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OPTIMAL REPLENISHMENT FOR EPQ MODELS UNDER CONDITIONS OF PERMISSIBLE DELAY IN PAYMENTS AND CASH DISCOUNT

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Abstract: In order to stimulate order quantity, the suppliers permit their retailers to delay the settlement of payments within a given period. Retailers can accumulate revenue and deposit it into an account for more profits before a permissible delay period. From the supplier's perspective, however, it is better to obtain payments from their retailers as soon as possible. This leads suppliers to implement the policy of a cash discount. When retailers utilize this policy and settle payments before the end of a given period, they can receive a cash discount. In general, retailers can only use one of these policies. In this paper, we propose a new economic production quantity (EPQ) model that considers both policies simultaneously. In this new model, retailers are allowed to separate payments into several parts, settling in the cash discount period and in a permissible delay period, and paying interest after the permissible delay period if necessary. This policy is called a two-stage payment. In addition to proposing this new model, this paper discusses the properties of its objective function. Based on these properties, the optimal solutions can be analytically determined. We also discuss some economic interpretations of the analytical method.

Key words: *EPQ, cash discount, permissible delay in payments*

Mathematics Subject Classification: *90B05*

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1 Introduction

In practice, suppliers often offer to their retailers two types of trade credit for their goods. If the retailer settles the payments within a permissible delay period, then no penalty will be charged by the supplier. Alternatively, in order to encourage retailers to settle payments as soon as possible, suppliers may offer a cash discount for retailers that settle their payment within the cash discount period, which is shorter than the permissible delay period. Retailers can accumulate the money obtained from selling goods and deposit it in a bank to earn interest before settling payments. However, retailers will be charged interest if the payments are settled after the permissible delay period. Hence, the cost function for retailers must consider setup costs and holding costs, as well as cash discounts, charged interest, and earned interest.

Although suppliers provide two kinds of trade credit, retailers can only choose one. If retailers choose the cash discount and the replenishment cycle time is less than the cash discount period M_1, then they must settle the full payment in M_1 in order to obtain as much benefit as possible. If retailers choose the permissible delay period M_2 for their

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payments and the replenishment cycle time is less than M_2, they must settle the full payment in M_2 in order to obtain as much benefit as possible. On the other hand, if retailers choose the cash discount and the replenishment cycle time is greater than M_1, then they settle the revenue of sold items in M_1 and pay the cost of interest charges for unsold items beyond M_1. Hence the payment is separated into two parts. If retailers choose M_2 for their payments and the replenishment cycle time is greater than M_2, they settle the revenue of sold items in M_2 and pay the cost of interest charges for unsold items beyond M_2. Again, the payment is separated into two parts. Based on these two periods, from the retailer's points of view, there are two policies. If the retailer chooses the permissible delay period for their payments, then the economic production quantity (EPQ) models under the condition of permissible delay in payment can be produced. Alternatively, if the retailers choose the cash discount period, then the EPQ models under condition of cash discount can be produced instead. Investigating these two models can help retailers to improve profit in comparison to traditional EPQ models. More details are given by Huang et al. [3].

In addition to the two policies discussed in [3], another policy is given in this paper. The retailers can obtain more profits, if they have the option of separating can separate the payment into two or three parts. If the replenishment cycle time is less than the permissible delay period M_2, then the first portion is settled in the cash discount period and the remaining portion is settled in the permissible delay period. This is in contrast to the policies of [3] that settle the whole payment in either M_1 or M_2. Our proposed policy separates the payment into two parts. In the case of the replenishment cycle time being greater than the permissible delay period M_2, then the first portion is settled in the cash discount period M_1, the second portion is settled in the permissible delay period M_2 and the retailer pays the cost of interest charges for unsold items beyond M_2. Both policies of [3] separate the payments into two portions, while our policy separates the payments into three parts. In order to distinguish the two policies and our given policy, we term our policy a two stage payment which was first discussed by [11] and refer to the the policies of [3] as single-stage payment.

Intuitively, if the unpaid payment after the cash discount period can generate more revenue from interest earned than from the cash discount rate, retailers should choose to separate the payments into more parts. This idea motivates the discussion of EPQ models under our policy. The first objective of this paper is to design an EPQ model with our proposed policy. The second objective is to discuss under what kind of situation the retailer should choose our policy. From the perspective of analytic decision making, a third objective is to find the minimum of the three models, that is, EPQ models under the condition of permissible delay in payment, EPQ models under the condition of cash discount, and EPQ models under our proposed policy.

Goyal [4] first developed the economic order quantity (EOQ) models under conditions of permissible delay in payments. Later, Aggarwal & Jaggi [1] and Chu et al. [2] discussed ordering policies of deteriorating items under permissible delay in payments. Jamal et al. [10] further generalized the models of deteriorating items under permissible delay in payments to allow for shortages. Huang [5] assumed that suppliers would offer the retailer a partially permissible delay in payments in the case of the order quantity being smaller than a predetermined quantity. Huang & Hsu [7] investigated the retailers' inventory policy under two levels of trade credit to reflect the supply chain management situation. More discussion related to EOQ models under conditions of permissible delay in payments can be found in Teng [13] and Ouyang et al.[12].

In addition to EOQ models, Huang & Chung[6] and Huang & Lai [8] discussed EPQ models under conditions of permissible delay in payments. They successfully extended

Goyal's model [4] to address the case where all items are replenished at a finite rate. Huang & Lin [9] and Huang et al. [3] discussed EPQ models under conditions of permissible delay in payments and cash discount.

The EPQ models under the two stage payment policy are given in Section 2. Some analysis of single-stage payment and two stage payment are given in Section 3. Furthermore, based on this analysis, we present an analytic method to find the minimum of these models. In Section 4, some numerical results are provided that show that the two stage payment is better than a single-stage payment in some cases. We also provide some economic interpretations in Section 4.

2 Optimal Replenishment Cycle of EPQ models under Conditions of Permissible Delay in Payments and Cash Discount

In this paper, we adopt the following notations and assumptions:
Notations:

D: demand rate(unit/years)

P: replenishment rate(unit/years)

s: selling price (dollars/unit)

p: purchasing price (dollars/unit)

A: setup cost (dollars/order)

h: stock holding cost per year excluding interest charges(dollars/unit)

I_c: interest charged on stock (dollars/year)

I_d: interest earned (dollars/year)

ρ: $\rho = \dfrac{D}{P}$

H: $H = h(1-\rho) = h\left(1 - \dfrac{D}{P}\right)$

r: cash discount rate

M_1: cash discount period(years)

M_2: permissible delay period(years)

T: cycle time(years)

Assumptions:

1. Both the demand rate and the replenishment rate are known and constant.
2. The demand rate D is smaller than the replenishment rate P, i.e. $D < P$ and $\rho < 1$.
3. Shortages are not allowed.
4. The unit selling price is strictly greater than the unit purchasing price, that is, $s > p$.

5. The supplier offers two periods for trade credit: the cash discount period M_1 and permissible delay period M_2. We assume the permissible delay period M_2 is strictly larger than the period M_1 for cash discount, that is, $M_1 < M_2$.

6. The interest rates I_c and I_d are constant and satisfy $0 < I_d < I_c$.

7. During the period in which the account is not settled, generated sales revenue is deposited in an interest-bearing account.

8. The rate r of cash discount is $r \in (0, 1)$.

9. No additional cash is added.

10. The time horizon is infinite.

11. The single-stage payment policy is defined as follows:

    (a) Suppose that the cycle time T satisfied $T \leq M_1$. Then the retailer has earned enough money to settle payments before the period M_1. If the retailer has chosen the cash discount policy, then it makes economic sense for them to delay the settlement of the replenishment account up to the last moment of the period M_1.

    (b) Suppose the cycle time T satisfied $T > M_1$. Then the retailer has not earned enough money to settle payments before the period M_1. If the retailer has chosen cash discount policy, then it makes economic sense for them to delay the settlement of the replenishment account up to the last moment of the period M_1. The cost of interest charged for the remaining amount needs to be considered.

    (c) Suppose the cycle time T satisfied $T \leq M_2$. Then the retailer has earned enough money to settle payments before the period M_2. If the retailer has chosen the permissible delay in payments policy, then it makes economic sense for them to delay the settlement of the replenishment account up to the last moment of the period M_2.

    (d) Suppose the cycle time T satisfied $T > M_2$. Then the retailer has not earned enough money to settle payments before the period M_2. If the retailer has chosen the permissible delay in payments policy, then it makes economic sense for them to delay the settlement of the replenishment account up to the last moment of the period M_2. The cost of interest charged for the remaining amount needs to be considered.

12. The two-stage payment policy is described as follows:
    The retailer settles part of the account at the end of the period M_1 and receives cash discount. The unpaid amount is paid at the end of the period M_2. If the cycle time T satisfied $T > M_2$, then the retailer has not earned enough money for the entire amount owed before the end of the period M_2 and the cost of interest charged for the remaining amount needs to be considered.

These two types of policies adopt different assumptions. In Section 2.1, we discuss the two-stage payment policy and adopt assumptions (1)-(10) and (12). In Section 2.2, we discuss the single-stage payment policy and adopt the assumptions (1)-(10) and (11).

2.1 Two-stage Payment Policy

In two stage payment policy, the following six cases (2.1a)-(2.1e) are considered.

$$M_1 \le \frac{DT}{P} \le T \le M_2, \tag{2.1a}$$

$$\frac{DT}{P} \le M_1 \le T \le M_2, \tag{2.1b}$$

$$M_1 \le M_2 \le \frac{DT}{P} \le T, \tag{2.1c}$$

$$M_1 \le \frac{DT}{P} \le M_2 \le T, \tag{2.1d}$$

$$\frac{DT}{P} \le M_1 \le M_2 \le T, \tag{2.1e}$$

$$\frac{DT}{P} \le T \le M_1 \le M_2. \tag{2.1f}$$

If the condition (2.1a) holds, then the retailer settles the payment pDM_1 at time M_1 and settle the remained payment $pD(T - M_1)$ at time M_2. Hence, if (2.1a) holds, then we define Case 1.1

$$\text{Case 1.1}: \frac{PM_1}{D} \le T \le M_2$$

and the average total cost function is defined by $ATC_{1.1}(T)$ which consists of average setup cost, average holding cost, average purchasing cost, average interest earned.

