A GOLDEN COMPLEMENTARY DUALITY IN QUADRATIC OPTIMIZATION PROBLEM

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Abstract. A real number $\phi$ is called Golden number, that is, $\phi \approx 1.618$. In this paper, we show the following triple Golden duality about a pair of primal and dual quadratic optimization problems. (i) The value of maximum and minimum are the identical Golden quadratic function (Golden dual). (ii) Both the minimum point and the maximum point constitute Golden path of rate $\phi^{-2}$ (Golden). (iii) An alternate sequence of both the Golden paths constitutes Golden path of rate $\phi^{-1}$ (Golden complementarity). This triplet is called Golden complementary duality.

1. ALTERNATELY FIBONACCI COMPLEMENTARY DUALITY

First, we consider a pair of primal and dual quadratic optimization problems

\[
\begin{align*}
\text{(P}_1\text{)} & \quad \text{minimize} \quad \sum_{k=0}^{n-1} [(x_k + x_{k+1})^2 + x_{k+1}^2] \\
\text{subject to} & \quad (i) \quad x \in \mathbb{R}^n \\
& \quad (ii) \quad x_0 = c,
\end{align*}
\]

where $c > 0$, $x = (x_1, x_2, \ldots, x_n)$, and

\[
\begin{align*}
\text{(D}_1\text{)} & \quad \text{maximize} \quad 2c\mu_0 - \mu_0^2 - \sum_{k=0}^{n-2} [(\mu_k + \mu_{k+1})^2 + \mu_{k+1}^2] - \mu_{n-1}^2 \\
\text{subject to} & \quad (i) \quad \mu \in \mathbb{R}^n
\end{align*}
\]

where $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$.

Theorem 1.1. For the problems (P$_1$) and (D$_1$), let $x = (x_0, x_1, \ldots, x_n)$ be feasible of the primal problem (P$_1$) and $\mu = (\mu_0, \mu_1, \ldots, \mu_{n-1})$ be feasible of the dual problem (D$_1$). Then, $\min(P_1) \geq \max(D_1)$.

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Definition 1.2. The Fibonacci sequence \( \{ F_n \} \) is defined as the solution to the second-order linear difference equation,
\[
F_{n+2} - F_{n+1} - F_n = 0, \quad F_1 = 1, \quad F_0 = 0.
\]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\( n \) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
\( F_n \) & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 & 144 & 233 & 377 & 610 & 987 \\
\hline
\end{tabular}
\caption{Fibonacci sequence \( \{ F_n \} \)}
\end{table}

Lemma 1.3 (Lucas formula). Let \( \{ F_k \} \) be the Fibonacci sequence. For any \( n \geq 1 \), it holds that
\[
\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.
\]

Theorem 1.4. The primal problem \((P_1)\) has the minimum value \( m_1 = \frac{F_{2n}}{F_{2n+1}} c^2 \) at the point
\[
\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{n-1}, \hat{x}_n)
= \frac{c}{F_{2n+1}} \left( F_{2n+1}, -F_{2n-1}, F_{2n-3}, \ldots, (-1)^k F_{2n-2k+1}, \ldots, (-1)^{n-1} F_3, (-1)^n F_1 \right).
\]
Proof. See [12].

Theorem 1.5. The dual problem \((D_1)\) has the maximum value \( M_1 = \frac{F_{2n}}{F_{2n+1}} c^2 \) at the point
\[
\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*, \ldots, \mu_{k-1}^*, \ldots, \mu_{n-1}^*)
= \frac{c}{F_{2n+1}} \left( F_{2n}, -F_{2n-2}, F_{2n-4}, \ldots, (-1)^k F_{2n-2k}, \ldots, (-1)^{n-1} F_2 \right).
\]
Proof. See [12].

