STRONG CONVERGENCE THEOREMS FOR FINITE FAMILIES OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In 1979, Ishikawa [12] presented an article “Common fixed points and iteration of commuting nonexpansive mappings”. The authors feel that it is not so easy to read the article. However, in the article, there are many interesting contents which are not arranged sufficiently. Therefore, we organize these contents with the hope that one can read it. Also, we refer to a part of Suzuki’s idea in [22].

1. Introduction

In 1963, DeMarr proved the following theorem.

Theorem 1.1 (DeMarr, 1963, [6]). Let \(D\) be a compact convex subset of a Banach space \(E\). Let \(\{T_j\}_{j \in J}\) be a family of commuting nonexpansive self-mappings on \(D\), that is, \(T_iT_j = T_jT_i\) for \(i, j \in J\). Then, \(\bigcap_{j \in J} F(T_j) \neq \emptyset\).

Motivated by DeMarr’s work, Ishikawa proved the following theorem and had another proof of Theorem 1.1.

Theorem 1.2 (Ishikawa, 1979, [12]). Let \(a\) be a real number belonging to \((0, 1)\). Let \(D\) be a compact convex subset of a Banach space \(E\). Let \(\{T_1, T_2, \cdots, T_k\}\) be a finite sequence of commuting nonexpansive self-mappings on \(D\). Let \(x_1 \in D\) and \(\{x_n\}\) be a sequence in \(D\) defined by

\[
x_n = \left[ \prod_{n_{k-1} \in S_1}^{n_k} \prod_{n_{k-2} \in S_2}^{n_{k-1}} \cdots \prod_{n_1 \in S_1}^{n_2} x_1 \right]
\]

for \(n \in N\), where \(S_i = aT_i + (1-a)I\) for \(i \in N(1, k)\). Then, \(\{x_n\}\) converges strongly to a common fixed point of \(\{T_1, T_2, \cdots, T_k\}\).

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We feel that contents in [12] are not arranged sufficiently. Then, we organize these with the hope that one can read it. Motivated by Ishikawa’s work, Suzuki [22] proved a strong convergence theorem to find a common fixed point for an infinite family of commuting nonexpansive mappings in a Banach space. We refer to a part of his idea. In the processes, we add something new.

2. Preliminaries

In this article, we denote by \( \mathbb{R} \) the set of real numbers, by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{N}_0 \) the set of nonnegative integers. For \( i, j \in \mathbb{N}_0 \) satisfying \( i \leq j \), \( N(i, j) \) denotes the set \( \{ k \in \mathbb{N}_0 : i \leq k \leq j \} \). We denote by \( E \) a real Banach space. For simplicity, we remove “real”. Let \( D \) be a nonempty subset of a Banach space \( E \). Let \( T \) be a mapping from \( D \) into \( E \). We denote by \( F(T) \) the set of fixed points of \( T \), that is, \( F(T) = \{ x \in D : Tx = x \} \). A mapping \( T \) is said to be nonexpansive if \( \| Tx - Ty \| \leq \| x - y \| \) for all \( x, y \in D \). We denote by \( I \) the identity mapping on \( E \).

Let \( D \) be a subset of a Banach space \( E \) and \( \{ T_i \} \) be a sequence of self-mappings on \( D \). For convenience, we introduce a symbol \( \Pi_{i=1}^{n} T_i \) such that \( \Pi_1 = T_1 \), \( \Pi_{n+1} = T_{n+1} \Pi_n \) for \( n \in \mathbb{N} \). For simplicity, we replace \( \Pi_n \) by \( \Pi_n \) if it does not cause confusion. Usually, \( \Pi_n \) is written by \( \prod_{i=1}^{n} T_i \). We present some examples:

\[
\begin{align*}
\pi(ST_n) &= (ST_n)(ST_{n-1}) \cdots (ST_1), \\
\pi(S_2 T_2) &= (S_2 T_2)(S_1 T_2), \\
\pi(S_k T_{(i, j)}) &= (S_k T_{(i,j)})(S_k T_{(i,4)}) \cdots (S_k T_{(1,1)}), \\
\pi(S_k T_{u_1} U_1) &= (S_k T_{u_1} U_1)(S_k T_{u_{n-1}} U_1) \cdots (S_k T_{1} U_1).
\end{align*}
\]

Here, we present basic facts as lemmas. Sometimes, we may use these facts without notice in later sections.

**Lemma 2.1.** Let \( D \) be a compact subset of a Banach space \( E \) and let \( T \) be a nonexpansive self-mapping of \( D \) into \( E \). Then, \( F(T) \) is compact.

**Proof.** Let \( \{ z_n \} \) be a sequence in \( F(T) \). Assume that \( \{ z_n \} \) converges strongly to some \( z \in E \). Then, \( Tz_n = z_n \) and \( \| Tz_n - Tz \| \leq \| z_n - z \| \) for any \( n \in \mathbb{N} \). We have \( z \in F(T) \) from the following inequality:

\[ \| z - Tz \| \leq \| z - Tz_n \| + \| Tz_n - Tz \| \leq 2 \| z - z_n \|. \]

Then, \( F(T) \) is closed subset of \( D \). That is, \( F(T) \) is also compact.

In the case that \( D \) is convex in Lemma 2.1, we cannot have that \( F(T) \) is convex in general; see Demarr [6].
Lemma 2.2. Let \( \{\alpha_n\} \) be a sequence in \([0, 1]\). Let \( D \) be a compact convex subset of a Banach space \( E \) and let \( T \) and \( P \) be nonexpansive self-mappings on \( D \). Let \( \{S_n\} \) and \( \{Q_n\} \) be sequences of mappings on \( D \) defined by

\[
S_n = \alpha_n T + (1 - \alpha_n) I, \quad Q_n = \pi(S_n P) \quad \text{for} \quad n \in N.
\]

Then the followings hold:

1. For \( i \in N \), \( S_i \) is a nonexpansive self-mapping on \( D \).
2. For \( i \in N \), \( F(T) \subset F(S_i) \) if \( \alpha_i > 0 \).
3. \( \{Q_n\} \) is a sequence of nonexpansive self-mappings such that

\[
F(T) \cap F(P) \subset \cap_n F(Q_n).
\]

Proof. Since \( D \) is convex, \( S_n \) and \( Q_n \) are self-mappings on \( D \) for \( n \in N \).

We prove (1). Since \( T \) is nonexpansive, we have that, for \( x, y \in D \),

\[
\|S_i x - S_i y\| = \|(\alpha_i T x + (1 - \alpha_i)x) - (\alpha_i T y + (1 - \alpha_i)y)\| \\
\leq \alpha_i \|T x - T y\| + (1 - \alpha_i)\|x - y\| \\
\leq \alpha_i \|x - y\| + (1 - \alpha_i)\|x - y\| = \|x - y\|.
\]

We prove (2). We have \( S_i x = \alpha_i T x + (1 - \alpha_i)x = \alpha_i x + (1 - \alpha_i)x = x \) if \( T x = x \). Suppose \( \alpha_i > 0 \) and \( S_i x = x \). Then, \( S_i x = \alpha_i T x + (1 - \alpha_i)x = x \) and \( \alpha_i T x = \alpha_i x \). Thus, \( T x = x \). We prove (3). By \( Q_n = \pi(S_n P) \) and (1), \( \{Q_n\} \) is a sequence of nonexpansive self-mappings on \( D \). Let \( x \in F(T) \cap F(P) \). Then,

\[
F(T) \cap F(P) \subset F(S_i) \cap F(P)
\]

for \( i \in N \). Then \( S_i P x = x \) for \( i \in N \). This implies that \( Q_n x = x \) for \( n \in N \). We have \( F(T) \cap F(P) \subset \cap_n F(Q_n) \).

Lemma 2.3. Let \( D \) be a subset of a Banach space \( E \). Let \( \{Q_n\} \) be a sequence of nonexpansive self-mappings on \( D \). Assume that, for any \( x \in D \), \( \{Q_n x\} \) converges strongly to some \( u_x \in D \). Let \( Q \) be a self-mapping on \( D \) defined by \( Q x = u_x \) for \( x \in D \). Then, the followings hold:

1. \( Q \) is nonexpansive.
2. \( \{Q_n\} \) converges uniformly to \( Q \) if \( D \) is compact.

Proof. We prove (1). Let \( y, z \in D \). Since \( Q_n \) is nonexpansive, we have

\[
\|Q y - Q z\| \leq \|Q y - Q_n y\| + \|Q_n y - Q_n z\| + \|Q_n z - Q z\| \\
\leq \|Q y - Q_n y\| + \|y - z\| + \|Q_n z - Q z\|
\]

for \( n \in N \). Then, by the definition of \( Q \), we have that \( Q \) is nonexpansive.

