{F} = \{1, 2, 12, 23, 135, 235\}$. $\mathcal{F}$ satisfies (AC) and (NO). There exists a spanning feasible network $\{(0, 1), (1, 3), (3, 4), (0, 2), (2, 3), (3, 5)\}$ but does not exist a feasible arborescence.

If $\mathcal{F}$ additionally satisfies the weakly-union closed property then we have $\mathcal{A}_\mathcal{F} \neq \emptyset$.

**Definition 4.6.** \cite{10} A feasible coalition system $\mathcal{F}$ is said to satisfy weak union closedness (WUC) if $S \cup T \in \mathcal{F}$ for all $S, T \in \mathcal{F}$ such that $S \cap T \neq \emptyset$.

This property implies the union of two sets which have intersection of them is included in $\mathcal{F}$. To show the existence of feasible arborescences, we use the following lemma.
Lemma 4.7. A feasible coalition system $\mathcal{F}$ satisfies (AC) and (WUC) then any two elements of $Q(\mathcal{F})$ are disjoint.

Proof. Since $\mathcal{F}$ satisfies (AC), $\mathcal{F}$ is included in $R(\mathcal{F})$. It is clear that $\mathcal{F} = R(\mathcal{F})$. Since $\mathcal{F}$ satisfies (WUC), any two maximal elements of $\mathcal{F}$ are disjoint. Hence any two maximal elements of $R(\mathcal{F})$, that is any two elements of $Q(\mathcal{F})$, are disjoint. \hfill \Box

Theorem 4.8. A feasible coalition system $\mathcal{F}$ satisfies (NO), (AC) and (WUC) then $A_{\mathcal{F}} \neq \emptyset$.

Proof. Since $\mathcal{F}$ satisfies (AC) and (WUC), from Lemma 4.7, all $S \in Q(\mathcal{F})$ are mutually disjoint. Since $\mathcal{F}$ satisfies (NO), elements of $Q(\mathcal{F})$ form a partition of $N$. For each $S \in Q(\mathcal{F})$ there exists a feasible 0-path which is a sequence of the numbers in $S$. The union of one of such feasible 0-paths for all $S \in Q(\mathcal{F})$ is a spanning feasible network. Moreover, since any two elements of $Q(\mathcal{F})$ is mutually disjoint, for each $i \in N$, there exists exactly one 0-path of $i$. Hence, it is a feasible arborescence. \hfill \Box

Now we give some examples which show that (NO) and (AC) cannot be dropped when $\mathcal{F}$ is (WUC) in Theorem 4.8.

Example 4.9. Let $(N_0, C)$ be such that $N = \{1, 2, 3, 4\}$, $\mathcal{F} = \{1, 2, 12, 23, 123\}$. $\mathcal{F}$ satisfies (AC) and (WUC). It does not satisfy (NO). There does not exist a spanning feasible network and an arborescence because an agent 4 is autonomous.

Example 4.10. Let $(N_0, C)$ be such that $N = \{1, 2, 3, 4\}$, $\mathcal{F} = \{12, 34, 1234\}$. $\mathcal{F}$ satisfies (NO) and (WUC). It does not satisfy (AC). There does not exist a spanning feasible network and an arborescence because there do not exist feasible 0-paths of all agents.

Now, we consider another sufficient condition for $A_{\mathcal{F}} \neq \emptyset$. As a consequence of Proposition 4.1, if $E_{\mathcal{F}}$ is a spanning strongly feasible network then $A_{\mathcal{F}} \neq \emptyset$. Therefore, a sufficient condition of $\mathcal{F}$ where $E_{\mathcal{F}}$ becomes a spanning strongly feasible network is also a sufficient condition for $A_{\mathcal{F}} \neq \emptyset$. Accordingly, we propose the following condition.

Definition 4.11. A feasible coalition system $\mathcal{F}$ is said to satisfy exchangeability (E) if $S, T \in \mathcal{F}$, $S \cap T \neq \emptyset$, $S \cup R \in \mathcal{F}$, and $S \cap R = \emptyset$ then $T \cup R \in \mathcal{F}$.

Consider the coalition when some members can join to one of the coalitions, then this property implies that new members can also join to the other coalition.

There is a closed relationship between the condition (E) and strong feasibility. Therefore we show another aspect of strong feasibility.
Definition 4.12. Let \( \mathcal{F} \) be a feasible coalition system. A network \( E \) is said to satisfy path exchangeability (PE) if for any two \( \mathcal{F} \)-feasible 0-paths \( 0, i_1, \ldots, i_k, i \) (\( k \geq 1 \)) and \( 0, j_1, j_2, \ldots, j_l \) (\( l \geq 1 \)) such that \( j_1, \ldots, j_l \) are all different from \( i \) and \( i_k = j_l \) the path \( 0, j_1, \ldots, j_l, i \) is also \( \mathcal{F} \)-feasible.

Proposition 4.13. Let \( \mathcal{F} \) be a feasible coalition system and \( E \) be an \( \mathcal{F} \)-feasible network. \( E \) is strongly \( \mathcal{F} \)-feasible if and only if \( E \) satisfies (PE).