Theorem 1.6. If \((P_1)\) has an optimal solution, then there is a feasible solution of \((D_1)\) and the two objectives have the same values. Moreover,
\[
\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{k-1}, \hat{x}_n)
= \frac{c}{F_{2n+1}} \left( F_{2n+1}, -F_{2n-1}, F_{2n-3}, \ldots, (-1)^k F_{2n-2k+1}, \ldots, (-1)^n F_1 \right)
is the optimal solution of \((P_1)\), and
\[
\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*, \ldots, \mu_k^*, \ldots, \mu_{n-1}^*)
\]
\[
= \frac{c}{F_{2n+1}} \left( F_{2n}, -F_{2n-2}, F_{2n-4}, \ldots,\ (-1)^k F_{2n-2k}, \ldots, (-1)^{n-1} F_2 \right)
\]
is the optimal solution of \((D_1)\). Hence, the alternately Fibonacci complementary duality holds between \((P_1)\) and \((D_1)\).

In [12], Iwamoto S. and Kimura Y. proved to satisfy the alternately Fibonacci complementary duality between the primal problem \((P_1)\) and its dual problem \((D_1)\). Both optimal solutions and optimal points are characterized by the Fibonacci sequence:

(i) (duality) The value of maximum and minimum are the same:
\[
m_1 = M_1 = \frac{F_{2n}}{F_{2n+1}} c^2.
\]
It is a quadratic function of \(c\), whose coefficient is ratio of adjacent Fibonacci number. This is the first alternately Fibonacci complementary duality.

(ii) (2-step alternately Fibonacci) Both the minimum point \(\hat{x}\) and the maximum point \(\mu^*\) are two-step alternate Fibonacci sequence:
\[
\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \hat{x}_3, \ldots, \hat{x}_k, \ldots, \hat{x}_{n-1}, \hat{x}_n)
\]
\[
= \frac{c}{F_{2n+1}} \left( F_{2n+1}, -F_{2n-1}, F_{2n-3}, \ldots,\ (-1)^k F_{2n-2k+1}, \ldots, (-1)^n F_1 \right)
\]
and
\[
\mu^* = (\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*, \ldots, \mu_k^*, \ldots, \mu_{n-1}^*)
\]
\[
= \frac{c}{F_{2n+1}} \left( F_{2n}, -F_{2n-2}, F_{2n-4}, \ldots,\ (-1)^k F_{2n-2k}, \ldots, (-1)^{n-1} F_2 \right).
\]
This is the second.

(iii) (alternately Fibonacci complementarity) Both the optimal points constitute alternately the (1-step) alternate two-run Fibonacci sequence:
\[
(x_0, \mu_0^*, \hat{x}_1, \mu_1^*, \ldots, \hat{x}_k, \mu_k^*, \ldots, \mu_{n-1}^*, \hat{x}_n)
\]
\[
= \frac{c}{F_{2n+1}} \left( F_{2n+1}, F_{2n}, -F_{2n-1}, -F_{2n-2}, \ldots,\ (-1)^k F_{2n-2k+1}, (-1)^k F_{2n-2k}, \ldots, (-1)^{n-1} F_2, (-1)^n F_1 \right).
\]
This is the third.
This triplet is called the \textit{alternately Fibonacci complementary duality}.

2. \textsc{Golden complementary duality}

In this section, we consider two pairs of primal and dual quadratic optimization problems,

\begin{align*}
\text{(P}_2\text{)} \quad & \text{minimize } \sum_{k=0}^{n-1} [(x_k - x_{k+1})^2 + x_{k+1}^2] + \phi^{-1} x_n^2 \\
& \text{subject to } (i) \ x \in \mathbb{R}^n \\
& \quad (ii) \ x_0 = c
\end{align*}

where $c \in \mathbb{R}^1$, $x = (x_1, x_2, \ldots, x_n)$, and for any $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n$,

\begin{align*}
\text{(D}_2\text{)} \quad & \text{maximize } 2c\mu_1 - \mu_1^2 - \sum_{k=1}^{n-1} [(\mu_k - \mu_{k+1})^2 + \mu_{k+1}^2] - \phi^{-1} \mu_n^2 \\
& \text{subject to } (i) \ \mu \in \mathbb{R}^n
\end{align*}

where $\phi$ is the \textit{Golden number}, that is,

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$ 

Note that $1 + \phi^{-1} = \phi$.