If the condition (2.1b) holds, then the retailer settles the payment pDM_1 at time M_1 and settle the remained payment $pD(T - M_1)$ at time M_2. Hence, if (2.1b) holds, then we define Case 1.2

$$\text{Case 1.2}: T \le \min\left\{\frac{PM_1}{D}, M_2\right\}$$

and the average total cost function is defined by $ATC_{1.2}(T)$ which consists of average setup cost, average holding cost, average purchasing cost, average interest earned.

If the condition (2.1c) holds, then the retailer settles the payment pDM_1 at time M_1, settle the payment $pD(M_2 - M_1)$ at time M_2. The retailer must pay the cost of interest charges for unsold items behind M_2. Hence, if (2.1c) holds, then we define Case 1.3

$$\text{Case 1.3}: M_2 \le \frac{PM_2}{D} \le T$$

and the average total cost function is defined by $ATC_{1.3}(T)$ which consists of average setup cost, average holding cost, average purchasing cost, average interest earned, average interest charges.

If the condition (2.1d) holds, then the retailer settles the payment pDM_1 at time M_1, settle the payment $pD(M_2 - M_1)$ at time M_2. The retailer must pay the cost of interest charges for unsold items behind M_2. Hence, if (2.1d) holds, then we define Case 1.4

$$\text{Case 1.4}: \max\left\{\frac{PM_1}{D}, M_2\right\} \le T \le \frac{PM_2}{D}$$

and the average total cost function is defined by $ATC_{1.4}(T)$ which consists of average setup cost, average holding cost, average purchasing cost, average interest earned, average interest charges.

If the condition (2.1e) holds, then the retailer settles the payment pDM_1 at time M_1, settle the payment $pD(M_2 - M_1)$ at time M_2. The retailer must pay the cost of interest charges for unsold items behind M_2. Hence, if (2.1e) holds, then we define Case 1.5

$$\text{Case 1.5}: \; M_2 \le T \le \frac{PM_1}{D}$$

and the average total cost function is defined by $ATC_{1.5}(T)$ which consists of average setup cost, average holding cost, average purchasing cost, average interest earned, average interest charges.

If the condition (2.1f) holds, then the retailer settles the whole payment at time M_1 in order to obtain the maximum benefit. Hence the value of the average total cost of (2.1f) is the same as the value of $ATC_{1.1}(M_1)$. Hence the condition (2.1f) is included by the condition (2.1a). In the rest of this subsection, we describe the explicit formulas of average total costs of the five cases: Case 1.1-Case 1.5.

(1) The sum of average setup cost and average holding cost is $\frac{A}{T} + \frac{DTH}{2}$.

(2) All the purchasing costs of Case 1.1-Case 1.5 are the same and the cost is $\frac{pD\left(T - rM_1\right)}{T}$ because all the cases settle the same payment at time M_1.

(3) Average Interest charges: Only Case 1.3-Case 1.5 need to consider interest charges.

Case 1.3 $\frac{pI_c(1-\rho)\left(DT^2 - PM_2^2\right)}{2T}$

Case 1.4 $\frac{pI_c(1-\rho)\left(DT^2 - PM_2^2\right)}{2T}$

Case 1.5 $\frac{pI_cD\left(T - M_2\right)^2}{2T}$

(3) Average Interest earned:

Case 1.1 $\frac{sI_dD\left(M_1\left(M_1 - M_2\right) - T\left(\frac{1}{2}T - M_2\right)\right)}{T}$

Case 1.2 $\frac{sI_dD\left(M_1\left(M_1 - M_2\right) - T\left(\frac{1}{2}T - M_2\right)\right)}{T}$

Case 1.3 $\frac{sI_dD\left(M_1\left(M_1 - M_2\right) + \frac{1}{2}M_2^2\right)}{T}$

Case 1.4 $\frac{sI_dD\left(M_1\left(M_1 - M_2\right) + \frac{1}{2}M_2^2\right)}{T}$

Case 1.5 $\frac{sI_dD\left(M_1\left(M_1 - M_2\right) + \frac{1}{2}M_2^2\right)}{T}$

The average total costs of Case 1.1-Case1.5 are defined as follows:

$$ATC_{1.1}\left(T\right) = \frac{A}{T} + \frac{DTH}{2} + \frac{pD\left(T - rM_1\right)}{T} - \frac{sI_dD\left(M_1\left(M_1 - M_2\right) - T\left(\frac{1}{2}T - M_2\right)\right)}{T}, \quad (2.2)$$

$$ATC_{1.2}\left(T\right) = \frac{A}{T} + \frac{DTH}{2} + \frac{pD\left(T - rM_1\right)}{T} - \frac{sI_dD\left(M_1\left(M_1 - M_2\right) - T\left(\frac{1}{2}T - M_2\right)\right)}{T},$$

$$
\begin{aligned}
ATC_{1.3}(T) = \; & \frac{A}{T} + \frac{DTH}{2} + \frac{pD(T - rM_1)}{T} - \frac{sI_dD\left(M_1(M_1 - M_2) + \frac{1}{2}M_2^2\right)}{T} \\
& + \frac{pI_c(1-\rho)\left(DT^2 - PM_2^2\right)}{2T},
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
ATC_{1.4}(T) = \; & \frac{A}{T} + \frac{DTH}{2} + \frac{pD(T - rM_1)}{T} - \frac{sI_dD\left(M_1(M_1 - M_2) + \frac{1}{2}M_2^2\right)}{T} \\
& + \frac{pI_cD(T - M_2)^2}{2T},
\end{aligned} \tag{2.4}
$$

and

$$
ATC_{1.5}(T) = \frac{A}{T} + \frac{DTH}{2} + \frac{pD(T - rM_1)}{T} - \frac{sI_dD\left(M_1(M_1 - M_2) + \frac{1}{2}M_2^2\right)}{T} + \frac{pI_cD(T - M_2)^2}{2T}.
$$

Therefore, the average total cost function $ATC_D(T)$ of EPQ models under conditions of cash discount and permissible delay in payment in a cycle time T is formulated as

$$
ATC_D(T) = \begin{cases} ATC_{1.1}(T) & \text{if } M_1 \le T \le M_2 \\ ATC_{1.3}(T) & \text{if } M_2 \le \frac{PM_2}{D} \le T \\ ATC_{1.4}(T) & \text{if } M_2 \le T \le \frac{PM_2}{D} \end{cases}
$$

Note, since the average total cost function $ATC_{1.2}(T)$ is the same as $ATC_{1.1}(T)$ and the average total cost function $ATC_{1.5}(T)$ is the same as $ATC_{1.4}(T)$, Case 1.2 is combined with Case 1.1 and Case 1.5 is combined with Case 1.4.

2.2 Single-stage Payment Policy

According to [3], we consider the following cases:

Case 2.1 : $0 \le T \le M_1$;
Case 2.2 : $M_1 \le T \le PM_1/D$;
Case 2.3 : $PM_1/D \le T$;
Case 2.4 : $0 \le T \le M_2$;
Case 2.5 : $M_2 \le T \le PM_2/D$;
Case 2.6 : $PM_2/D \le T$.

and re-define the cost functions in the follows. In Case 2.1-2.6, the average total cost functions are defined by

$$
\begin{aligned}
ATC_{2.1}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD(1-r) - sI_dD\left(M_1 - \frac{1}{2}T\right) \\
ATC_{2.2}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD(1-r) - \frac{sI_dDM_1^2}{2T} + \frac{p(1-r)I_cD(T - M_1)^2}{2T} \\
ATC_{2.3}(T) &= \frac{A}{T} + \frac{DTH}{2} + p(1-r)D + \frac{pI_c(1-r)(1-\rho)\left(DT^2 - PM_1^2\right)}{2T} - \frac{sI_dDM_1^2}{2T} \\
ATC_{2.4}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD - sI_dD\left(M_2 - \frac{1}{2}T\right) \\
ATC_{2.5}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD - \frac{sI_dDM_2^2}{2T} + \frac{pI_cD(T - M_2)^2}{2T} \\
ATC_{2.6}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD + \frac{pI_c(1-\rho)\left(DT^2 - PM_2^2\right)}{2T} - \frac{sI_dDM_2^2}{2T}.
\end{aligned}
$$

The detail of model formulation of the function $ATC_{2.1}(T)$ is referred to the function $TVC_{1.3}(T)$ of [3]. Similarly,

the function $ATC_{2.2}(T)$ is referred to the function $TVC_{1.2}(T)$ of [9];
the function $ATC_{2.3}(T)$ is referred to the function $TVC_{1.1}(T)$ of [9];
the function $ATC_{2.4}(T)$ is referred to the function $TVC_{2.3}(T)$ of [9];
the function $ATC_{2.5}(T)$ is referred to the function $TVC_{2.2}(T)$ of [9];
the function $ATC_{2.6}(T)$ is referred to the function $TVC_{2.1}(T)$ of [9].

Hence we define the following functions. The average total cost $ATC_{CD}(T)$ of EPQ models under conditions of cash discount is formulated as

$$ATC_{CD}(T) = \begin{cases} ATC_{2.1}(T) & \text{if } 0 \le T \le M_1 \\ ATC_{2.2}(T) & \text{if } M_1 \le T \le PM_1/D \\ ATC_{2.3}(T) & \text{if } PM_1/D \le T \end{cases}.$$

The average total cost $ATC_{PD}(T)$ of EPQ models under conditions of permissible delay in payments is formulated as

$$ATC_{PD}(T) = \begin{cases} ATC_{2.4}(T) & \text{if } 0 \le T \le M_2 \\ ATC_{2.5}(T) & \text{if } M_2 \le T \le PM_2/D \\ ATC_{2.6}(T) & \text{if } PM_2/D \le T \end{cases}.$$

The objective of this paper is to solve the problem

$$\min \left\{ \min_{T\ge 0} ATC_D(T), \min_{T\ge 0} ATC_{CD}(T), \min_{T\ge 0} ATC_{PD}(T) \right\}. \tag{2.5}$$

3 Theoretical Analysis

For short notation, we define some parameters

$$\begin{aligned} &\Delta = rp - sI_d\,(M_2 - M_1) \\ &k_1 = DM_1^2(H + sI_d), \\ &k_2 = DM_2^2(H + sI_d), \\ &k_3 = 2DM_1\Delta, \\ &k_4 = \left(\frac{P^2}{D^2} - 1\right) DM_1^2(H + pI_c(1-r)), \\ &k_5 = \left(\frac{P^2}{D^2} - 1\right) DM_2^2(H + pI_c). \end{aligned} \tag{3.1}$$

Then we can observe that

$$k_1 = DM_1^2(H + sI_d) < DM_2^2(H + sI_d) = k_2 \tag{3.2}$$

and

$$k_4 = \left(\frac{P^2}{D^2} - 1\right) DM_1^2(H + pI_c(1-r)) < \left(\frac{P^2}{D^2} - 1\right) DM_2^2(H + pI_c) = k_5. \tag{3.3}$$

Moreover, if $\Delta = rp - sI_d\,(M_2 - M_1) > 0$, then $k_3 = 2DM_1\Delta > 0$. Hence

$$k_1 < k_1 + k_3 < k_2 + k_3 < k_2 + k_3 + k_5 \text{ if } \Delta > 0. \tag{3.4}$$

Because of the inequalities (3.2), (3.3) and (3.4), we can simply determine the optimal solutions of average total cost functions $ATC_{i,j}(T)$, and then determine the optimal value of (2.5).