We prove (2). Let \( \varepsilon > 0 \). Since \( D \) is compact, there exists a finite set \( A_\varepsilon = \{z_1, \ldots, z_k\} \subset D \) such that \( \min\{\|x - z\| : z \in A_\varepsilon\} < \varepsilon/3 \) for \( x \in D \). Since \( A_\varepsilon \) is a finite set, there exists \( n_0 \in N \) such that, for \( n > n_0 \),

\[
\|Q_n z - Q z\| < \varepsilon/3 \quad \text{for} \quad z \in A_\varepsilon.
\]
Fix $n > n_0$ arbitrary. For any $x \in D$, there is $z_x \in A_x$ with $\|x - z_x\| < \varepsilon/3$. Since $Q_n$ and $Q$ are nonexpansive, we have

$$
\|Q_n x - Q x\| \leq \|Q_n x - Q_n z_x\| + \|Q_n z_x - Q z_x\| + \|Q z_x - Q x\|
$$

Thus $\{Q_n\}$ converges uniformly to $Q$. 

**Lemma 2.4.** Let $D$ be a subset of a Banach space $E$. Let $\{T_1, T_2, \cdots, T_k\}$ be a finite sequence of commuting self-mappings on $D$, that is, $T_i T_j = T_j T_i$ for $i, j \in N(1, k)$. Then, the following holds:

$$
T_{i+1} (\cap_{j=1}^i F(T_i)) \subseteq \cap_{j=1}^i F(T_j) \quad \text{for} \quad l \in N(1, k-1).
$$

**Proof.** Let $l \in N(1, k-1)$ and $x \in \cap_{j=1}^l F(T_i)$. Then, $x \in F(T_i)$ for $i \in N(1, l)$. It is easy to see that

$$
T_{i+1} x = T_{i+1} T_i x = T_i T_{i+1} x \quad \text{for} \quad i \in N(1, l).
$$

That is, $T_{i+1} x \in F(T_i)$ for any $i \in N(1, l)$. Thus, we have the result. 

3. Lemmas

We begin by preparing the following notations. Let $b$ be a real number belonging to $(0, 1)$ and $\{\alpha_m\}$ be a sequence in $[0, b]$. We set $\delta_i = 1 - \alpha_i$ for all $i \in N$ and $A = 1/(1 - b)$. For any $n, k \in N$, we define $\alpha_n(k)$ and $\delta_n(k)$ by

$$
\alpha_n(k) = \alpha_n + \alpha_{n+1} + \cdots + \alpha_{n+k-1}, \quad \delta_n(k) = 1/((\delta_n \cdots \delta_{n+k-1}).
$$

We present lemmas connected with a structure of sequences under study; refer to Goebel and Kirk [8], Ishikawa [11], Suzuki [22] and Takahashi [23].

**Lemma 3.1.** Let $b$ be a real number belonging to $(0, 1)$ and $\{\alpha_m\}$ be a sequence in $[0, b]$. For any $n, k \in N$, the following inequality holds:

$$
\delta_n(k) \leq e^{\alpha_n(k) A} < e^{(1 + \alpha_n(k)) A}.
$$

**Proof.** Let $h$ be a function on $[0, \infty)$ defined by $h(x) = x - \log(1 + x)$ for $x \in [0, \infty)$. Then, $h(x) \geq 0$ for all $x \in [0, \infty)$. We easily have

$$
\frac{1}{\alpha_i} = \frac{1}{\frac{1}{\alpha_i} - \frac{1}{1 - \alpha_i}} = 1 + \frac{\alpha_i}{1 - \alpha_i} \leq 1 + \frac{\alpha_i}{\frac{1}{A}} = 1 + \alpha_i A
$$

for $i \in N$. Thus we have the following inequality:

$$
\log \delta_n(k) = \log \frac{1}{\delta_n \cdots \delta_{n+k-1}} = \log \frac{1}{\delta_n} + \log \frac{1}{\delta_{n+1}} + \cdots + \log \frac{1}{\delta_{n+k-1}}
\leq \log(1 + \alpha_n A) + \log(1 + \alpha_{n+1} A) + \cdots + \log(1 + \alpha_{n+k-1} A)
\leq \alpha_n A + \alpha_{n+1} A + \cdots + \alpha_{n+k-1} A = \alpha_n(k) A.
$$

Since $e^x$ is strictly increasing, we have $\delta_n(k) \leq e^{\alpha_n(k) A} < e^{(1 + \alpha_n(k)) A}$. 

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Lemma 3.2. Let \( b \) be a real number belonging to \((0, 1)\) and \( \{\alpha_m\} \) be a sequence in \([0, b]\). Let \( \{u_m\} \), \( \{v_m\} \) and \( \{w_m\} \) be sequences in a Banach space \( E \). Suppose, for some \( n, k \in N \), the following conditions hold:

1. \( u_{i+1} = \alpha_i w_i + (1 - \alpha_i)v_i \) \( \forall i \in N(n, n+k-1) \).
2. There exists a nonnegative real number \( l_{(n,k)} \) such that
   \[ \|w_{i+1} - w_i\| \leq \alpha_i\|w_i - v_i\| + l_{(n,k)} \] \( \forall i \in N(n, n+k-1) \).
3. \( \|w_i - v_j\| \leq \|w_i - u_j\| \) \( \forall i, j \in N(n, n+k) \).

Then, the following inequality holds:

\[ (1 + \alpha_n(k))d_n \leq \|w_{n+k} - v_n\| + (\varepsilon_n(k) + k^3 l_{(n,k)})e^{(1+\alpha_n(k))A}, \]

where \( d_n = \|w_n - v_n\| \) and \( \varepsilon_n(k) = \|w_n - v_n\| - \|w_{n+k} - v_{n+k}\| \).

Proof. By (1) and (3), for \( i \in N(n, n+k-1), j \in N(n, n+k) \), we have

(a) \[ \|w_j - v_{i+1}\| \leq \|w_j - u_{i+1}\| = \|w_j - \alpha_i w_i - (1 - \alpha_i)v_i\| \leq \alpha_i\|w_j - w_i\| + (1 - \alpha_i)\|w_j - v_i\|. \]

Then, by (2) and (3), we have that, for \( i \in N(n, n+k-1), j \in N(n, n+k) \),

\[ \|w_{i+1} - v_{i+1}\| \leq \|w_{i+1} - u_{i+1}\| \leq \|w_{i+1} - w_i\| + \|w_i - u_{i+1}\| \leq \alpha_i\|w_i - v_i\| + l_{(n,k)} + (1 - \alpha_i)\|w_i - v_i\| \]

Thus, for \( i \in N(n, n+k-1) \), the following inequalities hold.

(b) \[ \|w_i - v_i\| \leq \|w_n - v_n\| + (k - 1)l_{(n,k)} = d_n + (k - 1)l_{(n,k)}. \]

(c) \[ \|w_{i+1} - w_i\| \leq \alpha_i\|w_i - v_i\| + l_{(n,k)} \leq \alpha_i d_n + kl_{(n,k)}. \]

By using (a),(c), we prove (*) for any \( j \in N(0, k-1) \).

(*) \[ \varepsilon_n(k) + (k-j)k^3 l_{(n,k)} + (1 + \alpha_n+j + \cdots + \alpha_{n+k-1})d_n \leq \|w_{n+k} - v_{n+j}\|. \]

We show (*) for \( j = k - 1 \). By \( \varepsilon_n(k) = d_n - \|w_{n+k} - v_n\| \), we have

\[ -\varepsilon_n(k) + d_n \leq \|w_{n+k} - v_n\| \leq \|w_{n+k} - w_{n+k-1}\| \]

\[ \leq \alpha_{n+k-1}\|w_{n+k} - w_{n+k-1}\| + (1 - \alpha_{n+k-1})\|w_{n+k} - v_{n+k-1}\| \]

\[ \leq \alpha_{n+k-1}(\alpha_{n+k-1}d_n + kl_{(n,k)}) + (1 - \alpha_{n+k-1})\|w_{n+k} - v_{n+k-1}\| \]

\[ \leq \alpha_{n+k-1}^2 d_n + (1 - \alpha_{n+k-1})\|w_{n+k} - v_{n+k-1}\| + kl_{(n,k)}. \]