\textbf{Lemma 2.1.} \textit{Let $n$ be a natural number. For the Golden number, it holds that}

\begin{enumerate}
    \item[(i)] $\sum_{k=1}^{n} \phi^{2k-1} = \phi^{2n} - 1$,
    \item[(ii)] $\sum_{k=1}^{n} \phi^{-2k} = \phi^{-1} - \phi^{-2n-1}$,
    \item[(iii)] $2 \sum_{k=1}^{n} \phi^{-3k-1} + \phi^{-3n-2} = \phi^{-2}$.
\end{enumerate}

\textbf{Lemma 2.2.} \textit{It holds that}

$$\phi^n + \phi^{n+1} = \phi^{n+2} \quad n = \cdots, -2, -1, 0, 1, 2, \cdots.$$ 

\textbf{Theorem 2.3.} \textit{For the problems (P}_2\text{) and (D}_2\text{), let $x = (x_0, x_1, \ldots, x_n)$ be feasible of the primal problem (P}_2\text{) and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ be feasible of the dual problem (D}_2\text{). Then, } \min(P_2) \geq \max(D_2).
Proof. Let \( x = (x_0, x_1, \ldots, x_n) \) be feasible of the primal problem \((P_2)\), and \( I(x) \) be the objective function of \((P_2)\), that is,

\[
I(x) := \sum_{k=0}^{n-1} [(x_k - x_{k+1})^2 + x_{k+1}^2] + \phi^{-1} x_n^2.
\]

Let

\[
u_k = x_k - x_{k+1} \quad 0 \leq k \leq n - 1
\]

Then we have for any \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n \),

\[
I(x) = \sum_{k=0}^{n-1} [u_k^2 + x_{k+1}^2 + 2\mu_{k+1} (x_k - x_{k+1})] + \phi^{-1} x_n^2.
\]

Since

\[
I(x) = 2c\mu_1 + \sum_{k=0}^{n-1} (u_k^2 - 2\mu_{k+1} u_k) + \sum_{k=1}^{n-1} [x_k^2 - 2(\mu_k - \mu_{k+1}) x_k] + (1 + \phi^{-1}) x_n^2 - 2\mu_n x_n,
\]

it holds that

\[
I(x) \geq 2c\mu_1 - \mu_1^2 - \sum_{k=1}^{n-1} [(\mu_k - \mu_{k+1})^2 + \mu_{k+1}^2] - \phi^{-1} \mu_n^2
\]

for any \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \), \( u = (u_0, \ldots, u_{n-1}) \in \mathbb{R}^n \) satisfying (2.2) and any \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n \). Let us take

\[
J(\mu) = 2c\mu_1 - \mu_1^2 - \sum_{k=1}^{n-1} [(\mu_k - \mu_{k+1})^2 + \mu_{k+1}^2] - \phi^{-1} \mu_n^2.
\]

Then it holds that

\[
I(x) \geq J(\mu)
\]

for any feasible \( x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^n \) of the primal problem \((P_2)\) and any feasible \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{R}^n \) of the dual problem \((D_2)\). \qed
Theorem 2.4. The primal problem (P_2) has the minimum value \( m_2 = \phi^{-1} c^2 \) at the point
\[ \hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{n-1}, \hat{x}_n) = c\left(\phi^{-2}, \phi^{-4}, \ldots, \phi^{-2n+2}, \phi^{-2n}\right). \]

Proof. Since the objective function \( I(x) \) (see (2.1)) for (P_2) is convex for any \( x = (x_1, x_2, \ldots, x_n) \), the minimum point for (P_2) satisfies the first order optimality condition,
\[ \frac{\partial I}{\partial x_k} = 0 \quad 1 \leq k \leq n. \]
From (2.6), that is,
\[ -(x_k - x_{k+1}) + x_{k+1} + (x_{k+1} - x_{k+2}) = 0 \quad k = 0, 1, \ldots, n - 2, \]
\[ -(x_{n-1} - x_n) + x_n + \phi^{-1} x_{n} = 0. \]
These conditions (2.7) and (2.8) are equivalent to the following \( n \) equations:
\[ \frac{x_k - x_{k+1}}{\phi} = \frac{x_{k+1}}{1} \quad 0 \leq k \leq n - 1. \]