First, we analyze the average total cost functions $ATC_{1.1}(T)$, $ATC_{1.3}(T)$ and $ATC_{1.4}(T)$ in Theorem 3.1, Theorem 3.2 and Theorem 3.3 respectively. We obtain that the function $ATC_{1.1}(T)$ is not always convex, although the average total cost function of traditional EPQ models is convex. We describe that if $A - DM_1\Delta > 0$, then $ATC_{1.1}(T)$ is convex. On the other hand, if $A - DM_1\Delta \le 0$, then $ATC_{1.1}(T)$ is increasing. Moreover, we use these two observations to obtain the optimum of $\min_{T\in[M_1,M_2]} ATC_{1.1}(T)$.

Theorem 3.1. *Let*

$$\lambda_{1.1} = A - DM_1\Delta. \tag{3.5}$$

If $\lambda_{1.1} > 0$, then we define

$$\overline{T_{1.1}} = \sqrt{\frac{2\lambda_{1.1}}{D(H + sI_d)}}. \tag{3.6}$$

Then we obtain the statements as follows:

(1) *If $\lambda_{1.1} > 0$, then $ATC_{1.1}(T)$ is strictly convex. The stationary point of $ATC_{1.1}(T)$ is $\overline{T_{1.1}}$.*

(2) *If $\lambda_{1.1} = 0$, then $ATC_{1.1}(T)$ is strictly increasing.*

(3) *If $\lambda_{1.1} < 0$, then $ATC_{1.1}(T)$ is strictly increasing and concave.*

(4) *If $2A < k_3$, then $\lambda_{1.1} < 0$.*

(5) *If $2A > k_3$, then $\lambda_{1.1} > 0$.*

(6) *If $2A \in (k_3, k_1 + k_3)$, then $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} < M_1$.*

(7) *If $2A \ge k_1 + k_3$, then $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} \ge M_1$.*

(8) *If $2A \in (k_3, k_2 + k_3)$, then $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} < M_2$.*

(9) *If $2A \ge k_2 + k_3$, then $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} \ge M_2$.*

(10) *An optimal solution of $\min_{T\in[M_1,M_2]} ATC_{1.1}(T)$ is $T^*_{1.1}$ which is defined as*

$$T^*_{1.1} = \begin{cases} M_1 & \text{if } 2A < k_1 + k_3 \\ \overline{T_{1.1}} & \text{if } 2A \in [k_1 + k_3, k_2 + k_3) \\ M_2 & \text{if } 2A \ge k_2 + k_3 \end{cases}.$$

In Theorem 3.2, we discuss some properties of the function $ATC_{1.3}(T)$. Similar to Theorem 3.1, the convex and increasing properties of the function $ATC_{1.3}(T)$ are discussed. Also we find out the optimum of $\min_{T\in[\frac{PM_2}{D},\infty]} ATC_{1.3}(T)$ in Theorem 3.2.

Theorem 3.2. *Let*

$$\lambda_{1.3} = 2\lambda_{1.1} - M_2^2\left(pI_c(1-\rho)P + sI_dD\right). \tag{3.7}$$

If $\lambda_{1.3} > 0$, then we define

$$\overline{T_{1.3}} = \sqrt{\frac{2\lambda_{1.3}}{D(1-\rho)(h + pI_c)}} \tag{3.8}$$

Then we obtain the statements as follows:

(1) *If* $\lambda_{1.3} > 0$, *then* $ATC_{1.3}(T)$ *is strictly convex. The stationary point of* $ATC_{1.3}(T)$ *is* $\overline{T_{1.3}}$.

(2) *If* $\lambda_{1.3} = 0$, *then* $ATC_{1.3}(T)$ *is strictly increasing.*

(3) *If* $\lambda_{1.3} < 0$, *then* $ATC_{1.3}(T)$ *is strictly increasing and concave.*

(4) *If* $2A < k_2 + k_3 + k_5$ *and* $\lambda_{1.3} > 0$, *then* $\overline{T_{1.3}} < \dfrac{PM_2}{D}$.

(5) *If* $2A \geq k_2 + k_3 + k_5$, *then* $\lambda_{1.3} > 0$ *and* $\overline{T_{1.3}} \geq \dfrac{PM_2}{D}$.

(6) *An optimal solution of* $\min\limits_{T \in \left[\frac{PM_2}{D}, \infty\right]} ATC_{1.3}(T)$ *is* $T^*_{1.3}$ *which is defined as*

$$T^*_{1.3} = \begin{cases} \dfrac{PM_2}{D} & if\ 2A < k_2 + k_3 + k_5 \\ \overline{T_{1.3}} & if\ 2A \geq k_2 + k_3 + k_5 \end{cases}.$$

In Theorem 3.3, we discuss the function $ATC_{1.4}(T)$. Again, the optimum of $\min\limits_{T \in [M_1, M_2]} ATC_{1.4}(T)$ is stated in (8) of Theorem 3.3.

Theorem 3.3. *Let*

$$\lambda_{1.4} = 2\lambda_{1.1} - DM_2^2 \left(sI_d - pI_c\right). \tag{3.9}$$

If $\lambda_{1.4} > 0$, *then we define*

$$\overline{T_{1.4}} = \sqrt{\frac{2\lambda_{1.1} - DM_2^2 \left(sI_d - pI_c\right)}{D(H + pI_c)}} \tag{3.10}$$

Then we obtain the statements as follows:

(1) *If* $\lambda_{1.4} > 0$, *then* $ATC_{1.4}(T)$ *is strictly convex. The stationary point of* $ATC_{1.4}(T)$ *is* $\overline{T_{1.4}}$.

(2) *If* $\lambda_{1.4} = 0$, *then* $ATC_{1.4}(T)$ *is strictly increasing.*

(3) *If* $\lambda_{1.4} < 0$, *then* $ATC_{1.4}(T)$ *is strictly increasing and concave.*

(4) *If* $2A < k_2 + k_3$ *and* $\lambda_{1.4} > 0$, *then* $\overline{T_{1.4}} < M_2$.

(5) *If* $2A \geq k_2 + k_3$, *then* $\lambda_{1.4} > 0$ *and* $\overline{T_{1.4}} \geq M_2$.

(6) *If* $2A \in [k_2 + k_3, k_2 + k_3 + k_5)$, *then* $\lambda_{1.4} > 0$, $\overline{T_{1.4}} \geq M_2$ *and* $\overline{T_{1.4}} < \dfrac{PM_2}{D}$.

(7) *If* $2A \geq k_2 + k_3 + k_5$, *then* $\lambda_{1.4} > 0$ *and* $\overline{T_{1.4}} \geq \dfrac{PM_2}{D}$.

(8) *An optimal solution of* $\min\limits_{T \in \left[M_2, \frac{PM_2}{D}\right]} ATC_{1.4}(T)$ *is* $T^*_{1.4}$ *which is defined as*

$$T^*_{1.4} = \begin{cases} M_2 & if\ 2A < k_2 + k_3 \\ \overline{T_{1.4}} & if\ 2A \in [k_2 + k_3, k_2 + k_3 + k_5) \\ \dfrac{PM_2}{D} & if\ 2A \geq k_2 + k_3 + k_5 \end{cases}.$$

Besides to discuss the functions $ATC_{1.1}(T)$, $ATC_{1.3}(T)$, $ATC_{1.4}(T)$, we also need some properties of $ATC_{2.1}(T)$, $ATC_{2.2}(T)$, $\ldots$, $ATC_{2.6}(T)$ in order to solve the problem (2.5). Hence we re-stated and re-proved the properties of $ATC_{2.1}(T)$, $ATC_{2.2}(T)$, $\ldots$, $ATC_{2.6}(T)$ in the following Theorem 3.4, Theorem 3.5, Theorem 3.6, Theorem 3.7 and Theorem 3.8. Similar results can be found in [3].