Then, by \( k \leq k^2 \), we have

\[ -\varepsilon_n(k) - k^3 l_{(n,k)} + (1 - \alpha_{n+k-1}^2)d_n \leq (1 - \alpha_{n+k-1})\|w_{n+k} - v_{n+k-1}\|. \]
Dividing by \((1 - \alpha_{n+k-1}) = \delta_{n+k-1}\), we have
\[
-\frac{\varepsilon_n(k) + 1 + k^2 \ell(n,k)}{\delta_{n+k-1}} + (1 + \alpha_{n+k-1}) d_n \leq \|w_{n+k} - v_{n+k-1}\|.
\]
For \(j \in N(1, k - 1)\), assume that (*) holds. By (c), we have
\[
\|w_{n+k} - w_{n+j-1}\| \leq \|w_{n+j} - w_{n+j-1}\| + \cdots + \|w_{n+k} - w_{n+j-1}\| \\
\leq (\alpha_{n+j-1} d_n + k\ell(n,k)) + \cdots + (\alpha_{n+k-1} d_n + k\ell(n,k)) \\
\leq (\alpha_{n+j-1} + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) d_n + k^2 \ell(n,k).
\]
Then, we have
\[
-\frac{\varepsilon_n(k) + (j-1) k^2 \ell(n,k)}{\delta_{n+j-1} \cdot \delta_{n+k-1}} + (1 + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) d_n \\
\leq \|w_{n+k} - v_{n+j}\| \\
\leq \alpha_{n+j-1} \|w_{n+k} - w_{n+j-1}\| + (1 - \alpha_{n+j-1}) \|w_{n+k} - v_{n+j-1}\| \\
\leq \alpha_{n+j-1} (\alpha_{n+j-1} + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) d_n + k^2 \ell(n,k) \\
+ (1 - \alpha_{n+j-1}) \|w_{n+k} - v_{n+j-1}\|.
\]
Transpose \(\alpha_{n+j-1} (\alpha_{n+j-1} + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) d_n, k^2 \ell(n,k)\) to the left side and divide by \((1 - \alpha_{n+j-1}) = \delta_{n+j-1}\). We note that
\[
(1 + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) - \alpha_{n+j-1}(\alpha_{n+j-1} + \alpha_{n+j} + \cdots + \alpha_{n+k-1}) \\
= (1 - \alpha_{n+j-1}^2) + (\alpha_{n+j} + \cdots + \alpha_{n+k-1}) - \alpha_{n+j-1}(\alpha_{n+j} + \cdots + \alpha_{n+k-1}) \\
= (1 - \alpha_{n+j-1}^2)(1 + \alpha_{n+j-1} + \alpha_{n+j} + \cdots + \alpha_{n+k-1})
\]
and \(1/\delta_{n+j-1} \leq 1/(\delta_{n+j-1} \cdot \delta_{n+k-1})\). Thus we have
\[
-\frac{\varepsilon_n(k) + (j-1) k^2 \ell(n,k)}{\delta_{n+j-1} \cdot \delta_{n+k-1}} + (1 + \alpha_{n+j-1} + \cdots + \alpha_{n+k-1}) d_n \\
\leq \|w_{n+k} - v_{n+j-1}\|.
\]
By induction, (*) holds for all \(j \in N(0, k - 1)\). We set \(j = 0\) in (*). By Lemma 3.1, we have the inequality:
\[
(1 + \alpha_n(k)) d_n \leq \|w_{n+k} - v_n\| + (\varepsilon_n(k) + k^3 \ell(n,k)) e^{(1 + \alpha_n(k)) A}.
\]
\(\square\)

**Lemma 3.3.** Let \(b\) be a real number belonging to \((0, 1)\) and \(\{\alpha_n\}\) be a sequence in \([0, b]\). Let \(\{u_n\}, \{v_n\}\) and \(\{w_n\}\) be sequences in a Banach space \(E\). Assume that, for any \(i, j \in N\), the following conditions hold:
\[
(1) \ u_{i+1} = \alpha_i w_i + (1 - \alpha_i) v_i. \quad (2) \ |w_{i+1} - w_i| \leq \alpha_i |w_i - v_i|.
\]
\[
(3) \ |v_i - w_j| \leq |u_i - w_j|.
\]
Then, the followings hold:
This is a contradiction. Therefore, we obtain

\[
\lim_{n} \|w_n - v_n\| = 0 \quad \text{and} \quad \lim_{n} \|w_n - u_n\| = 0.
\]

**Proof.** For any \(n, k \in N\), Lemma 3.2 (1),(2),(3) hold as \(l_{(n,k)} = 0\).

We prove (a). By our assumptions, it is obvious that, for \(i \in N\),

\[
\begin{align*}
(i) \quad & ||w_{i+1} - v_{i+1}|| \leq ||w_{i+1} - u_{i+1}|| \leq ||w_{i+1} - w_i|| + ||w_i - u_{i+1}|| \\
& \leq \alpha_i ||wi - vi|| + (1 - \alpha_i) ||wi - vi|| = ||w_i - vi||.
\end{align*}
\]

Then, \(\{||w_n - v_n||\}\) is non-increasing. Thus \(\lim_{n} ||w_n - v_n||\) exists.

Assume that either \(\{v_n\}\) or \(\{w_n\}\) is bounded and \(\sum_{n=1}^{\infty} \alpha_n = \infty\). Then, it follows from (a) that \(M = \sup \{||w_n - v_m|| : m, n \in N\} < \infty\). We set

\[
c = \lim_{n} ||w_n - v_n||.
\]

Then, \(0 \leq c \leq M\). For any \(n, k \in N\), set

\[
d_n = ||w_n - v_n||, \quad \varepsilon(n) = ||w_n - v_n|| - c,
\]

\[
\varepsilon_n(k) = ||w_n - v_n|| - ||w_{n+k} - v_{n+k}||.
\]

By (a), we have \(c \leq d_n, 0 \leq \varepsilon_n(k) \leq \varepsilon(n)\) for \(n, k \in N\) and \(\lim_n \varepsilon(n) = 0\).

We prove (b). Arguing by contradiction, we assume \(c > 0\). Let \(\varepsilon \in (0,1)\).

Then, there exists \(n_0 \in N\) such that

\[
\varepsilon(n_0) < \varepsilon/ \exp ((M + 1 + c)A/c) \quad (A = 1/(1 - b)).
\]

Since \(0 < c \leq M\) and \(\sum_{n=1}^{\infty} \alpha_n = \infty\), there exists the smallest positive integer \(k_0\) satisfying \(M + 1 < c + \alpha_{n_0}(k_0)c\). By \(\alpha_{n_0+k_0-1}c < c\), we have

\[
M + 1 < (1 + \alpha_{n_0}(k_0))c < M + 1 + c, \quad (1 + \alpha_{n_0}(k_0)) < (M + 1 + c)/c.
\]

We already know that \(\varepsilon_{n_0}(k_0) \leq \varepsilon(n_0), l_{n_0,k_0} = 0\) and \(c \leq d_{n_0}\). Since \(e^x\) is strictly increasing and Lemma 3.2, the following inequality holds:

\[
M + 1 < (1 + \alpha_{n_0}(k_0))c \leq (1 + \alpha_{n_0}(k_0))d_{n_0} \leq ||w_{n_0+k_0} - v_{n_0}|| + \varepsilon_{n_0}(k_0) \exp ((1 + \alpha_{n_0}(k_0))A) \leq ||w_{n_0+k_0} - v_{n_0}|| + \varepsilon(n_0) \exp ((M + 1 + c)A/c) < M + \varepsilon < M + 1.
\]

This is a contradiction. Therefore, we obtain \(c = \lim_{n} ||w_n - v_n|| = 0\). It follows from (i) that \(\lim_{n} ||w_n - u_n|| = 0\).

\[
\square
\]

4. THEOREMS CONNECTED WITH ISHIKAWA’S IDEA

In this section, we study the structure of Ishikawa’s approximate procedure to find a common fixed point for a finite family of nonexpansive mappings.
Lemma 4.1. Let $b$ be a real number in $(0, 1)$ and $\{\alpha_n\}$ be a sequence in $[0, b]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T$ and $P$ be nonexpansive self-mappings on $D$ such that

\[(I)\quad P(D) = F(P)\quad \text{and}\quad TP(D) \subset P(D) = F(P).\]

For each $n \in \mathbb{N}$, let $S_n$ be a mapping on $D$ defined by $S_n = \alpha_n T + (1 - \alpha_n) I$. Let $u_1 \in D$, $v_1 = P u_1$ and $w_1 = T v_1$. Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences in $D$ such that, for $n \in \mathbb{N}$,

\[u_{n+1} = \alpha_n w_n + (1 - \alpha_n) v_n, \quad v_{n+1} = Pu_{n+1}, \quad w_{n+1} = Tv_{n+1}.\]

That is, $u_{n+1} = S_n Pu_n = \pi(S_n P) u_1$ for $n \in \mathbb{N}$. Then,

\[\lim_n \|w_n - v_n\| = \lim_n \|w_n - u_n\| = 0.\]

Proof. Since $D$ is compact, it is obvious that $\{v_n\}$ is bounded. We show that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ satisfy the conditions Lemma 3.3 (1),(2),(3).