We note that \( 1 + \phi^{-1} = \phi \). The condition \((GC)_P\) is called the Golden condition for (P_2). From the condition \((GC)_P\),
\[ x_{k+1} = \phi^{-2} x_k \quad 0 \leq k \leq n - 1. \]
Thus, we have
\[ \hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k, \hat{x}_k, \ldots, \hat{x}_{n-1}, \hat{x}_n) \]
\[ = c\left(1, \phi^{-2}, \phi^{-4}, \ldots, \phi^{-2k}, \ldots, \phi^{-2n+2}, \phi^{-2n}\right). \]

Next, we prove the minimum value \( m_2 = \phi^{-1} c^2 \). From (2.1) and (2.11),
\[ I(\hat{x}) = (1 - \phi^{-2})^2 c^2 + (\phi^{-2})^2 c^2 + (\phi^{-4})^2 c^2 + (\phi^{-2} - \phi^{-4})^2 c^2 + \cdots \]
\[ + \left(\phi^{-2n-1} - \phi^{-2n}\right)^2 c^2 + \phi^{-2n} c^2 + \phi^{-1} \left(\phi^{-2n} c^2 \right) + \cdots \]
\[ = c^2 \left[ (\phi^{-1})^2 + \phi^{-4} + (\phi^{-3})^2 + \phi^{-8} + \cdots \right. \]
\[ + \left. \left(\phi^{-2n-1}\right)^2 + \phi^{-4n} + \phi^{-4n-1} \right] \]
\[ = c^2 \left[ \sum_{k=1}^{2n} \phi^{-2k} + \phi^{-4n-1} \right] \]
\[ = c^2 \left[ (\phi^{-1} - \phi^{-4n-1}) + \phi^{-4n-1} \right] \]
Consequently, it holds that
\[ m_2 = I(\hat{x}) = \phi^{-1}c^2. \]

- **Theorem 2.5.** The dual problem (D_2) has the maximum value \( M_2 = \phi^{-1}c^2 \) at the point
\[
\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_k^*, \ldots, \mu_{n-1}^*, \mu_n^*)
= c \left( \phi^{-1}, \phi^{-3}, \ldots, \phi^{-2k+1}, \ldots, \phi^{-2n+3}, \phi^{-2n+1} \right).
\]

- **Proof.** Since the objective function \( J(\mu) \) (see (2.4)) for (D_2) is concave for any \( \mu = (\mu_1, \mu_2, \ldots, \mu_n) \), the maximum point for (D_2) satisfies the first order optimality condition,
\[ \frac{\partial J}{\partial \mu_k} = 0 \quad 1 \leq k \leq n. \]

These conditions (2.13) are equivalent to the following \( n \) equations:
\[
\begin{align*}
&c - \mu_1 - (\mu_1 - \mu_2) = 0, \\
&(\mu_k - \mu_{k+1}) - (\mu_{k+1} - \mu_{k+2}) = 0 \quad k = 1, 2, \ldots, n - 2, \\
&(\mu_n - \mu_n) - \phi^{-1} \mu_n = 0.
\end{align*}
\]

From these equations, we have the following condition (GC)_D,
\[ \frac{c - \mu_1}{\phi} = \frac{\mu_1 - \mu_2}{\phi} = \frac{\mu_2}{1}, \]
\[ \frac{\mu_{k+1} - \mu_k}{\phi} = \frac{\mu_k}{1} \quad k = 2, 3, \ldots, n - 1. \]

The condition (GC)_D is Golden condition for (D_2). From (GC)_D, we have the solutions
\[ \mu_k^* = \phi^{-2k+1}c \quad 1 \leq k \leq n \]
for dual problem (D_2), that is,
\[ \mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_k^*, \ldots, \mu_{n-1}^*, \mu_n^*)
= c \left( \phi^{-1}, \phi^{-3}, \ldots, \phi^{-2k+1}, \ldots, \phi^{-2n+3}, \phi^{-2n+1} \right). \]