Theorem 3.4. *Let*

$$\overline{T_{2.1}} = \sqrt{\frac{2A}{D(H+sI_d)}}. \tag{3.11}$$

Then we obtain the statements as follows:

(1) *The function $ATC_{2.1}(T)$ and $ATC_{2.4}(T)$ are strictly convex. Both of their stationary point are $\overline{T_{2.1}}$.*

(2) *If $2A < k_1$, then $\overline{T_{2.1}} < M_1$.*

(3) *If $2A \geq k_1$, then $\overline{T_{2.1}} \geq M_1$.*

(4) *An optimal solution of $\min_{T\in(0,M_1]} ATC_{2.1}(T)$ is $T^*_{2.1}$ which is defined as*

$$T^*_{2.1} = \begin{cases} \overline{T_{2.1}} & if\ 2A < k_1 \\ M_1 & if\ 2A \geq k_1 \end{cases}.$$

(5) *If $2A < k_2$, then $\overline{T_{2.4}} < M_2$.*

(6) *If $2A \geq k_2$, then $\overline{T_{2.4}} \geq M_2$.*

(7) *An optimal solution of $\min_{T\in(0,M_2]} ATC_{2.4}(T)$ is $T^*_{2.4}$ which is defined as*

$$T^*_{2.4} = \begin{cases} \overline{T_{2.4}} & if\ 2A < k_2 \\ M_2 & if\ 2A \geq k_2 \end{cases}.$$

Theorem 3.5. *Let*

$$\lambda_{2.2} = 2A - DM_1^2\left(sI_d - pI_c(1-r)\right). \tag{3.12}$$

If $\lambda_{2.2} > 0$, then we define

$$\overline{T_{2.2}} = \sqrt{\frac{\lambda_{2.2}}{D(H+pI_c(1-r))}}. \tag{3.13}$$

Then we obtain the statements as follows:

(1) *If $\lambda_{2.2} > 0$, then $ATC_{2.2}(T)$ is strictly convex. The stationary point of $ATC_{2.2}(T)$ is $\overline{T_{2.2}}$.*

(2) *If $\lambda_{2.2} = 0$, then $ATC_{2.2}(T)$ is strictly increasing.*

(3) *If $\lambda_{2.2} < 0$, then $ATC_{2.2}(T)$ is strictly increasing and concave.*

(4) *If $2A < k_1$ and $\lambda_{2.2} > 0$, then $\overline{T_{2.2}} < M_1$.*

(5) *If $2A \geq k_1$, then $\lambda_{2.2} > 0$ and $\overline{T_{2.2}} \geq M_1$.*

(6) *If $2A \in [k_1, k_1+k_4)$, then $\lambda_{2.2} > 0$, $\overline{T_{2.2}} > M_1$ and $\overline{T_{2.2}} < \dfrac{PM_1}{D}$.*

(7) *If* $2A \geq k_1 + k_4$*, then* $\lambda_{2.2} > 0$ *and* $\overline{T_{2.2}} \geq \dfrac{PM_1}{D}$.

(8) *An optimal solution of* $\min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T)$ *is*

$$T_{2.2}^* = \begin{cases} M_1 & if\ 2A < k_1 \\ \overline{T_{2.2}} & if\ 2A \in [k_1, k_1 + k_4) \\ \frac{PM_1}{D} & if\ 2A \geq k_1 + k_4 \end{cases}.$$

Theorem 3.6. *Let*

$$\lambda_{2.3} = 2A - M_1^2 \left(pI_c(1-r)(1-\rho)P + sI_dD\right). \tag{3.14}$$

If $\lambda_{2.3} > 0$*, then we define*

$$\overline{T_{2.3}} = \sqrt{\frac{\lambda_{2.3}}{D(H + pI_c(1-r)(1-\rho))}}. \tag{3.15}$$

Then we obtain the statements as follows:

1. *If* $\lambda_{2.3} > 0$*, then* $ATC_{2.3}(T)$ *is strictly convex. The stationary point of* $ATC_{2.3}(T)$ *is* $\overline{T_{2.3}}$.
2. *If* $\lambda_{2.3} = 0$*, then* $ATC_{2.3}(T)$ *is strictly convex.*
3. *If* $\lambda_{2.3} < 0$*, then* $ATC_{2.3}(T)$ *is strictly increasing and concave.*
4. *If* $2A < k_1 + k_4$ *and* $\lambda_{2.3} > 0$*, then* $\overline{T_{2.3}} < \dfrac{PM_1}{D}$.
5. *If* $2A \geq k_1 + k_4$*, then* $\lambda_{2.3} > 0$ *and* $\overline{T_{2.3}} \geq \dfrac{PM_1}{D}$.
6. *An optimal solution of* $\min_{T \in \left[\frac{PM_1}{D}, +\infty\right)} ATC_{2.3}(T)$ *is*

$$T_{2.3}^* = \begin{cases} \dfrac{PM_1}{D} & if\ 2A < k_1 + k_4 \\ \overline{T_{2.3}} & if\ 2A \geq k_1 + k_4 \end{cases}.$$

Theorem 3.7. *Let*

$$\lambda_{2.5} = 2A - DM_2^2 \left(sI_d - pI_c\right). \tag{3.16}$$

If $\lambda_{2.5} > 0$*, then we define*

$$\overline{T_{2.5}} = \sqrt{\frac{\lambda_{2.5}}{D(H + pI_c)}}. \tag{3.17}$$

Then we obtain the statement as follows:

1. *If* $\lambda_{2.5} > 0$*, then* $ATC_{2.5}(T)$ *is strictly convex. The stationary point of of* $ATC_{2.3}(T)$ *is* $\overline{T_{2.5}}$.
2. *If* $\lambda_{2.5} = 0$*, then* $ATC_{2.5}(T)$ *is strictly increasing.*
3. *If* $\lambda_{2.5} < 0$*, then* $ATC_{2.5}(T)$ *is strictly increasing and concave.*

4. *If* $2A \le k_2$ *and* $\lambda_{2.5} > 0$*, then* $\overline{T_{2.5}} \le M_2$.
5. *If* $2A > k_2$*, then* $\lambda_{2.5} > 0$ *and* $\overline{T_{2.5}} > M_2$.
6. *If* $2A \in (k_2, k_2 + k_5)$*, then* $\lambda_{2.5} > 0$ *and* $\overline{T_{2.5}} < \dfrac{PM_2}{D}$.
7. *If* $2A \ge k_2 + k_5$*, then* $\lambda_{2.5} > 0$ *and* $\overline{T_{2.5}} \ge \dfrac{PM_2}{D}$.
8. *An optimal solution of* $\min_{T \in \left[M_2, \frac{PM_2}{D}\right]} ATC_{2.5}(T)$ *is*

$$T_{2.5}^* = \begin{cases} M_2 & if\ 2A \le k_2 \\ \overline{T_{2.5}} & if\ 2A \in (k_2, k_2+k_5) \\ \dfrac{PM_2}{D} & if\ 2A \ge k_2 + k_5 \end{cases}.$$

Theorem 3.8. *Let*

$$\lambda_{2.6} = 2A - M_2^2 \left(pI_c(1-\rho)P + sI_d D\right). \tag{3.18}$$

If $\lambda_{2.6} > 0$*, then we define*

$$\overline{T_{2.6}} = \sqrt{\frac{\lambda_{2.6}}{D(H + pI_c(1-\rho))}}. \tag{3.19}$$

Then we obtain the statements as follows:

1. *If* $\lambda_{2.6} > 0$*, then* $ATC_{2.6}(T)$ *is strictly convex. The stationary point of* $ATC_{2.6}(T)$ *is* $\overline{T_{2.6}}$.
2. *If* $\lambda_{2.6} = 0$*, then* $ATC_{2.6}(T)$ *is strictly increasing.*
3. *If* $\lambda_{2.6} < 0$*, then* $ATC_{2.6}(T)$ *is strictly increasing and concave.*
4. *If* $2A < k_2 + k_5$ *and* $\lambda_{2.6} > 0$ *then* $\overline{T_{2.6}} < \frac{PM_2}{D}$.
5. *If* $2A \ge k_2 + k_5$*, then* $\lambda_{2.6} > 0$ *and* $\overline{T_{2.6}} \ge \dfrac{PM_2}{D}$.
6. *An optimal solution of* $\min_{T \in \left[\frac{PM_2}{D}, +\infty\right)} ATC_{2.6}(T)$ *is*

$$T_{2.6}^* = \begin{cases} \frac{PM_2}{D} & if\ 2A < k_2 + k_5 \\ \overline{T_{2.6}} & if\ 2A \ge k_2 + k_5 \end{cases}.$$

From Theorem 3.1 - Theorem 3.8, we state the optimal conditions of

$$\min_{T \in [M_1, M_2]} ATC_{1.1}(T), \min_{T \in \left[\frac{PM_2}{D}, \infty\right]} ATC_{1.3}(T), \min_{T \in \left[M_2, \frac{PM_2}{D}\right]} ATC_{1.4}(T)$$

$$\min_{T \in (0, M_1]} ATC_{2.1}(T), \min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T), \min_{T \in \left[\frac{PM_1}{D}, +\infty\right)} ATC_{2.3}(T)$$

$$\min_{T \in (0, M_2]} ATC_{2.4}(T), \min_{T \in \left[M_2, \frac{PM_2}{D}\right]} ATC_{2.5}(T) \text{ and } \min_{T \in \left[\frac{PM_2}{D}, +\infty\right)} ATC_{2.6}(T).$$

In the following theorem, we state some properties of the average total cost functions $ATC_{i,j}(T)$.

Theorem 3.9. *Let* $\Delta = rp - sI_d\,(M_2 - M_1)$.

1. *The condition* $\Delta > 0$ *is necessary and sufficient for the followings:*

   a) *For all* $T < M_1$, $ATC_{2.1}(T) < ATC_{2.4}(T)$ *holds.*
   b) *For all* $T \in [M_1, M_2]$, $ATC_{1.1}(T) < ATC_{2.4}(T)$ *holds.*
   c) *For all* $T > M_2$, $ATC_{1.3}(T) < ATC_{2.6}(T)$ *and* $ATC_{1.4}(T) < ATC_{2.5}(T)$ *hold.*

2. *The condition* $\Delta < 0$ *is necessary and sufficient for the followings:*

   a) *For all* $T < M_1$, $ATC_{2.4}(T) < ATC_{2.1}(T)$ *holds.*
   b) *For all* $T \in [M_1, M_2]$, $ATC_{2.4}(T) < ATC_{1.1}(T)$ *holds.*
   c) *For all* $T > M_2$, $ATC_{2.6}(T) < ATC_{1.3}(T)$ *and* $ATC_{2.5}(T) = ATC_{1.4}(T)$ *holds.*

3. *The condition* $\Delta = 0$ *is necessary and sufficient for the followings:*

   a) *For all* $T < M_1$, $ATC_{2.1}(T) = ATC_{2.4}(T)$ *holds.*
   b) *For all* $T \in [M_1, M_2]$, $ATC_{1.1}(T) = ATC_{2.4}(T)$ *holds.*
   c) *For all* $T > M_2$, $ATC_{1.3}(T) = ATC_{2.6}(T)$ *and* $ATC_{1.4}(T) = ATC_{2.5}(T)$ *holds.*

Note: even the condition $T < M_1$, $T \in [M_1, M_2]$ *and* $T > M_2$ *are not satisfied, the results still hold.*

With Theorem 3.9, the problem (2.5) can be reduced. If $\Delta > 0$, then the optimal value ATC^* of (2.5) is the minimal value of

$$\min_{T\in[M_1,M_2]} ATC_{1.1}(T), \quad \min_{T\in\left[\frac{PM_2}{D},\infty\right]} ATC_{1.3}(T), \quad \min_{T\in\left[M_2,\frac{PM_2}{D}\right]} ATC_{1.4}(T)$$

$$\min_{T\in(0,M_1]} ATC_{2.1}(T), \quad \min_{T\in\left[M_1,\frac{PM_1}{D}\right]} ATC_{2.2}(T) \text{ and } \min_{T\in\left[\frac{PM_1}{D},+\infty\right)} ATC_{2.3}(T).$$

Moreover, some simplified results are listed in Theorem 3.10.