It is obvious that (1) holds. Fix $i, j \in \mathbb{N}$ arbitrary. By $v_i \in F(P)$ and (I), we have $w_i = Tv_i \in F(P)$. Then, $PT v_i = T v_i = TP v_i$. It is easy to see that

\[\|w_{i+1} - v_i\| = \|Tv_{i+1} - T v_i\| = \|TPu_{i+1} - TP v_i\| \leq \|u_{i+1} - v_i\| = \alpha_i \|w_i - v_i\|,\]

\[\|v_i - w_j\| = \|Pu_i - P w_j\| \leq \|u_i - w_j\|.\]

Thus, Lemma 3.3 (2),(3) hold. By Lemma 3.3, we have the results. \hfill \square

Theorem 4.2 (Ishikawa, [12], 1979). Let $b \in (0, 1)$ and $\{\alpha_n\}$ be a sequence in $[0, b]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T$ and $P$ be nonexpansive self-mappings on $D$ such that

\[(I)\quad P(D) = F(P)\quad \text{and}\quad TP(D) \subset P(D) = F(P).\]

Let $\{S_n\}$ and $\{Q_n\}$ be sequences of self-mappings on $D$ defined by

\[S_n = \alpha_n T + (1 - \alpha_n) I, \quad Q_n = \pi(S_n P) \quad \text{for} \quad n \in \mathbb{N}.\]

Let $u_1 \in D$, $v_1 = P u_1$ and $w_1 = T v_1$. Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be sequences in $D$ such that, for $n \in \mathbb{N}$,

\[u_{n+1} = \alpha_n w_n + (1 - \alpha_n) v_n, \quad v_{n+1} = Pu_{n+1}, \quad w_{n+1} = Tv_{n+1}.\]

That is, $u_{n+1} = S_n Pu_n = \pi(S_n P) u_1 = Q_n u_1$. Then, the followings hold:

1. $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ converges strongly to a point in $F(T) \cap F(P)$.
2. There exists a nonexpansive self-mapping $Q$ on $D$ such that
   - (a) $\{Q_n\}$ converges uniformly to $Q$,
   - (b) $Q(D) = F(Q) = F(T) \cap F(P)$.\hfill \square
Theorem 4.3. Then, $P(I)$ is more general than original one. By setting $P = I$ in Theorem 4.2, we have

$$\lim_{n \to \infty} w_n = v$$

for some $v \in F(P)$. We easily have that, for $j \in N$,

$$\|v - w_{n_j}\| \leq \|v - v_n\| + \|v_n - w_{n_j}\|.$$  

Since $\lim_j \|w_{n_j} - v_n\| = 0$, $\{w_n\}$ also converges strongly to $v$. It is also easy to see that, for $j \in N$,

$$\|Tv - v\| \leq \|Tv - Tv_{n_j}\| + \|Tv_{n_j} - v\| \leq \|Tv - Tv_{n_j}\| + \|w_{n_j} - v\|.$$  

Since $T$ is continuous, we have $Tv = v$. That is, $v \in F(T) \cap F(P)$. We also proved $F(T) \cap F(P) \neq \emptyset$. By $v \in F(T) \cap F(P)$, we have that, for $i \in N$,

$$\|v_{i+1} - v\| = \|Pu_{i+1} - Pv\| \leq \|u_{i+1} - v\| \leq \alpha_i \|Tv_i - v\| + (1 - \alpha_i) \|v_i - v\| \leq \alpha_i \|v_i - v\| + (1 - \alpha_i) \|v_i - v\| = \|v_i - v\|.$$  

Then, $\{\|v_n - v\|\}$ is non-increasing and converges. By $\lim_j \|v_{n_j} - v\| = 0$, we have $\lim_n \|v_n - v\| = 0$. By Lemma 4.1, $\{u_n\}$ and $\{w_n\}$ also converge strongly to the point $v \in F(T) \cap F(P)$.

We prove (2). Recall that $\{Q_n\}$ is a sequence of nonexpansive self-mappings on $D$ by Lemma 2.2. By (1), for any $x \in D$, there is $u_x \in F(T) \cap F(P)$ such that $\{Q_n x\}$ converges strongly to $u_x$. We define a mapping $Q$ by $Qx = u_x$ for $x \in D$. By Lemma 2.3, $Q$ is a nonexpansive self-mapping on $D$ such that $\{Q_n\}$ converges uniformly to $Q$. By the definition of $Q$, it is obvious that $Q(D) \subset F(T) \cap F(P)$. By Lemma 2.2, we know that $F(T) \cap F(P) \subset \cap_n F(Q_n).$ It follows that $Q_n x = x$ for $x \in F(T) \cap F(P)$, $n \in N$. This implies $Qx = x$ for $x \in F(T) \cap F(P)$. Therefore we have $F(T) \cap F(P) \subset F(Q) \subset Q(D)$. Consequently, we have $Q(D) = F(Q) = F(T) \cap F(P).$  

The following theorem is a direct consequence of Theorem 4.2

**Theorem 4.3.** Let $D$ be a compact convex subset of a Banach space $E$. Let $T$ and $P$ be nonexpansive self-mappings on $D$ such that

(I) \hspace{1cm} $P(D) = F(P)$ and $TP(D) \subset P(D) = F(P)$.

Then, $F(T) \cap F(P) \neq \emptyset$. Moreover, there exists a nonexpansive self-mapping $Q$ on $D$ such that $Q(D) = F(Q) = F(T) \cap F(P)$.

Theorem 4.2 is essentially proved by Ishikawa. The control sequence $\{\alpha_n\}$ is more general than original one. By setting $P = I$ in Theorem 4.2, we have Theorem 4.4. Ishikawa's original theorem has not the description of $Q$.  

Proof. We prove (1). By Lemma 4.1 and (I), we know that $\lim_n \|w_n - v_n\| = 0$ and $\{v_n\} \subset F(P)$. Since $F(P)$ is compact, there is a subsequence $\{v_{n_j}\}$ which converges strongly to some $v \in F(P)$. We easily have that, for $j \in N$, 

$$\|v - w_{n_j}\| \leq \|v - v_{n_j}\| + \|v_{n_j} - w_{n_j}\|.$$  

Since $\lim_j \|w_{n_j} - v_{n_j}\| = 0$, $\{w_n\}$ also converges strongly to $v$. It is also easy to see that, for $j \in N$,

$$\|Tv - v\| \leq \|Tv - Tv_{n_j}\| + \|Tv_{n_j} - v\| \leq \|Tv - Tv_{n_j}\| + \|w_{n_j} - v\|.$$  

Since $T$ is continuous, we have $Tv = v$. That is, $v \in F(T) \cap F(P)$. We also proved $F(T) \cap F(P) \neq \emptyset$. By $v \in F(T) \cap F(P)$, we have that, for $i \in N$,

$$\|v_{i+1} - v\| = \|Pu_{i+1} - Pv\| \leq \|u_{i+1} - v\| \leq \alpha_i \|Tv_i - v\| + (1 - \alpha_i) \|v_i - v\| \leq \alpha_i \|v_i - v\| + (1 - \alpha_i) \|v_i - v\| = \|v_i - v\|.$$  

Then, $\{\|v_n - v\|\}$ is non-increasing and converges. By $\lim_j \|v_{n_j} - v\| = 0$, we have $\lim_n \|v_n - v\| = 0$. By Lemma 4.1, $\{u_n\}$ and $\{w_n\}$ also converge strongly to the point $v \in F(T) \cap F(P)$.

We prove (2). Recall that $\{Q_n\}$ is a sequence of nonexpansive self-mappings on $D$ by Lemma 2.2. By (1), for any $x \in D$, there is $u_x \in F(T) \cap F(P)$ such that $\{Q_n x\}$ converges strongly to $u_x$. We define a mapping $Q$ by $Qx = u_x$ for $x \in D$. By Lemma 2.3, $Q$ is a nonexpansive self-mapping on $D$ such that $\{Q_n\}$ converges uniformly to $Q$. By the definition of $Q$, it is obvious that $Q(D) \subset F(T) \cap F(P)$. By Lemma 2.2, we know that $F(T) \cap F(P) \subset \cap_n F(Q_n).$ It follows that $Q_n x = x$ for $x \in F(T) \cap F(P)$, $n \in N$. This implies $Qx = x$ for $x \in F(T) \cap F(P)$. Therefore we have $F(T) \cap F(P) \subset F(Q) \subset Q(D)$. Consequently, we have $Q(D) = F(Q) = F(T) \cap F(P).$  

The following theorem is a direct consequence of Theorem 4.2

**Theorem 4.3.** Let $D$ be a compact convex subset of a Banach space $E$. Let $T$ and $P$ be nonexpansive self-mappings on $D$ such that

(I) \hspace{1cm} $P(D) = F(P)$ and $TP(D) \subset P(D) = F(P)$.

Then, $F(T) \cap F(P) \neq \emptyset$. Moreover, there exists a nonexpansive self-mapping $Q$ on $D$ such that $Q(D) = F(Q) = F(T) \cap F(P)$.
Theorem 4.4 (Ishikawa, [11], 1976). Let $b$ be a real number belonging to $(0, 1)$ and $\{\alpha_n\}$ be a sequence in $[0, b]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T$ be a nonexpansive self-mapping on $D$. Let $\{S_n\}$ and $\{Q_n\}$ be sequences of nonexpansive self-mappings on $D$ defined by

$$S_n = \alpha_n T + (1 - \alpha_n) I, \quad Q_n = \pi(S_n) \quad \text{for } n \in \mathbb{N}.$$ 

Let $u_1 \in D$ and $\{u_n\}$ be a sequence defined by

$$u_{n+1} = \alpha_n T u_n + (1 - \alpha_n) u_n = S_n u_n = \pi(S_n) u_1 = Q_n u_1 \quad \text{for } n \in \mathbb{N}.$$

Then, the following hold:

(a) $\{u_n\}$ converges strongly to a fixed point of $T$.

(b) There is a nonexpansive self-mapping $Q$ on $D$ such that $\{Q_n\}$ converges uniformly to $Q$ and $Q(D) = F(Q) = F(T)$.

By using Theorem 4.3, we have another proof of Theorem 1.1.