Proof. (only if part) This is obvious. (if part) We assume that \( E \) is not strongly \( \mathcal{F} \)-feasible, that is, there exists some infeasible 0-path in \( E \). Notice that any edge \((0, i)\) in \( E \) is feasible. Therefore we can take a feasible 0-path \( 0, j_1, \ldots, j_l \) (\( l \geq 1 \)) and an infeasible 0-path \( 0, j_1, \ldots, j_l, i \). Since \((j_l, i)\) in \( E \), there exists a feasible 0-path containing the edge. We assume that the 0-path is \( 0, i_1, \ldots, i_k, i \) with \( i_k = j_l \). Since \( E \) satisfies (PE), a 0-path \( 0, j_1, \ldots, j_l, i \) is also feasible. This contradicts that a 0-path \( 0, j_1, \ldots, j_l, i \) is not feasible. \( \square \)

Theorem 4.14. If a feasible coalition system \( \mathcal{F} \) satisfies (E) then the \( \mathcal{F} \)-induced network \( E_\mathcal{F} \) satisfies (PE). Moreover, if \( \mathcal{F} \) satisfies (NO), (AC) and (E) then \( E_\mathcal{F} \) is a spanning strongly \( \mathcal{F} \)-feasible network.

Proof. First, we show that if \( \mathcal{F} \) satisfies (E) then \( E_\mathcal{F} \) satisfies (PE). We consider \( \mathcal{F} \)-feasible paths \( 0, i_1, \ldots, i_k, i \) (\( k \geq 1 \)) and \( 0, j_1, \ldots, j_l \) (\( l \geq 1 \)) such that \( j_1, \ldots, j_l \) are all different from \( i \) and \( i_k = j_l \). We call the path \( i_1, \ldots, i_k \) as the set \{\( i_1, \ldots, i_k \)\} when there is no ambiguity. We take \( S = \{i_1, \ldots, i_k\} \), \( T = \{j_1, \ldots, j_l\} \) and \( R = \{i\} \) in the assumption of (E). Since \( \mathcal{F} \) satisfies (E), \( \{j_1, \ldots, j_l, i\} \in \mathcal{F} \). Therefore the path \( 0, j_1, \ldots, j_l, i \) is feasible and hence \( E_\mathcal{F} \) satisfies (PE).

Second, suppose that if \( \mathcal{F} \) satisfies (NO), (AC) and (E). We prove that \( E_\mathcal{F} \) is a spanning strongly feasible network. Since \( \mathcal{F} \) satisfies (E) then \( E_\mathcal{F} \) satisfies (PE) from the first part of this theorem. Additionally from Proposition 4.13, since \( E_\mathcal{F} \) satisfies (PE), \( E_\mathcal{F} \) is strongly \( \mathcal{F} \)-feasible. Therefore since \( \mathcal{F} \) satisfies (E) then \( E_\mathcal{F} \) is strongly \( \mathcal{F} \)-feasible. Finally, since \( \mathcal{F} \) satisfies (NO) and (AC), from Theorem 4.4 and Proposition 4.1 (a), \( E_\mathcal{F} \) is spanning. Hence, \( E_\mathcal{F} \) is a spanning strongly feasible network. \( \square \)

Corollary 4.15. If a feasible coalition system \( \mathcal{F} \) satisfies (NO), (AC) and (E) then \( \mathcal{A}_\mathcal{F} \neq \emptyset \).

Proof. Since \( \mathcal{F} \) satisfies (NO), (AC) and (E), from Theorem 4.14, \( E_\mathcal{F} \) is a spanning strongly feasible network. Therefore from Proposition 4.1 (b), \( \mathcal{A}_\mathcal{F} \neq \emptyset \). \( \square \)

Finally we consider a relationship between (E) and (WUC). Let \((N_0, C)\) be such that \( N = \{1, 2, 3, 4\} \), \( \mathcal{F} = \{1, 2, 13, 23, 134\} \). \( \mathcal{F} \) satisfies (NO), (AC) and
(E) but not (WUC). It is more reasonable to suppose (E) than (WUC) when we consider the arborescences.

5. Feasible minimum cost arborescences

In this section, we study cost allocation for an mst (N₀, C) with a feasible coalition system F. However, for a general coalition system, it is difficult to propose rational allocation rules, because we do not have clear perspectives to minimum cost spanning feasible networks. Hence, we deal with a coalition system whose induced network is spanning strongly feasible. In that case, the following proposition shows that we can easily deal with minimum cost spanning feasible networks.

**Proposition 5.1.** Let F be a feasible coalition system and E be a spanning network. Suppose that the F-induced network E_F is spanning strongly feasible. E is a subnetwork of E_F if and only if E is an F-feasible network. Therefore, for any cost matrix C, all of minimum cost spanning F-feasible networks are arborescences which are subnetworks of E_F.