Theorem 3.10. *Suppose* $\Delta > 0$.

(a) *The condition* $2A < k_1$ *implies that*

$$ATC^* = \quad ATC_{2.1}(\overline{T_{2.1}}).$$

(b) *The condition* $k_1 \le 2A < \min\{k_1 + k_3, k_2 + k_3, k_1 + k_4\}$ *implies that*

$$ATC^* = ATC_{2.2}(\overline{T_{2.2}}).$$

(c) *The condition* $k_1 + k_3 \le 2A < \min\{k_2 + k_3, k_1 + k_4\}$ *implies that*

$$ATC^* = \min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.2}(\overline{T_{2.2}})\}.$$

(d) *The condition* $\max\{k_1+k_3, k_1+k_4\} \le 2A < k_2+k_3$ *implies that*

$$ATC^* = \min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.3}(\overline{T_{2.3}})\}.$$

(e) *The condition* $k_2+k_3 \le 2A < k_1+k_4$ *implies that*

$$ATC^* = \min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.2}(\overline{T_{2.2}})\}.$$

(f) *The condition* $k_1+k_4 \le 2A < k_1+k_3$ *implies that*

$$ATC^* = ATC_{2.3}(\overline{T_{2.3}}).$$

(g) *The condition* $\max\{k_1+k_3, k_2+k_3, k_1+k_4\} \le 2A < k_2+k_3+k_5$ *implies that*

$$ATC^* = \min\{ATC_{1.4}(\overline{T_{1.4}}), ATC_{2.3}(\overline{T_{2.3}})\}.$$

(h) *The condition* $k_2+k_3+k_5 \le 2A$ *implies that*

$$ATC^* = \min\{ATC_{1.3}(\overline{T_{1.3}}), ATC_{2.3}(\overline{T_{2.3}})\}.$$

If $\Delta \le 0$, then, from Theorem 3.9, the optimal value ATC^* of (2.5) is the minimal value of

$$\min_{T\in\left[M_1,\frac{PM_1}{D}\right]} ATC_{2.2}(T), \min_{T\in\left[\frac{PM_1}{D},+\infty\right)} ATC_{2.3}(T), \min_{T\in(0,M_2]} ATC_{2.4}(T),$$
$$\min_{T\in\left[M_2,\frac{PM_2}{D}\right]} ATC_{2.5}(T) \text{ and } \min_{T\in\left[\frac{PM_2}{D},+\infty\right)} ATC_{2.6}(T).$$

This result is very important, since it implies that if $\Delta \le 0$, retailers should choose single-stage payment instead of two stage payment. Hence we re-state Theorem 1 of [3] in Theorem 3.11.

Theorem 3.11. *Suppose* $\Delta \le 0$

(a) *The condition* $2A < k_1$ *implies that*

$$ATC^* = \min\{ATC_{2.1}(M_1), ATC_{2.4}(\overline{T_{2.4}})\}.$$

(b) *The condition* $k_1 \le 2A < \min\{k_2, k_1+k_4\}$ *implies that*

$$ATC^* = \min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.4}(\overline{T_{2.4}})\}.$$

(c) *Suppose* $k_1+k_4 \le k_2$. *The condition* $k_1+k_4 \le 2A \le k_2$ *implies that*

$$ATC^* = \min\{ATC_{2.3}(\overline{T_{2.3}}), ATC_{2.4}(\overline{T_{2.4}})\}.$$

(d) *Suppose* $k_2 < k_1+k_4$. *The condition* $k_2 < 2A < k_1+k_4$ *implies that*

$$ATC^* = \min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.5}(\overline{T_{2.5}})\}.$$

(e) *The condition* $\max\{k_2, k_1+k_4\} < 2A < k_2+k_5$ *implies that*

$$ATC^* = \min\{ATC_{2.3}(\overline{T_{2.3}}), ATC_{2.5}(\overline{T_{2.5}})\}.$$

(f) *The condition* $k_2+k_5 \le 2A$ *implies that*

$$ATC^* = \min\{ATC_{2.3}(\overline{T_{2.3}}), ATC_{2.6}(\overline{T_{2.6}})\}.$$

With Theorem 3.10 and Theorem 3.11, we can find the optimal replenishment of the problem (2.5).

4 Numerical Experiments

We design the following example based on the numerical example in [3]. In the following testing problem, the minimum cost of the two-stage payment is smaller than that of the single-stage payment.

Example 4.1. Let $A = 100$ dollars/order, $P = 1500$ units/year, $D = 1000$ units/year, $r = 0.01$, $h = 12$ dollars/unit/year, $p = 20$ dollars/unit, $s = 25$ dollars/unit, $I_c = 0.15$ dollars/year, $I_d = 0.07$ dollars/year, $M_1 = 0.1$ years, and $M_2 = 0.15$ years.

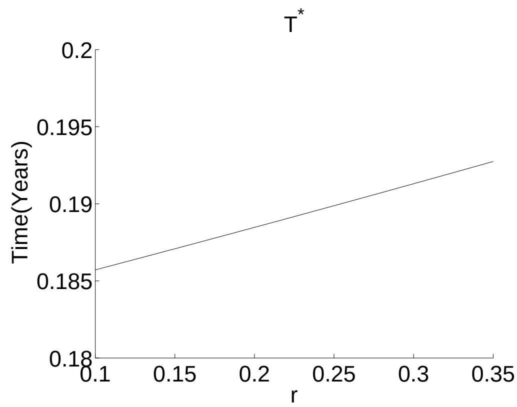

Figure 1: The optimal replenishment cycle times of Problem 4.1 with $r \in [0.1, 0.35]$

By Theorem 3.10, since

$$\Delta = rp - sI_d(M_2 - M_1) = 0.01125 > 0,$$

$$k_1 + k_3 = 80 < k_1 + k_4 = 144.625 < k_2 + k_3 = 151.875 < 2A = 180 < 348.75 = k_2 + k_3 + k_5,$$

and

$$\min\left\{ATC_{1.4}(\overline{T_{1.4}}), ATC_{2.3}(\overline{T_{2.3}})\right\} = ATC_{1.4}(\overline{T_{1.4}}) = 21435.305,$$

we obtain the optimal replenishment cycle time $\overline{T_{1.4}} = 0.171391$ years and the minimized average total cost $ATC_{1.4}(\overline{T_{1.4}}) = 21435.305$ dollars/year. The best objective value for a single-stage payment is $ATC_{2.3}(\overline{T_{2.3}}) = 21447.825$ dollars/year.

In an economic sense, the discount rate can stimulate the order quantity. This situation can be observed from Figure 1. When the discount rate r is increased, the optimal replenishment cycle time increases. Thus the optimal order quantity also increases. In Figure 2, we consider Problem 4.1 and modify the production rate P and the selling price s. From Figure 2, when P is fixed, the selling price s increases implying a decrease in the optimal replenishment cycle time. This means that, if they wish to obtain the same benefit, retailers must order less often as the selling price.

Conclusion

In this paper, we proposed a new policy involving a two-stage payment. Under this new policy, the payments are separated into more portions than a single-stage payment. The

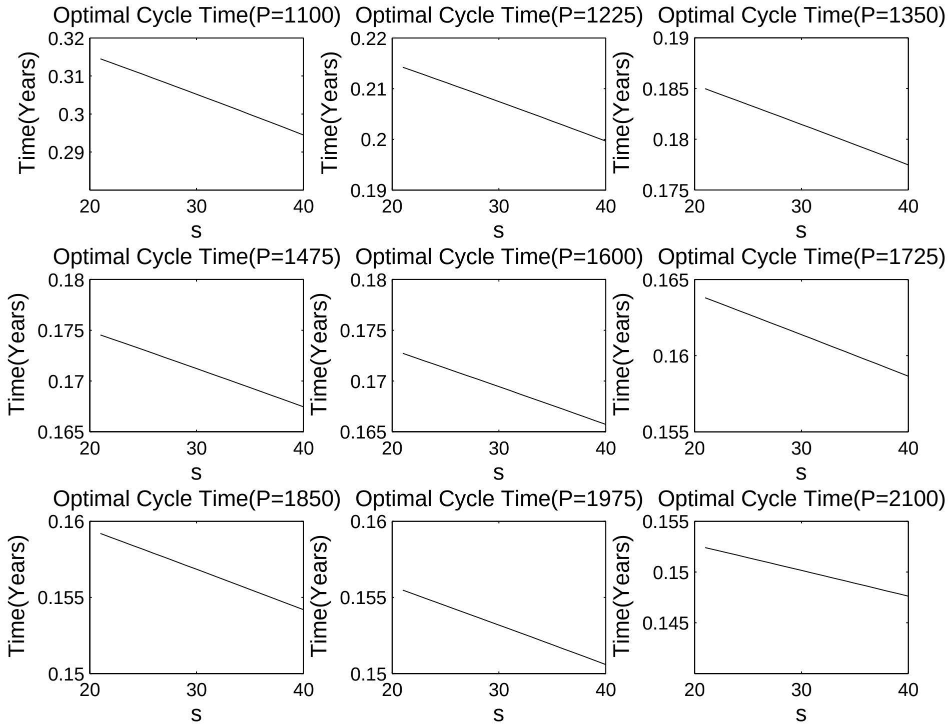

Figure 2: The optimal replenishment cycle times of Example 4.1 with $P \in \{1100, 1225, \ldots, 2100\}$ and $s \in \{21, 22, \ldots, 40\}$

mathematical representation of the economic production quantity (EPQ) models under a two-stage payment policy was given and an analytical method for solving the given models was stated and proved. Finally, we computed the value Δ of the difference between the cash discount of the whole payment and the interest earned from the unpaid payment after the cash discount period. If $\Delta > 0$, retailers may choose a two-stage payment to obtain a better profit than would result from a single-stage payment. In addition to providing the economic interpretation of Δ, we discussed the relationships between the discount rate r and the optimal cycle time, and between the selling price s and the optimal cycle time. In future, we can extend our EPQ models to a new Economic Lot Scheduling Problem, using the results of this paper.