Proof of Theorem 1.1. We know that $\{F(T_j)\}_{j \in J}$ is a family of closed sets in a compact set $D$. To have the result, it is sufficient to show $\cap_{i=1}^{k} F(T_{h(i)}) \neq \emptyset$ for any finite subfamily $\{F(T_{h(i)})\}_{i=1}^{k}$, where $h$ is an injection from $N(1, k)$ into index set $J$. We fix $\{F(T_{h(i)})\}_{i=1}^{k}$ arbitrary.

(1) It is obvious that $T_{h(1)}$ and $P_0 = I$ satisfy assumptions of Theorem 4.3. Then, $F(T_{h(1)}) \neq \emptyset$. And, there exists a nonexpansive self-mapping $P_1$ such that $P_1(D) = F(P_1) = F(T_{h(1)})$.

(2) Since $\{T_{h(i)}\}_{i=1}^{k}$ is commuting and (1), we have

$$T_{h(2)}(F(P_1)) = T_{h(2)}(F(T_{h(1)})) \subset F(T_{h(1)}) = F(P_1) = P_1(D).$$

We confirmed that $T_{h(2)}$ and $P_1$ satisfy assumptions of Theorem 4.3. Then, we have $F(T_{h(2)}) \cap F(P_1) = F(T_{h(2)}) \cap F(T_{h(1)}) \neq \emptyset$. Further, there exists a nonexpansive self-mapping $P_2$ such that

$$P_2(D) = F(P_2) = F(T_{h(2)}) \cap F(P_1) = F(T_{h(2)}) \cap F(T_{h(1)}).$$

(3) Since $\{T_{h(i)}\}_{i=1}^{k}$ is commuting and (2), we have

$$T_{h(3)}(F(P_2)) = T_{h(3)}(F(T_{h(2)}) \cap F(T_{h(1)})) \subset F(T_{h(2)}) \cap F(T_{h(1)}) = F(P_2) = P_2(D).$$

Then, $T_{h(3)}$ and $P_2$ satisfy assumptions of Theorem 4.3.

Continuing this process, we have $\cap_{i=1}^{k} F(T_{h(i)}) \neq \emptyset$. This completes the proof. Moreover, there is a nonexpansive self-mapping $P_k$ satisfying

$$P_k(D) = F(P_k) = F(T_{h(k)}) \cap \cdots \cap F(T_{h(1)}).$$
We can continue our argument preserving past conditions of the control sequence \( \{\alpha_n\} \). However, for simplicity, we consider the case that terms of \( \{\alpha_n\} \) are a constant \( a \in (0, 1) \). By using Theorem 4.2, we prove Theorem 4.5. Isikawa’s approximate procedure depends on the theorem.

**Theorem 4.5** (Ishikawa, [12], 1979). Let \( a \) be a real number belonging to \( (0, 1) \). Let \( D \) be a compact convex subset of a Banach space \( E \). Let \( T \) and \( P \) be nonexpansive self-mappings on \( D \) such that

\[
P(D) = F(P) \quad \text{and} \quad TP(D) \subset P(D) = F(P).
\]

Let \( \{M_n\} \) be a sequence of nonexpansive self-mappings on \( D \). Assume that \( \{M_n\} \) converges uniformly to \( P \) and the following holds:

\[
P(D) = F(P) \subset \cap_n F(M_n).
\]

Let \( S \) be a nonexpansive self-mapping on \( D \) defined by \( S = aT + (1 - a)I \). Let \( \{L_n\} \) be a sequence of nonexpansive self-mappings on \( D \) defined by

\[
L_n = \pi(SM_n) \quad \text{for} \quad n \in N.
\]

Then, there exists a nonexpansive self-mapping \( L \) on \( D \) such that

(1) \( \{L_n\} \) converges uniformly to \( L \),

(2) \( L(D) = F(L) = F(T) \cap F(P) \subset \cap_n F(L_n) \).

That is, for any \( x \in D \), \( \{L_nx\} \) converges to a point in \( F(T) \cap F(P) \).

**Proof.** Since \( T \) and \( P \) satisfy (1), we know that \( F(T) \cap F(P) \neq \emptyset \). Let \( \{Q_n\} \) be a sequence of nonexpansive self-mappings on \( D \) defined by \( Q_n = (SP)^n \) for \( n \in N \). It is obvious that, for each \( n \in N \), \( Q_n = \pi(S_nP) \) as \( S_n = S \). Then, by Theorem 4.2, there exists a nonexpansive self-mapping \( Q \) on \( D \) such that

\[
\{Q_n\} \text{ converges uniformly to } Q, \quad Q(D) = F(Q) = F(T) \cap F(P).
\]

**Step 1.** We show that, for \( x \in D \), we can take a subsequence \( \{L_nx\} \) of \( \{L_nx\} \) and a sequence \( \{u_{nk}\} \) in \( D \) such that

\[
u_{nk} \in F(T) \cap F(P), \quad \|L_{nk}x - u_{nk}\| < 1/k \quad \text{for} \quad k \in N.
\]

Let \( k \in N \) and set \( \varepsilon = 1/k \). Since \( \{Q_n\} \) converges uniformly to \( Q \), there exists \( n_1 \in N \) such that

\[
\|Q_nz - Qz\| < \varepsilon/2 \quad \text{for} \quad n \geq n_1, \quad z \in D.
\]

Since \( \{M_n\} \) converges uniformly to \( P \), there is \( n_2 \in N \) such that

\[
\|M_nz - Pz\| < \varepsilon/(2n_1) \quad \text{for} \quad m \geq n_2, \quad z \in D.
\]

Fix \( x \in D \) arbitrary and set

\[
y_1 = L_{n_2+n_1-1}x = \pi(SM_{n_2+n_1-1})x, \quad z_1 = Q_{n_1-1}L_{n_2}x = (SP)^{n_1-1}L_{n_2}x.
\]
By (B), we have
\[
\|L_{n_2+n_1}x - Q_{n_1}L_{n_2}x\| = \|\pi(SM_{n_2+n_1})x - (SP)^{n_1}L_{n_2}x\|
\leq \|SM_{n_2+n_1}y_1 - SPz_1\| \leq \|M_{n_2+n_1}y_1 - Pz_1\|
\leq \|M_{n_2+n_1}y_1 - y_1\| + \|Py_1 - Pz_1\|
\leq \|y_1 - z_1\| + \frac{\varepsilon}{2n_1} = \|L_{n_2+n_1-1}x - Q_{n_1-1}L_{n_2}x\| + \frac{\varepsilon}{2n_1}.
\]
Inductively, we have that
\[
\|L_{n_2+n_1}x - Q_{n_1}L_{n_2}x\| < \|L_{n_2+1}x - Q_1L_{n_2}x\| + (n_1 - 1)\frac{\varepsilon}{2n_1}
\leq \|SM_{n_2+1}(L_{n_2}x) - SP(L_{n_2}x)\| + (n_1 - 1)\frac{\varepsilon}{2n_1}
\leq \|M_{n_2+1}(L_{n_2}x) - P(L_{n_2}x)\| + (n_1 - 1)\frac{\varepsilon}{2n_1} < \frac{\varepsilon}{2}.
\]
Then, by (A), we have
\[
\|L_{n_2+n_1}x - QL_{n_2}x\|
\leq \|L_{n_2+n_1}x - Q_{n_1}L_{n_2}x\| + \|Q_{n_1}(L_{n_2}x) - Q(L_{n_2}x)\| < \varepsilon.
\]
Set \(n_k = n_2 + n_1\) and \(u_{n_k} = QL_{n_2}x\). By \(Q(D) = F(T) \cap F(P)\) and (C), we have that \(u_{n_k} = QL_{n_2}x \in F(T) \cap F(P)\) and \(\|L_{n_k}x - u_{n_k}\| < 1/k\).

By the argument as above, there are a subsequence \(\{L_{n_k}x\}\) of \(\{L_nx\}\) and a sequence \(\{u_{n_k}\}\) in \(D\) which satisfy the condition (*).

**Step 2.** Fix \(x \in D\) arbitrary. By Step 1, there are a subsequence \(\{L_{n_k}x\}\) of \(\{L_nx\}\) and a sequence \(\{u_{n_k}\}\) in \(D\) which satisfy (*). We know that \(F(T) \cap F(P)\) is compact. Then, there is a subsequence of \(\{u_{n_k}\}\) which converges strongly to some \(u_x \in F(T) \cap F(P)\). No loss of generality, we can assume that \(\{u_{n_k}\}\) itself converges strongly to \(u_x\). We easily have that, for \(k \in N\),
\[
\|L_{n_k}x - u_x\| \leq \|L_{n_k}x - u_{n_k}\| + \|u_{n_k} - u_x\| < \|u_{n_k} - u_x\| + 1/k.
\]
Then, \(\lim_k \|L_{n_k}x - u_x\| = 0\). Thus \(\{L_{n_k}x\}\) converges strongly to \(u_x\).