**Proof.** Since E is a spanning subnetwork of E_F, for each edge in E there exists a 0-path in E which includes the edge. It is feasible because E_F is strongly feasible. The union of those 0-paths recovers E. Therefore, E is feasible. The converse is obvious.

We can apply Dutta and Mishra’s approach [6] by modifying the cost matrix C such that if (i, j) ∉ E_F then the value of c_{ij} is replaced some large number. Because Edmonds’ algorithm shown above successfully finds an MCA avoiding the edges excluded by E_F, and then we can have the irreducible cost matrix C^R.

**Example 5.2.** Let (N₀, C) be such that N = {1, 2, 3, 4} and c_{01} = 2, c_{02} = 3, c_{13} = 4, c_{14} = 5, c_{24} = 6, c_{34} = c_{43} = 1 otherwise c = ∞. Figure 3(a) illustrates mst (N₀, C). Let feasible coalition system be F = {1, 2, 13, 14, 24, 134, 234} and F satisfies (NO), (AC) and (E). It should be emphasized that E_F is spanning strongly F-feasible. We consider a network connection problem. We use Edmonds’ algorithm to compute the MCA.

**Stage(1)-0:** Set N₀ = N₀, Δ₁ = 2, Δ₂ = 3, Δ₃ = 1, Δ₄ = 1

\[ \tilde{c}_{01} = 0, \tilde{c}_{02} = 0, \tilde{c}_{13} = 3, \tilde{c}_{14} = 4, \tilde{c}_{24} = 5, \tilde{c}_{34} = \tilde{c}_{43} = 0 \text{ otherwise} \]

\[ \tilde{c}^1 = \infty \]

**Stage(1)-1:** Construct a partition N¹ = {{0}, {1}, {2}, {3, 4}}.

\[ \tilde{c}^1_{01} = 0, \tilde{c}^1_{02} = 0, \tilde{c}^1_{1[3,4]} = 3, \tilde{c}^1_{2[3,4]} = 5 \]

\[ \Delta^1_0 = 0, \Delta^1_2 = 0, \Delta^1_4 = 3 \]
Update $\tilde{C}^1$ by $\tilde{c}^1_{01} = 0$, $\tilde{c}^1_{02} = 0$, $\tilde{c}^1_{1(3,4)} = 0$, $\tilde{c}^1_{2(3,4)} = 2$.

Stage(2)-1: Set $E^1 = \{(0, 1), (0, 2), (1, 3), (1, 4), (2, 3), (3, 4)\}$.

Stage(2)-0: $E^0 = \{(0, 1), (0, 2), (1, 3), (3, 4)\}$.

Hence, the MCA corresponding to the original cost matrix is $\{(0, 1), (0, 2), (1, 3), (3, 4)\}$. Figure 3(c) show the network before constructing a partition $N^1$. Figure 3(d) show the network before update $\tilde{C}^1$. Figure 3(e) show the network after update $\tilde{C}^1$. Figure 3(f) show the MCA corresponding to the original cost matrix.

Next we consider cost allocation problems. We use the cost allocation rules $f^*$ presented by Dutta and Mishra [6]. In consequence

$$f^*_1 = 2, \quad f^*_2 = 3, \quad f^*_3 = 2.5, \quad f^*_4 = 2.5$$

However, the question remains as allocating costs regardless of distance of source. Also we need to study characterization of their cost allocation rule with restrictions on coalitions.

6. Conclusions

This paper considered minimum cost spanning tree problems with restrictions on coalitions. We expressed restrictions by a set system $\mathcal{F}$, that was called a feasible coalition system. In addition, we defined feasible 0-paths, feasible networks, strongly feasible networks, and induced networks. Although an mctp was discussed on undirected networks in conventional studies, when we considered for $\mathcal{F}$, undirected networks could not express restrictions. Hence we considered an mctp on directed networks for $\mathcal{F}$. Also since we dealt with a network that supplies of service from 0 to all agents, this paper considered MCA rooted at 0.

For an mctp with restrictions on coalitions, feasible arborescences did not necessarily exist and even if there existed some feasible arborescences, minimum cost feasible networks were not arborescences. Furthermore spanning feasible networks may not exist. Therefore we consider existence of spanning feasible networks as well as feasible arborescences. We introduce several properties of $\mathcal{F}$, that is normality, accessibility and weakly union closed. Using these properties, we show sufficient conditions for the existence of spanning feasible networks and arborescences for mctp with restrictions on coalitions. Next we introduce a new property of $\mathcal{F}$, that is exchangeability. Furthermore we introduce path exchangeability. Then we give a different sufficient condition that existence of feasible arborescences.

Finally, we applied Dutta and Mishra’s approach by modifying the cost matrix. Also we gave the example then we consider an MCA via Edmonds’ algorithm and Dutta and Mishra’s cost allocation rule.
As future research, we consider different feasibilities from this paper. Also this paper has not studied cost allocation problems enough. Therefore we need to propose characterizations of cost allocation rules with restrictions on coalitions. Furthermore we study an extension of MCA into the game theory.
References


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