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Appendix

A.1 The proof of Theorem 3.1

Proof. We have

$$\begin{aligned}
ATC_{1.1}(T) &= \frac{A}{T} + \frac{DTH}{2} + \frac{pD(T - rM_1)}{T} - \frac{sI_dD\left(M_1(M_1 - M_2) - T\left(\frac{1}{2}T - M_2\right)\right)}{T}, \\
ATC'_{1.1}(T) &= -\frac{A}{T^2} + \frac{DH}{2} + \frac{rpDM_1}{T^2} - sI_dD\left(\frac{-M_1(M_1 - M_2)}{T^2} - \frac{1}{2}\right) \\
&= \frac{D(H + sI_d)}{2} - \frac{A - DM_1(rp - sI_d(M_2 - M_1))}{T^2} \\
&= \frac{D(H + sI_d)}{2} - \frac{\lambda_{1.1}}{T^2}, \\
ATC''_{1.1}(T) &= \frac{2\lambda_{1.1}}{T^3}.
\end{aligned}$$

and

$$\begin{aligned} 2A - k_1 + k_3 &= 2A - DM_1^2(H + sI_d) - 2DM_1\left(rp - sI_d(M_2 - M_1)\right) \\ &= 2\left\{A - DM_1\left(rp - sI_d(M_2 - M_1)\right)\right\} - DM_1^2(H + sI_d) \\ &= 2\lambda_{1.1} - DM_1^2(H + sI_d) \\ 2A - k_2 + k_3 &= 2A - DM_2^2(H + sI_d) - 2DM_1\left(rp - sI_d(M_2 - M_1)\right) \\ &= 2\left\{A - DM_2\left(rp - sI_d(M_2 - M_1)\right)\right\} - DM_2^2(H + sI_d) \\ &= 2\lambda_{1.1} - DM_2^2(H + sI_d). \end{aligned}$$

(1) Since $\lambda_{1.1} > 0$ and $T > 0$ implies $ATC''_{1.1}(T) > 0$, the function $ATC_{1.1}(T)$ is strictly convex on $\{T > 0\}$. The stationary point $\overline{T_{1.1}}$ of $ATC_{1.1}(T)$ can be obtained from the following computation

$$\begin{aligned} ATC'_{1.1}(T) = 0 \quad &\Rightarrow \quad \frac{D(H + sI_d)}{2} - \frac{\lambda_{1.1}}{T^2} = 0 \\ &\Rightarrow \quad T = \sqrt{\frac{2\lambda_{1.1}}{D(H + sI_d)}}. \end{aligned}$$

(2) If $\lambda_{1.1} = 0$, then $ATC'_{1.1}(T) = \dfrac{D(H + sI_d)}{2} > 0$ which implies that the function $ATC_{1.1}(T)$ is strictly increasing.

(3) Since $\lambda_{1.1} < 0$ and $T > 0$ implies $ATC'_{1.1}(T) > 0$ and $ATC''_{1.1}(T) < 0$, the function $ATC_{1.1}(T)$ is strictly increasing and concave on $\{T > 0\}$.

(4) $k_3 > 2A \Rightarrow 2A - k_3 < 0 \Rightarrow 2A - 2DM_1\Delta < 0 \Rightarrow \lambda_{1.1} < 0$.

(5) $k_3 < 2A \Rightarrow 2A - k_3 > 0 \Rightarrow 2A - 2DM_1\Delta > 0 \Rightarrow \lambda_{1.1} > 0$.

(6) $2A \in (k_3, k_1 + k_3)$ implies $2A > k_3$, hence $\lambda_{1.1} > 0$. On the other hand,

$$\begin{aligned} 2A \in (k_3, k_1 + k_3) \Rightarrow 2A < k_1 + k_3 \Rightarrow 2\lambda_{1.1} - DM_1^2(H + sI_d) < 0 \\ \Rightarrow \overline{T_{1.1}} = \sqrt{\frac{2\lambda_{1.1}}{D(H + sI_d)}} < M_1. \end{aligned}$$

(7) $2A \geq k_1 + k_3$ implies $2A > k_3$, hence $\lambda_{1.1} > 0$.

$$2A \geq k_1 + k_3 \Rightarrow 2\lambda_{1.1} - DM_1^2(H + sI_d) \geq 0 \Rightarrow \overline{T_{1.1}} = \sqrt{\frac{2\lambda_{1.1}}{D(H + sI_d)}} \geq M_1.$$

(8) $2A \in (k_3, k_2 + k_3)$ implies $2A > k_3$, hence $\lambda_{1.1} > 0$. On the other hand,

$$\begin{aligned} 2A \in (k_3, k_2 + k_3) \Rightarrow 2A < k_2 + k_3 \Rightarrow 2\lambda_{1.1} - DM_2^2(H + sI_d) < 0 \\ \Rightarrow \overline{T_{1.1}} = \sqrt{\frac{2\lambda_{1.1}}{D(H + sI_d)}} < M_2. \end{aligned}$$

(9) $2A \geq k_2 + k_3$ implies $2A > k_3$, hence $\lambda_{1.1} > 0$. On the other hand,

$$2A \geq k_2 + k_3 \Rightarrow 2\lambda_{1.1} - DM_2^2(H + sI_d) \geq 0 \Rightarrow \overline{T_{1.1}} = \sqrt{\frac{2\lambda_{1.1}}{D(H + sI_d)}} \geq M_2.$$

(10) (a) Suppose $2A < k_1 + k_3$. Then we need to consider three cases: $2A < k_3$, $2A = k_3$, and $2A \in (k_3, k_1 + k_3)$.

(i) $2A < k_3 \Rightarrow \lambda_{1.1} < 0 \Rightarrow ATC_{1.1}(T)$ is strictly increasing $\Rightarrow ATC_{1.1}(M_1) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T)$.

(ii) $2A = k_3 \Rightarrow \lambda_{1.1} = 0 \Rightarrow ATC_{1.1}(T)$ is strictly increasing $\Rightarrow ATC_{1.1}(M_1) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T)$.

(ii) $2A \in (k_3, k_1 + k_3) \Rightarrow \lambda_{1.1} > 0 \Rightarrow ATC_{1.1}(T)$ is strictly convex. Moreover, $2A \in (k_3, k_1 + k_3)$ implies $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} < M_1$. Hence, $ATC_{1.1}(M_1) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T)$.

Therefore, if $2A < k_1 + k_3$, then $ATC_{1.1}(M_1) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T)$.

(b) Suppose $2A \in [k_1 + k_3, k_2 + k_3)$. Then $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} \in [M_1, M_2]$. Moreover, $\lambda_{1.1} > 0$ implies $ATC_{1.1}(T)$ is strictly convex. Hence, $ATC_{1.1}(\overline{T_{1.1}}) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T)$,

(c) Suppose $2A \geq k_2 + k_3$. Then $\lambda_{1.1} > 0$ and $\overline{T_{1.1}} \geq M_2$. Moreover, $\lambda_{1.1} > 0$ implies $ATC_{1.1}(T)$ is strictly convex. Hence, $ATC_{1.1}(\overline{M_2}) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T)$,

□

A.2 The proof of Theorem 3.2

Proof. We have

$$\begin{aligned}
ATC_{1.3}(T) &= \frac{A}{T} + \frac{DTH}{2} + \frac{pD(T - rM_1)}{T} - \frac{sI_dD\left(M_1(M_1 - M_2) + \frac{1}{2}M_2^2\right)}{T} \\
&\quad + \frac{pI_c(1-\rho)\left(DT^2 - PM_2^2\right)}{2T} \\
ATC'_{1.3}(T) &= -\frac{A}{T^2} + \frac{DH}{2} + \frac{rpDM_1}{T^2} + \frac{sI_dD\left(M_1(M_1 - M_2) + \frac{1}{2}M_2^2\right)}{T^2} \\
&\quad + pI_c(1-\rho)\left(\frac{D}{2} + \frac{PM_2^2}{2T}\right) \\
&= \frac{D(1-\rho)(h + pI_c)}{2} \\
&\quad - \frac{2\{A - DM_1(rp - sI_d(M_2 - M_1))\} - M_2^2(pI_c(1-\rho)P + sI_dD)}{2T^2} \\
&= \frac{D(1-\rho)(h + pI_c)}{2} - \frac{\lambda_{1.3}}{2T^2} \\
ATC''_{1.3}(T) &= \frac{\lambda_{1.3}}{T^3}.
\end{aligned}$$

and

$$\begin{aligned}
2A-k_2-k_3-k_5 &= 2A-k_3-DM_2^2(H+sI_d)-\left(\tfrac{P^2}{D^2}-1\right)DM_2^2(H+pI_c)\\
&= 2A-k_3-DM_2^2(sI_d-pI_c)-\left(\tfrac{P^2}{D^2}\right)DM_2^2(H+pI_c)\\
&= 2A-k_3-M_2^2(pI_c(1-\rho)P+sDI_d-pI_c(1-\rho)P-pDI_c)\\
&\quad -\left(\tfrac{P^2}{D^2}\right)DM_2^2(H+pI_c)\\
&= 2A-k_3-M_2^2(pI_c(1-\rho)P+sDI_d-pI_cP)\\
&\quad -\left(\tfrac{P^2}{D^2}\right)DM_2^2(H+pI_c)\\
&= 2A-k_3-M_2^2(pI_c(1-\rho)P+sDI_d)+M_2^2pI_cP\\
&\quad -\left(\tfrac{P^2}{D^2}\right)DM_2^2(H+pI_c)\\
&= \lambda_{1.3}-\left(\tfrac{P^2}{D^2}\right)DM_2^2\left(H+pI_c-\tfrac{pI_cD}{P}\right)\\
&= \lambda_{1.3}-\left(\tfrac{P^2}{D^2}\right)DM_2^2\left(h(1-\rho)+(1-\rho)pI_c\right)\\
&= \lambda_{1.3}-\left(\tfrac{P^2}{D^2}\right)DM_2^2(1-\rho)\left(h+pI_c\right).
\end{aligned}$$

(1) Since $\lambda_{1.3}>0$ and $T>0$ implies $ATC''_{1.3}(T)>0$, the function $ATC_{1.3}(T)$ is strictly convex on $\{T>0\}$. The stationary point $\overline{T_{1.3}}$ of $ATC_{1.3}(T)$ can be obtained from the following computation

$$\begin{aligned}
ATC'_{1.3}(T)=0 \quad &\Rightarrow \quad \frac{D(1-\rho)(h+pI_c)}{2}-\frac{\lambda_{1.3}}{2T^2}=0\\
&\Rightarrow \quad T=\sqrt{\frac{2\lambda_{1.3}}{D(1-\rho)(h+pI_c)}}.
\end{aligned}$$

(2) If $\lambda_{1.3}=0$, then $ATC'_{1.3}(T)=\dfrac{D(1-\rho)(h+pI_c)}{2}>0$ which implies that the function $ATC_{1.3}(T)$ is strictly increasing.