Since \(F(T) \subset F(S)\) and assumptions of \(\{M_n\}\), we have
\[
F(T) \cap F(P) \subset F(S) \cap (\cap \{M_n\}).
\]
This implies that \(F(T) \cap F(P) \subset F(SM_n)\) for \(n \in N\). We have
\[
F(T) \cap F(P) \subset \cap_n F(\pi(SM_n)) = \cap_n F(L_n).
\]
By \(u_x \in F(T) \cap F(P)\), we have
\[
\|L_{n+1}x - u_x\| = \|SM_{n+1}L_nx - SM_{n+1}u_x\| \leq \|L_nx - u_x\|.
\]
Then, we have that \(\{\|L_nx - u_x\|\}\) is non-increasing and converges. From \(\lim_k \|L_{n_k}x - u_x\| = 0\), we have \(\lim_n \|L_nx - u_x\| = 0\). Thus \(\{L_nx\}\) also converges strongly to \(u_x \in F(T) \cap F(P)\).

**Step 3.** By Step 2, for any \(x \in D\), \(\{L_nx\}\) converges to some \(u_x \in F(T) \cap F(P)\).
We define a mapping $L$ from $D$ into $F(T) \cap F(P)$ by $Lx = u_x$. By Lemma 2.3, we know that $L$ is a nonexpansive self-mapping on $D$ such that $\{L_n\}$ converges uniformly to $L$. By the definition of $L$, it is obvious that $L(D) \subset F(T) \cap F(P)$. In Step 2, we proved $F(T) \cap F(P) \subset \cap_n F(L_n)$. Then, it is obvious that $L_n x = x$ for $x \in F(T) \cap F(P)$, $n \in N$. This implies that $L x = x$ for $x \in F(T) \cap F(P)$. Therefore we have $F(T) \cap F(P) \subset F(L) \subset L(D)$. Consequently, we have $L(D) = F(L) = F(T) \cap F(P) \subset \cap_n F(L_n)$.

5. ISIKAWA’S APPROXIMATE PROCEDURE

**Theorem 5.1.** Let $a$ be a real number belonging to $(0, 1)$. Let $D$ be a compact convex subset of a Banach space $E$. Let $\{T_1, T_2, \ldots, T_k\}$ be a finite sequence of nonexpansive self-mappings on $D$ such that

$$T_{l+1}(\cap_{i=1}^l F(T_i)) \subset \cap_{i=1}^l F(T_i) \quad \text{for } l \in N(1, k-1).$$

Let $S_i$ be a nonexpansive self-mapping on $D$ defined by $S_i = aT_i + (1-a)I$ for $i \in N(1, k)$. Let $L_{(1,n)} = S_i^n$ for $n \in N$ and $\{L_{(i,n)}\}$ be a double sequence of nonexpansive self-mappings on $D$ defined by

$$L_{(i+1,n)} = \pi(S_{i+1} L_{(i,n)}) \quad \text{for } i \in N(1, k-1), \ n \in N.$$  

Then, there exists a nonexpansive self-mapping $P$ on $D$ such that

(a) $\{L_{(k,n)}\}$ converges uniformly to $P$,

(b) $P(D) = F(P) = F(T_k) \cap \cdots \cap F(T_2) \cap F(T_1)$.

**Proof.** (1) Let $P_0 = I$ and let $L_{(0,n)} = I$ for $n \in N$. Then, it is obvious that $\{L_{(0,n)}\}$ is a sequence of nonexpansive self-mappings on $D$ which converges uniformly to $P_0$. Obviously, the following conditions hold:

$$P_0(D) = F(P_0) = D, \ T_1 P_0(D) \subset P_0(D) = F(P_0) \subset \cap_n F(L_{(0,n)}).$$

We confirmed that $P_0, T_1$ and $\{L_{(0,n)}\}$ satisfy assumptions of Theorem 4.5. It is also obvious that $L_{(1,n)} = S_i^n = \pi(S_i L_{(0,n)})$ for $n \in N$. By Theorem 4.5, there exists a nonexpansive self-mapping $P_1$ on $D$ such that

- $\{L_{(1,n)}\}$ converges uniformly to $P_1$,
- $P_1(D) = F(P_1) = F(T_1) \cap F(P_0) = F(T_1) \subset \cap_n F(L_{(1,n)}).$

(2) By (1) and (II), we know that

- $\{L_{(1,n)}\}$ converges uniformly to $P_1$,
- $P_2(D) = F(P_2) = F(T_1)$,
- $T_2 P_1(D) \subset P_2(D) = F(P_1) \subset \cap_n F(L_{(1,n)})$,
- $L_{(2,n)} = \pi(S_2 L_{(1,n)})$ for $n \in N$.

By Theorem 4.5, there is a nonexpansive self-mapping $P_2$ on $D$ such that

- $\{L_{(2,n)}\}$ converges uniformly to $P_2$, 

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• $P_2(D) = F(P_2) = F(T_2) \cap F(P_1) = F(T_2) \cap F(T_1) \subset \cap_n F(L(2,n))$.

(3) By (2) and (II), we know that
• $\{L(2,n)\}$ converges uniformly to $P_2$,
• $P_2(D) = F(P_2) = F(T_2) \cap F(T_1)$,
• $T_3P_2(D) \subset P_2(D) = F(P_2) \subset \cap_n F(L(2,n))$.
• $L_{(3,n)} = \pi(S_3L_{(2,n)})$ for $n \in N$.

By Theorem 4.5, there is a nonexpansive self-mapping $P_3$ on $D$ such that
• $\{L_{(3,n)}\}$ converges uniformly to $P_3$,
• $P_3(D) = F(P_3) = F(T_3) \cap F(P_2) = F(T_3) \cap F(T_2) \cap F(T_1) \subset \cap_n F(L_{(3,n)})$.

Inductively, we have a nonexpansive self-mapping $P = P_k$ on $D$ such that
(a) $\{L_{(k,n)}\}$ converges uniformly to $P_k$,
(b) $P_k(D) = F(P_k) = F(T_k) \cap F(P_{k-1}) = F(T_k) \cap \cdots \cap F(T_2) \cap F(T_1)$.

\[
\square
\]

Remark 5.2. In Theorem 5.1, the conditions (a) and (b) implies that, for any $x_1 \in D$, $\{L_{(k,n)}x_1\}$ converges strongly to a common fixed point of $\{T_1, T_2, \ldots, T_k\}$. We will not notice the fact any more. We can rewrite the approximate procedure as the following iteration:

\[
x_1 \in D, \quad x_{n+1} = S_kL_{(k-1,n)}x_n \quad \text{for} \quad n \in N.
\]

For reference, in the case of $k = 2$ or $k = 4$, we show $x_2$ and $x_3$.

\[
L_{(1,n)} = S_1^n \quad \text{for} \quad n \in N,
\]
\[
L_{(2,1)} = \pi(S_2L_{(1,1)}) = S_2L_{(1,1)} = S_2S_1,
\]
\[
L_{(2,2)} = \pi(S_2L_{(1,2)}) = (S_2L_{(1,2)})(S_2L_{(1,1)}) = (S_2S_1^2)(S_2S_1),
\]
\[
L_{(3,1)} = \pi(S_3L_{(2,1)}) = S_3L_{(2,1)} = S_3(S_2S_1),
\]
\[
L_{(3,2)} = \pi(S_3L_{(2,2)}) = (S_3L_{(2,2)})(S_3L_{(2,1)}) = (S_3(S_2S_1^2)(S_2S_1))(S_3(S_2S_1)),
\]
\[
L_{(4,1)} = \pi(S_4L_{(3,1)}) = S_4L_{(3,1)} = S_4(S_3(S_2S_1)),
\]
\[
L_{(4,2)} = \pi(S_4L_{(3,2)}) = (S_4L_{(3,2)})(S_4L_{(3,1)}) = (S_4(S_3(S_2S_1^2)(S_2S_1)))(S_4(S_3(S_2S_1)))(S_4(S_3(S_2S_1))).
\]

In the case of $k = 2$,
\[
x_2 = L_{(2,1)}x_1 = (S_2S_1)x_1,
\]
\[
x_3 = L_{(2,2)}x_1 = (S_2S_1^2)(S_2S_1)x_1 = (S_2S_1^2)L_{(2,1)}x_1 = (S_2S_1^2)x_2.
\]
In the case of \( k = 4 \),
\[
x_2 = L_{(4,1)} x_1 = (S_1 S_2 S_3 S_4) x_1
\]
\[
x_3 = L_{(4,2)} x_1 = (S_1 S_2 S_3 S_4^2 S_5 S_4 S_3 S_2 S_1)(S_4 S_3 S_2 S_1) x_1
\]
\[
= (S_1 S_3 S_2 S_4^2 S_5 S_4 S_3 S_2 S_1) L_{(4,1)} x_1 = (S_1 S_3 S_2 S_4^2 S_5 S_4 S_3 S_2 S_1) x_2.
\]

By Lemma 2.4, Theorem 5.3 is a direct consequence of Theorem 5.1. Theorem 5.3 is another expression of Ishikawa’s Theorem 1.2

**Theorem 5.3.** Let \( a \in (0, 1) \). Let \( D \) be a compact convex subset of a Banach space \( E \). Let \( \{T_1, T_2, \ldots, T_k\} \) be a finite sequence of commuting nonexpansive self-mappings on \( D \). Let \( S_i \) be a nonexpansive self-mapping on \( D \) defined by
\[
S_i = a T_i + (1 - a) I \quad \text{for} \quad i \in N(1, k).
\]
Let \( L_{i(1,n)} = S_i^n \) for \( n \in N \) and \( \{L_{i(1,n)}\} \) be a double sequence of nonexpansive self-mappings on \( D \) defined by
\[
L_{i(1,n)} = \pi(S_i^{n+1} L_{i(1,n)}) \quad \text{for} \quad i \in N(1, k - 1), \quad n \in N.
\]
Then, there exists a nonexpansive self-mapping \( P \) on \( D \) such that

(a) \( \{L_{i(1,n)}\} \) converges uniformly to \( P \),

(b) \( P(D) = F(P) = F(T_k) \cap \cdots \cap F(T_2) \cap F(T_1) \).