Next, we prove the maximum value \( M_2 = \phi^{-1}c^2 \). From (2.4) and (2.19),
\[ J(\mu^*) = 2\phi^{-1}c^2 - \phi^{-2}c^2 \]
\[ - \sum_{k=1}^{n-1} \left( (\phi^{-2k+1} - \phi^{-2k-1})^2 c^2 + \phi^{2(-2k-1)} c^2 \right) = \phi^{-1} \cdot \phi^{2(-2n+1)} c^2 \]

\[ = c^2 \left[ 2\phi^{-1} - \phi^{-2} - \sum_{k=1}^{n-1} \left( (\phi^{-2k})^2 + \phi^{-4k-2} \right) - \phi^{-4n+1} \right] \]

\[ = c^2 \left[ 2\phi^{-1} - \sum_{k=1}^{2n-1} \phi^{-2k} - \phi^{-4n+1} \right] \]

\[ = c^2 \left[ 2\phi^{-1} - \left( \phi^{-1} - \phi^{-2(2n-1)-1} \right) - \phi^{-4n+1} \right] \]

\[ = \phi^{-1} c^2. \]

Consequently, it holds that

(2.20) \[ M_2 = J(\mu^*) = \phi^{-1} c^2. \]

\[ \square \]

**Definition 2.6.** Let \( \{x_n\}_{n \geq 1} \) be a sequence. \( \{x_n\} \) is called a *Golden path* (GP) if

\[ x_n = c\phi^{-2n} \text{ or } x_n = c\phi^{-n}, \]

where \( c \) is a constant. In particular, \( \{x_n\} \) is called Golden path of 1 : \( \phi \) if

\[ x_n = c\phi^{-2n}, \]

and \( \{x_n\} \) is called Golden path of \( \phi : 1 \) if

\[ x_n = c\phi^{-n}. \]

There are the following triplet relations between the minimum point \( \hat{x} \) of the primal problem (P\(_2\)) and the maximum point \( \mu^* \) of the dual problem (D\(_2\)).

(i) (Golden dual) The minimum value of the maximum value are the same.

\[ m_2 = M_2 = \phi^{-1} c^2. \]

Both the minimum value function and the maximum value function are the identical inverse-Golden quadratic (inverse-Golden dual). This is the first Golden complementary duality.
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(ii) (Golden) Both the minimum point $\hat{x}$ and the maximum point $\mu^*$ constitute Golden paths of the type of $1 : \phi$, as was shown. This is the second.

\[
\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k, \ldots, \hat{x}_n)
\]

\[
= c \left( 1, \phi^{-2}, \phi^{-4}, \ldots, \phi^{-2k}, \ldots, \phi^{-2n} \right)
\]

and

\[
\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \ldots, \mu_k^*, \ldots, \mu_n^*)
\]

\[
= c \left( \phi^{-1}, \phi^{-3}, \phi^{-5}, \ldots, \phi^{-2k+1}, \ldots, \phi^{-2n+1} \right).
\]

(iii) (Golden complementarity) Both the optimal points constitute alternately the Golden paths of the type of $\phi : 1$.

\[
(x_0, \mu_1^*, \hat{x}_1, \mu_2^*, \ldots, \hat{x}_k, \mu_{k+1}^*, \ldots, \mu_n^*, \hat{x}_n)
\]

\[
= c \left( 1, \phi^{-1}, \phi^{-2}, \phi^{-3}, \ldots, \phi^{-2k}, \phi^{-2k-1}, \ldots, \phi^{-2n+1}, \phi^{-2n} \right).
\]

This triplet is called the Golden complementary duality.

Theorem 2.7. If $(P_2)$ has an optimal solution, then there is a feasible solution of $(D_2)$ and the two objectives have the same values. Moreover,

\[
\hat{x} = (x_0, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_k, \ldots, \hat{x}_n)
\]

\[
= c \left( 1, \phi^{-2}, \phi^{-4}, \ldots, \phi^{-2k}, \ldots, \phi^{-2n} \right)
\]

is the optimal solution of $(P_2)$, and

\[
\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*, \ldots, \mu_k^*, \ldots, \mu_n^*)
\]

\[
= c \left( \phi^{-1}, \phi^{-3}, \phi^{-5}, \ldots, \phi^{-2k+1}, \ldots, \phi^{-2n+1} \right).
\]

is the optimal solution of $(D_n)$. Hence, the Golden complementary duality holds between $(P_2)$ and $(D_2)$.

Proof. It is obvious to prove this theorem from the proofs of theorem 2.4 and 2.5. \qed

References


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