(3) Since $\lambda_{1.3}<0$ and $T>0$ implies $ATC'_{1.3}(T)>0$ and $ATC''_{1.3}(T)<0$, the function $ATC_{1.3}(T)$ is strictly increasing and concave on $\{T>0\}$.

(4) $2A<k_2+k_3+k_5$ and $\lambda_{1.3}>0$ implies $\overline{T_{1.3}}<\frac{PM_2}{D}$ since

$$2A<k_2+k_3+k_5 \Rightarrow \lambda_{1.3}-\left(\frac{P^2}{D^2}\right)DM_2^2(1-\rho)\left(h+pI_c\right)<0 \Rightarrow \overline{T_{1.3}}<\frac{PM_2}{D}.$$

(5) $2A\geq k_2+k_3+k_5$ implies $\lambda_{1.3}>0$ and $\overline{T_{1.3}}\geq\frac{PM_2}{D}$ since

$$\begin{aligned}
2A\geq k_2+k_3+k_5 &\Rightarrow \lambda_{1.3}-\left(\frac{P^2}{D^2}\right)DM_2^2(1-\rho)\left(h+pI_c\right)\geq 0\\
&\Rightarrow \lambda_{1.3}\geq\left(\frac{P^2}{D^2}\right)DM_2^2(1-\rho)\left(h+pI_c\right)>0
\end{aligned}$$

and

$$2A\geq k_2+k_3+k_5 \Rightarrow \lambda_{1.3}-\left(\frac{P^2}{D^2}\right)DM_2^2(1-\rho)\left(h+pI_c\right)\geq 0 \Rightarrow \overline{T_{1.3}}\geq\frac{PM_2}{D}.$$

(6) (a) Suppose $2A < k_2+k_3+k_5$ and $\lambda_{1.3} < 0$. Then $ATC_{1.3}(T)$ is strictly increasing, Hence $ATC_{1.3}\left(\frac{PM_2}{D}\right) = \min_{T\in[\frac{PM_2}{D},+\infty)} ATC_{1.3}(T)$,

(b) Suppose $2A < k_2+k_3+k_5$ and $\lambda_{1.3} = 0$. Then $ATC_{1.3}(T)$ is strictly increasing, Hence $ATC_{1.3}\left(\frac{PM_2}{D}\right) = \min_{T\in[\frac{PM_2}{D},+\infty)} ATC_{1.3}(T)$,

(c) Suppose $2A < k_2+k_3+k_5$ and $\lambda_{1.3} > 0$. Then $ATC_{1.3}(T)$ is strictly convex and $\overline{T_{1.3}} < \frac{PM_2}{D}$. Hence $ATC_{1.3}\left(\frac{PM_2}{D}\right) = \min_{T\in[\frac{PM_2}{D},+\infty)} ATC_{1.3}(T)$,

(d) Suppose $2A \geq k_2+k_3+k_5$. Then $\lambda_{1.3} > 0$, $ATC_{1.3}(T)$ is strictly convex and $\overline{T_{1.3}} \geq \frac{PM_2}{D}$. Hence $ATC_{1.3}(\overline{T_{1.3}}) = \min_{T\in[\frac{PM_2}{D},+\infty)} ATC_{1.3}(T)$.

□

A.3 The proof of Theorem 3.3

Proof. We have

$$\begin{aligned}
ATC_{1.4}(T) &= \frac{A}{T}+\frac{DTH}{2}+\frac{pD(T-rM_1)}{T}-\frac{sI_dD\left(M_1(M_1-M_2)+\frac{1}{2}M_2^2\right)}{T}\\
&\quad+\frac{pI_cD(T-M_2)^2}{2T}\\
ATC'_{1.4}(T) &= -\frac{A}{T^2}+\frac{DH}{2}+\frac{rpDM_1}{T^2}+\frac{sI_dD\left(M_1(M_1-M_2)+\frac{1}{2}M_2^2\right)}{T^2}\\
&\quad+pI_cD\left(\frac{1}{2}-\frac{M_2^2}{2T^2}\right)\\
&= \frac{D(H+pI_c)}{2}-\frac{2\{A-DM_1(rp-sI_d(M_2-M_1))\}-DM_2^2(sI_d-pI_c)}{2T^2}\\
&= \frac{D(H+pI_c)}{2}-\frac{\lambda_{1.4}}{2T^2}\\
ATC''_{1.4}(T) &= \frac{\lambda_{1.4}}{T^3},\\
2A-k_2-k_3 &= 2A-DM_2^2(H+sI_d)-2DM_1\Delta\\
&= 2A-2DM_1\Delta-DM_2^2(sI_d-pI_c)+DM_2^2(sI_d-pI_c)-DM_2^2(H+sI_d)\\
&= \lambda_{1.4}-DM_2^2(H+pI_c)
\end{aligned}$$

and

$$\begin{aligned}
2A-k_2-k_3-k_5 &= \lambda_{1.4}-DM_2^2(H+pI_c)-\left(\frac{P^2}{D^2}-1\right)DM_2^2(H+pI_c)\\
&= \lambda_{1.4}-\frac{P^2M_2^2(H+pI_c)}{D}.
\end{aligned}$$

(1) Since $\lambda_{1.4} > 0$ and $T > 0$ implies $ATC''_{1.4}(T) > 0$, the function $ATC_{1.4}(T)$ is strictly convex on $\{T > 0\}$. The stationary point $\overline{T_{1.4}}$ of $ATC_{1.4}(T)$ can be obtained from the following computation

$$\begin{aligned}
ATC'_{1.4}(T)=0 \quad &\Rightarrow \quad \frac{D(H+pI_c)}{2}-\frac{\lambda_{1.4}}{2T^2}=0\\
&\Rightarrow \quad T=\sqrt{\frac{2\lambda_{1.4}}{D(H+pI_c)}}.
\end{aligned}$$

(2) If $\lambda_{1.4} = 0$, then $ATC'_{1.4}(T) = \dfrac{D(H+pI_c)}{2} > 0$ which implies that the function $ATC_{1.4}(T)$ is strictly increasing.

(3) Since $\lambda_{1.4} < 0$ and $T > 0$ implies $ATC'_{1.4}(T) > 0$ and $ATC''_{1.4}(T) < 0$, the function $ATC_{1.4}(T)$ is strictly increasing and concave on $\{T > 0\}$.

(4) $2A < k_2 + k_3$ and $\lambda_{1.4} > 0$ implies $\overline{T_{1.4}} < M_2$ since

$$2A < k_2 + k_3 \Rightarrow \lambda_{1.4} - DM_2^2(H + pI_c) < 0 \Rightarrow \overline{T_{1.4}} < M_2.$$

(5) $2A \geq k_2 + k_3$ implies $\lambda_{1.4} > 0$ and $\overline{T_{1.4}} \geq M_2$ since

$$2A \geq k_2 + k_3 \Rightarrow \lambda_{1.4} - DM_2^2(H + pI_c) \geq 0 \Rightarrow \lambda_{1.4} \geq DM_2^2(H + pI_c) > 0$$

and

$$2A \geq k_2 + k_3 \Rightarrow \lambda_{1.4} - DM_2^2(H + pI_c) \geq 0 \Rightarrow \overline{T_{1.4}} \geq M_2.$$

(6) $2A \in [k_2 + k_3, k_2 + k_3 + k_5)$ implies $\lambda_{1.4} > 0$, $\overline{T_{1.4}} \geq M_2$ and $\overline{T_{1.4}} < \frac{PM_2}{D}$ because $2A \geq k_2 + k_3$ and

$$2A \in [k_2 + k_3, k_2 + k_3 + k_5) \Rightarrow \lambda_{1.4} - \frac{P^2M_2^2(H + pI_c)}{D} < 0 \Rightarrow \overline{T_{1.4}} < \frac{PM_2}{D}.$$

(7) $2A \geq k_2 + k_3 + k_5$ implies $\lambda_{1.4} > 0$ and $\overline{T_{1.4}} \geq \frac{PM_2}{D}$ because $2A > k_2 + k_3$ and

$$2A \geq k_2 + k_3 + k_5 \Rightarrow \lambda_{1.4} - \frac{P^2M_2^2(H + pI_c)}{D} \geq 0 \Rightarrow \overline{T_{1.4}} \geq \frac{PM_2}{D}.$$

(8) (a) Suppose $2A < k_2 + k_3$ and $\lambda_{1.4} < 0$. Then $ATC_{1.4}(T)$ is strictly increasing. Hence $ATC_{1.4}(M_2) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{1.4}(T)$.

(b) Suppose $2A < k_2 + k_3$ and $\lambda_{1.4} = 0$. Then $ATC_{1.4}(T)$ is strictly increasing. Hence $ATC_{1.4}(M_2) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{1.4}(T)$.

(c) Suppose $2A < k_2 + k_3$ and $\lambda_{1.4} > 0$. Then $ATC_{1.4}(T)$ is strictly convex and $\overline{T_{1.4}} \leq M_2$. Hence $ATC_{1.4}(M_2) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{1.4}(T)$.

(d) Suppose $2A \in [k_2 + k_3, k_2 + k_3 + k_5)$. Then $\lambda_{1.4} > 0$, $ATC_{1.4}(T)$ is strictly convex and $\overline{T_{1.4}} \in \left[M_2, \frac{PM_2}{D}\right]$. Hence $ATC_{1.4}\left(\overline{T_{1.4}}\right) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{1.4}(T)$.