**Remark 5.4.** The condition (II) in Theorem 5.1 is strictly weaker than the condition that \( \{T_1, T_2, \ldots, T_k\} \) is commuting. We give a typical example.

Let \( R^2 \) be 2-dimensional Euclidean space. Let \( D = [-1, 1]^2 \subset R^2 \). Let \( T_1 \) and \( T_2 \) be nonexpansive self-mappings on \( D \) defined by
\[
T_1(a, b) = (-b, a), \quad T_2(a, b) = (a, 0) \quad \text{for} \quad (a, b) \in D.
\]
It is obvious that \( F(T_1) = \{(0, 0)\}, \quad F(T_2) = \{(a, 0) \mid a \in [-1, 1]\}. \) Then, \( T_2(F(T_1)) \subset F(T_1) \). On the other hand, we easily have
\[
T_2 T_1(a, b) = T_2(-b, a) = (-b, 0), \quad T_1 T_2(a, b) = T_1(a, 0) = (0, a).
\]
Then, \( \{T_1, T_2\} \) is not commuting. We note that, for each \( i \in N(1, k) \), Ishikawa used \( a_i \in (0, 1) \) in place of \( a \) in Theorem 1.2.

6. **Suzuki’s approximate procedure**

To study Suzuki’s idea in [22], we need Suzuki’s lemma; see [21], [22], [14].

**Lemma 6.1** (Suzuki, 2005, [22]). Let \( \{\alpha_n\} \) be a sequence in \([0,1]\). Let \( \{u_n\} \) and \( \{w_n\} \) be bounded sequences in a Banach space \( E \). Assume that

1. \( u_{i+1} = \alpha_i w_i + (1 - \alpha_i) u_i \quad \text{for} \quad i \in N, \)
2. \( 0 < \lim \inf \alpha_n \leq \lim \sup \alpha_n < 1, \)
3. \( \lim \sup \alpha_n \|w_{n+1} - w_n\| - \|u_{n+1} - u_n\| \leq 0. \)

Then, \( \lim_n \|w_n - u_n\| = 0. \)
From now on, we limit our argument to finding a common fixed point for two nonexpansive mappings $T_1$ and $T_2$ which satisfy $T_1T_2 = T_2T_1$.

Consider a sequence $\{b_n\}$ in $[0, 1]$ defined by
\[
b_n = \frac{n-(2^k-2)}{2^{k}} \quad \text{for} \quad n \in N(2^k-1, 2^{(k+1)}-2), \quad k = 1, 2, \ldots.
\]
That is, we consider the sequence:
\[
1/2, 1, 1/4, 2/4, 3/4, 1, 1/8, 2/8, \ldots, 7/8, 1, \ldots.
\]
We define a sequence $\{\alpha_n\}$ in $[0, 1]$ by
\[
(S) \quad \alpha_n = \sin (b_n \pi) \quad \text{for} \quad n \in N.
\]
Then, the following lemma is obvious.

**Lemma 6.2.** Let $\{\alpha_n\}$ be a sequence defined by $(S)$. Then, the followings hold:
1. $\alpha_n \in [0, 1]$ for $n \in N$, $\lim_n |\alpha_{n+1} - \alpha_n| = 0$.
2. For any $\alpha \in [0, 1]$, there is a subsequence $\{\alpha_{n_k}\}$ such that
   \[
   \alpha_{n_k} \in (0, 1), \quad \text{and} \quad \lim_k |\alpha_{n_k} - \alpha| = 0.
   \]

The following results are presented in [22] under more general conditions.

**Theorem 6.3.** Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ satisfying $\lim_n \alpha_n = 0$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T_1$ and $T_2$ be nonexpansive self-mappings on $D$ with $T_1T_2 = T_2T_1$. For any $n \in N$, let $U_n$ be a nonexpansive mapping defined by,
\[
U_n x = (1 - \alpha_n)T_1 x + \alpha_n T_2 x \quad \text{for} \quad x \in D.
\]
Suppose $\{w_n\}$ is a sequence in $D$ such that $U_n w_n = w_n$ for $n \in N$ and $\{w_n\}$ converges strongly to some $w \in D$. Then, $w \in F(T_1) \cap F(T_2)$.

**Proof.** We set $M = 2 \sup\{\|x\| : x \in D\} < \infty$. Let $\varepsilon > 0$.

Fix $n \in N$ arbitrary. By $U_n w_n = w_n$, we have
\[
\|T_1 w_n - w\| = \|T_1 w_n - ((1 - \alpha_n)T_1 w_n + \alpha_n T_2 w_n)\| + \|w_n - w\|
\leq \alpha_n \|T_1 w_n - T_2 w_n\| + \|w_n - w\| < \alpha_n M + \|w_n - w\|.
\]
From the inequality, it follows that
\[
\|T_1 w_n - w\| \leq \|T_1 w_n - T_1 w_n\| + \|T_1 w_n - w\| < \alpha_n M + 2\|w_n - w\|.
\]
This implies $w \in F(T_1)$. By $T_1T_2 = T_2T_1$, we have
\[
\|T_2 w - w_n\| = \|T_2 w - ((1 - \alpha_n)T_1 w_n + \alpha_n T_2 w_n)\|
\leq (1 - \alpha_n)\|T_2 T_1 w - T_1 w_n\| + \alpha_n \|T_2 w - T_2 w_n\|
\leq (1 - \alpha_n)\|T_1 T_2 w - T_1 w_n\| + \alpha_n \|w_n - w\|
\leq (1 - \alpha_n)\|T_2 w - w_n\| + \alpha_n \|w_n - w\|.
\]
From $\alpha_n \in (0, 1)$, it follows that $\|T_2w - w_n\| \leq \|w_n - w\|$. Then,
\[\|T_2w - w\| \leq \|T_2w - w_n\| + \|w_n - w\| \leq 2\|w_n - w\|.
\]
This implies $w \in F(T_2)$. Thus, we have $w \in F(T_1) \cap F(T_2)$.

\textbf{Lemma 6.4.} Let $c \in (0, 1)$ and $\{\alpha_n\}$ be a sequence defined by \((S)\). Let $D$ be a compact convex subset of a Banach space $E$. Let $T_1$ and $T_2$ be nonexpansive self-mappings on $D$ such that $T_1T = T_2T$. Let $x_1 \in D$ and $\{x_n\}$ be a sequence in $D$ defined by
\[x_{n+1} = c(1 - \alpha_n)T_1x_n + \alpha_nT_2x_n + (1 - c)x_n \text{ for } n \in N.
\]
Then, for any $j \in N$, there is $w_j \in D$ such that $(1 - \alpha_j)T_1w_j + \alpha_Tw_j = w_j$.

Further, for any $\varepsilon > 0$, there is a term $x_{(j)}$ such that $\|x_{(j)} - w_j\| < \varepsilon$.

\textbf{Proof.} We set $M = 2\sup\{\|x\| : x \in D\} < \infty$ and
\[y_n = (1 - \alpha_n)T_1x_n + \alpha_nT_2x_n \text{ for } n \in N.
\]
We note that $x_{n+1} = cy_n + (1 - c)x_n$ for $n \in N$. It is easy to see that
\[\|y_{n+1} - y_n\| \leq (1 - \alpha_{n+1})\|T_1x_{n+1} - T_1x_n\| + |\alpha_{n+1} - \alpha_n| \|T_1x_n\|
+ |\alpha_{n+1} - \alpha_n| \|T_2x_n\|
\]
\[\leq \|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n| M
\]
for $n \in N$. From Lemma 6.2 (1), it follows that
\[\limsup_n(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.
\]
Then, by $c \in (0, 1)$ and Suzuki’s lemma, we have $\lim_n \|x_n - y_n\| = 0$.

We fix a term $\alpha_j$ arbitrary. By Lemma 6.2 (2), there is a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ satisfying $\alpha_{n_k} \in (0, 1)$ and $\lim_k |\alpha_{n_k} - \alpha_j| = 0$. We also take the subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Since $D$ is compact, $\{x_{n_k}\}$ has a subsequence which converges to some $w_j \in D$. No loss of generality, we can assume that $\{x_{n_k}\}$ converges to $w_j \in D$. We know $\lim_k \|x_{n_k} - w_j\| = 0$ and $\lim_k \|x_{n_k} - y_{n_k}\| = 0$.

We have that, for any $n_k$,
\[||(1 - \alpha_j)T_1w_j + \alpha_jT_2w_j - w_j||
\]
\[\leq \|(1 - \alpha_j)T_1w_j + \alpha_jT_2w_j - ((T_1x_{n_k} + \alpha_Tx_{n_k}))
+ \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - w_j\|
\]
\[\leq (1 - \alpha_j)\|T_1w_j - T_1x_{n_k}\| + \alpha_j\|T_2w_j - T_2x_{n_k}\|
+ \|y_{n_k} - x_{n_k}\| + \|x_{n_k} - w_j\| + |\alpha_{n_k} - \alpha_j|\|T_1x_{n_k}\| + \|T_2x_{n_k}\|
\]
\[\leq 2\|x_{n_k} - w_j\| + \|y_{n_k} - x_{n_k}\| + |\alpha_{n_k} - \alpha_j| M.
\]
This implies that $(1 - \alpha_j)T_1w_j + \alpha_jT_2w_j = w_j$.\]
Let $\varepsilon > 0$ arbitrary. Then, by $\lim_{k} \|x_{n_k} - w_j\| = 0$, there exists some term $x_{(j)} = x_{n_{k_{j}}}$ of $\{x_n\}$ such that $\|x_{(j)} - w_j\| < \varepsilon$. □

**Theorem 6.5.** Let $c \in (0, 1)$ and $\{\alpha_n\}$ be a sequence defined by $(S)$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T_1$ and $T_2$ be nonexpansive self-mappings on $D$ such that $T_1T_2 = T_2T_1$. Let $x_1 \in D$ and $\{x_n\}$ be a sequence in $D$ defined by

$$x_{n+1} = c((1 - \alpha_n)T_1x_n + \alpha_nT_2x_n) + (1 - c)x_n \quad \text{for} \quad n \in \mathbb{N}.$$ 

Then $\{x_n\}$ converges strongly to a common fixed point of $T_1$ and $T_2$.

**Proof.** By Lemma 6.2 (2), there is a subsequence $\{\alpha_{n_j}\}$ such that $\alpha_{n_j} \in (0, 1)$ and $\lim n_j = \infty$.

By Lemma 6.4, we can take a sequence $\{w_{n_j}\}$ in $D$ and a subsequence $\{x_{(n_j)}\}$ of $\{x_n\}$ such that, for any $n_j$,

$$(1 - \alpha_{n_j})T_1w_{n_j} + \alpha_{n_j}T_2w_{n_j} = w_{n_j}, \quad \|x_{(n_j)} - w_{n_j}\| < 1/n_j.$$ 

Since $D$ is compact, there is a subsequence of $\{w_{n_j}\}$ which converges strongly to some $w \in D$. No loss of generality, we can assume that $\{w_{n_j}\}$ itself converges strongly to $w \in D$. By Theorem 6.3, we know $w \in F(T_1) \cap F(T_2)$.

It is easy to see that, for any $n_j$,

$$\|x_{(n_j)} - w\| \leq \|x_{(n_j)} - w_{n_j}\| + \|w_{n_j} - w\| < \|w_{n_j} - w\| + 1/n_j.$$ 

Then, $\{\|x_{(n_j)} - w\|\}$ converges to 0, that is, $\lim n_j \|x_{(n_j)} - w\| = 0$.

On the other hand, by $w \in F(T_1) \cap F(T_2)$, we have that, for $n \in \mathbb{N}$,

$$\|x_{n+1} - w\| = \|c((1 - \alpha_n)T_1x_n + \alpha_nT_2x_n) + (1 - c)x_n - w\| 
\leq c(1 - \alpha_n)\|T_1x_n - w\| + \alpha_n\|T_2x_n - w\| + (1 - c)\|x_n - w\| 
\leq c(1 - \alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| + (1 - c)\|x_n - w\| = \|x_n - w\|.$$ 

Then, $\{\|x_{(n)} - w\|\}$ is non-increasing and converges. Since $\{\|x_{(n)} - w\|\}$ converges to 0, $\{\|x_n - w\|\}$ also converges to 0. □

**Remark 6.6.** In Theorem 6.5, we can replace the condition $T_1T_2 = T_2T_1$ by the condition $T_2(F(T_1)) \subset F(T_1)$.

7. **Common fixed points for some kind of two mappings**

In this section, we consider common fixed points of two mappings $T_1$ and $T_2$. In preparation for our argument, we need Lemma 7.1.
Lemma 7.1. Let $D$ be a subset of a Banach space $E$. Let $T_1$ be a self-mapping on $D$ and let $T_2$ be a mapping from $D$ into $E$. Then,

$$F(T_1) \cap F(T_2) = F(T_1) \cap F(T_2T_1) \subset F(T_2T_1).$$

Proof. If $x \in F(T_1) \cap F(T_2)$ then $T_2T_1x = T_2x = x$. This implies that

$$F(T_1) \cap F(T_2) \subset F(T_2T_1),$$

$$F(T_1) \cap F(T_2) = F(T_1) \cap F(T_1) \subset F(T_1) \cap F(T_2T_1) \subset F(T_2T_1).$$

Assume that $x \in F(T_1) \cap F(T_2T_1)$. It is obvious that

$$T_1x = x, \quad F(T_1) \cap F(T_2T_1) \subset F(T_1).$$

It is also easy to see that

$$T_2x = T_2T_1x = x \quad \text{and} \quad F(T_1) \cap F(T_2T_1) \subset F(T_2).$$

Then, we have $F(T_1) \cap F(T_2T_1) \subset F(T_1) \cap F(T_2)$. \hfill $\Box$

The following theorem is derived from Lemma 7.1 and Theorem 5.1.

Theorem 7.2. Let $a$ be a real number belonging to $(0, 1)$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T_1$ be a nonexpansive self-mapping on $D$ and let $T_2$ be a mapping from $D$ into $E$. Assume that the mapping $T_2T_1$ is a nonexpansive self-mapping on $D$ such that

$$(III) \quad T_1(F(T_2T_1)) \subset F(T_2T_1).$$

Let $S_1 = aT_2T_1 + (1 - a)I$ and $S_2 = aT_1 + (1 - a)I$. For each $n \in N$, define nonexpansive self-mappings $L_{(1,n)}$ and $L_{(2,n)}$ on $D$ by

$$L_{(1,n)} = S_1^n, \quad L_{(2,n)} = \pi(S_2L_{(1,2)}).$$

Then, there exists a nonexpansive self-mapping $P$ on $D$ such that

(a) \{ $L_{(2,n)}$ $\}$ converges uniformly to $P$,
(b) $P(D) = F(P) = F(T_2) \cap F(T_1)$.

Proof. By our assumptions and Theorem 5.1, there exists a nonexpansive self-mapping $P$ on $D$ such that

(a) \{ $L_{(2,n)}$ $\}$ converges uniformly to $P$,
(b) $P(D) = F(P) = F(T_1) \cap F(T_2T_1)$.

By Lemma 7.1, we have $F(T_1) \cap F(T_2T_1) = F(T_2) \cap F(T_1)$. \hfill $\Box$

Remark 7.3. In Theorem 7.2, it is obvious that we can replace the condition (III) by the condition $T_1(T_2T_1) = (T_2T_1)T_1$.

The following theorem is derived from Lemma 7.1 and Theorem 6.5.
Theorem 7.4. Let $c \in (0, 1)$ and $\{\alpha_n\}$ be a sequence defined by $(S)$. Let $D$ be a compact convex subset of a Banach space $E$. Let $T_1$ be a nonexpansive self-mapping on $D$ and $T_2$ be a mapping from $D$ into $E$. Assume that the mapping $T_2T_1$ is a nonexpansive self-mapping on $D$ satisfying $T_1(T_2T_1) = (T_2T_1)T_1$. Let $x_1 \in D$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = c((1 - \alpha_n)T_2T_1x_n + \alpha_nT_1x_n) + (1 - c)x_n \quad \text{for } n \in \mathbb{N}.$$ 

Then, $\{x_n\}$ converges strongly to a common fixed point of $T_1$ and $T_2$.

Proof. By our assumptions and Theorem 6.5, $\{x_n\}$ converges strongly to some point $w \in F(T_1) \cap F(T_2T_1)$. By Lemma 7.1, we have the desired result.

Remark 7.5. Let $R^2$ be 2-dimensional Euclidean space. Let $D = [0, 1]^2 \subset R^2$. Let $T_1$ and $T_2$ be mappings defined by

$$T_1(a, b) = (a/2, b), \quad T_2(a, b) = (2a, b) \quad \text{for } (a, b) \in D.$$ 

Then, we have $T_2T_1(a, b) = (a, b) = I(a, b)$ and

$$(T_2T_1T_1(a, b) = (a/2, b) = T_1(T_2T_1)(a, b).$$

This implies that $T_1$ and $T_2T_1$ are nonexpansive self-mappings on $D$ such that $(T_2T_1T_1 = T_1(T_2T_1)).$ However, $T_2$ is neither nonexpansive nor self-mapping.

References


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