(e) Suppose $2A \geq k_2 + k_3 + k_5$. Then $\lambda_{1.4} > 0$, $ATC_{1.4}(T)$ is strictly convex and $\overline{T_{1.4}} \geq \frac{PM_2}{D}$. Hence $ATC_{1.4}\left(\frac{PM_2}{D}\right) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{1.4}(T)$.

□

A.4 The proof of Theorem 3.4

Proof. We have

$$\begin{aligned}
ATC_{2.1}(T) &= \frac{A}{T}+\frac{DTH}{2}+pD(1-r)-sI_dD\left(M_1-\frac{1}{2}T\right)\\
ATC'_{2.1}(T) &= -\frac{A}{T^2}+\frac{DH}{2}+\frac{sI_dD}{2}\\
ATC''_{2.1}(T) &= \frac{2A}{T^3}\\
ATC_{2.4}(T) &= \frac{A}{T}+\frac{DTH}{2}+pD-sI_dD\left(M_2-\frac{1}{2}T\right)\\
ATC'_{2.4}(T) &= -\frac{A}{T^2}+\frac{DH}{2}+\frac{sI_dD}{2}\\
ATC''_{2.4}(T) &= \frac{2A}{T^3}
\end{aligned}$$

(1) Since $ATC''_{2.1}(T)>0$ and $ATC''_{2.4}(T)>0$ for all $T>0$, both of $ATC_{2.1}(T)$ and $ATC_{2.4}(T)$ are strictly convex. Since $ATC'_{2.1}(T)=-\frac{A}{T^2}+\frac{DH}{2}+\frac{sI_dD}{2}=ATC'_{2.4}(T)$, both of their stationary points are $\overline{T_{2.1}}$.

(2) $\overline{T_{2.1}}<M_1$ holds because

$$2A<k_1\Rightarrow 2A<DM_1^2(H+sI_d)\Rightarrow\frac{2A}{D(H+sI_d)}<M_1^2\Rightarrow\overline{T_{2.1}}<M_1.$$

(3) $\overline{T_{2.1}}\geq M_1$ holds because

$$2A\geq k_1\Rightarrow 2A\geq DM_1^2(H+sI_d)\Rightarrow\frac{2A}{D(H+sI_d)}\geq M_1^2\Rightarrow\overline{T_{2.1}}\geq M_1.$$

(4) Suppose $2A<k_1$. Then $\overline{T_{2.1}}<M_1$. Hence $ATC_{2.1}\left(\overline{T_{2.1}}\right)=\min_{T\in(0,M_1]}ATC_{2.1}(T)$.

Suppose $2A\geq k_1$. Then $\overline{T_{2.1}}\geq M_1$. Hence $ATC_{2.1}(M_1)=\min_{T\in(0,M_1]}ATC_{2.1}(T)$.

(5) $\overline{T_{2.4}}<M_2$ holds because

$$2A<k_2\Rightarrow 2A<DM_2^2(H+sI_d)\Rightarrow\frac{2A}{D(H+sI_d)}<M_2^2\Rightarrow\overline{T_{2.4}}<M_2.$$

(6) $\overline{T_{2.4}}\geq M_2$ holds because

$$2A\geq k_2\Rightarrow 2A\geq DM_2^2(H+sI_d)\Rightarrow\frac{2A}{D(H+sI_d)}\geq M_2^2\Rightarrow\overline{T_{2.4}}\geq M_2.$$

(7) Suppose $2A<k_2$. Then $\overline{T_{2.4}}<M_2$. Hence $ATC_{2.4}\left(\overline{T_{2.4}}\right)=\min_{T\in(0,M_2]}ATC_{2.4}(T)$.

Suppose $2A\geq k_2$. Then $\overline{T_{2.4}}\geq M_2$. Hence $ATC_{2.4}(M_2)=\min_{T\in(0,M_2]}ATC_{2.4}(T)$.

□

A.5 The proof of Theorem 3.5

Proof. We have

$$
\begin{aligned}
ATC_{2.2}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD(1-r) - \frac{sI_dDM_1^2}{2T} + \frac{p(1-r)I_cD(T-M_1)^2}{2T} \\
ATC'_{2.2}(T) &= -\frac{A}{T^2} + \frac{DH}{2} + \frac{sI_dDM_1^2}{2T^2} + \frac{p(1-r)I_cD}{2} - \frac{p(1-r)I_cDM_1^2}{2T^2} \\
&= \frac{D(H+pI_c(1-r))}{2} - \frac{2A - DM_1^2(sI_d - pI_c(1-r))}{2T^2} \\
&= \frac{D(H+pI_c(1-r))}{2} - \frac{\lambda_{2.2}}{2T^2} \\
ATC''_{2.2}(T) &= \frac{\lambda_{2.2}}{T^3},
\end{aligned}
$$

$$
\begin{aligned}
2A - k_1 &= 2A - DM_1^2(H + sI_d) \\
&= 2A - DM_1^2(sI_d - pI_c(1-r)) + DM_1^2(sI_d - pI_c(1-r)) - DM_1^2(H + sI_d) \\
&= 2A - DM_1^2(sI_d - pI_c(1-r)) - DM_1^2(H + pI_c(1-r)) \\
&= \lambda_{2.2} - DM_1^2(H + pI_c(1-r))
\end{aligned}
$$

and

$$
\begin{aligned}
2A - k_1 - k_4 &= \lambda_{2.2} - DM_1^2(H + pI_c(1-r)) - \left(\tfrac{P^2}{D^2} - 1\right) DM_1^2(H + pI_c(1-r)) \\
&= \lambda_{2.2} - \tfrac{P^2}{D^2} DM_1^2(H + pI_c(1-r)).
\end{aligned}
$$

(1) Since $\lambda_{2.2} > 0$ and $T > 0$ implies $ATC''_{2.2}(T) > 0$, the function $ATC_{2.2}(T)$ is strictly convex on $\{T > 0\}$. The stationary point $\overline{T_{2.2}}$ of $ATC_{2.2}(T)$ can be obtained from the following computation

$$
\begin{aligned}
ATC'_{2.2}(T) = 0 \quad &\Rightarrow \quad \frac{D(H+pI_c(1-r)}{2} - \frac{\lambda_{2.2}}{2T^2} = 0 \\
&\Rightarrow \quad T = \sqrt{\frac{\lambda_{2.2}}{D(H+pI_c(1-r))}}.
\end{aligned}
$$

(2) If $\lambda_{2.2} = 0$, then $ATC'_{2.2}(T) = \dfrac{D(H+pI_c(1-r))}{2} > 0$ which implies that the function $ATC_{2.2}(T)$ is strictly increasing.

(3) Since $\lambda_{2.2} < 0$ and $T > 0$ implies $ATC'_{2.2}(T) > 0$ and $ATC''_{2.2}(T) < 0$, the function $ATC_{2.2}(T)$ is strictly increasing and concave on $\{T > 0\}$.

(4) $2A < k_1$ and $\lambda_{2.2} > 0$ implies $\overline{T_{2.2}} < M_1$ since

$$2A < k_1 \Rightarrow \lambda_{2.2} - DM_1^2(H + pI_c(1-r)) < 0 \Rightarrow \overline{T_{2.2}} < M_1.$$

(5) $2A \geq k_1$ implies $\lambda_{2.2} > 0$ and $\overline{T_{2.2}} \geq M_1$ since

$$2A \geq k_1 \Rightarrow \lambda_{2.2} - DM_1^2(H + pI_c(1-r)) \geq 0 \Rightarrow \lambda_{2.2} \geq DM_1^2(H + pI_c(1-r)) > 0.$$

and

$$2A \geq k_1 \Rightarrow \lambda_{2.2} - DM_1^2(H + pI_c(1-r)) \geq 0 \Rightarrow \overline{T_{2.2}} \geq M_1.$$

(6) $2A \in [k_1, k_1 + k_4)$ implies $\lambda_{2.2} > 0$, $\overline{T_{2.2}} \geq M_1$ and $\overline{T_{2.2}} < \frac{PM_1}{D}$ because $2A \geq k_1$ and

$$2A \in [k_1, k_1 + k_4) \Rightarrow \lambda_{2.2} - \frac{P^2}{D^2} DM_1^2 (H + pI_c(1-r)) < 0 \Rightarrow \overline{T_{2.2}} < \frac{PM_1}{D}.$$

(7) $2A \geq k_1 + k_4$ implies $\lambda_{2.2} > 0$ and $\overline{T_{2.2}} \geq \frac{PM_1}{D}$ since $2A > k_1$ and

$$2A \geq k_1 + k_4 \Rightarrow \lambda_{2.2} - \frac{P^2}{D^2} DM_1^2 (H + pI_c(1-r)) \geq 0 \Rightarrow \overline{T_{2.2}} \geq \frac{PM_1}{D}.$$

(8) Suppose $2A < k_1$ and $\lambda_{2.2} < 0$. Then $ATC_{2.2}(T)$ is strictly increasing. Hence $ATC_{2.2}(M_1) = \min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T)$.

Suppose $2A < k_1$ and $\lambda_{2.2} = 0$. Then $ATC_{2.2}(T)$ is strictly increasing. Hence $ATC_{2.2}(M_1) = \min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T)$.

Suppose $2A < k_1$ and $\lambda_{2.2} > 0$. Then $ATC_{2.2}(T)$ is strictly convex and $\overline{T_{2.2}} < M_1$. Hence $ATC_{2.2}(M_1) = \min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T)$.

Suppose $2A \in [k_1, k_1 + k_4)$. Then $\lambda_{2.2} > 0$, $ATC_{2.2}(T)$ is strictly convex and $\overline{T_{2.2}} \in \left[M_1, \frac{PM_1}{D}\right]$. Hence $ATC_{2.2}(\overline{T_{2.2}}) = \min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T)$.

Suppose $2A \geq k_1 + k_4$. Then $\lambda_{2.2} > 0$, $ATC_{2.2}(T)$ is strictly convex and $\overline{T_{2.2}} \geq \frac{PM_1}{D}$. Hence $ATC_{2.2}(\frac{PM_1}{D}) = \min_{T \in \left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T)$.

□

A.6 The proof of Theorem 3.6

Proof. We have

$$\begin{aligned}
ATC_{2.3}(T) &= \frac{A}{T} + \frac{DTH}{2} + p(1-r)D + \frac{pI_c(1-r)(1-\rho)\left(DT^2 - PM_1^2\right)}{2T} - \frac{sI_dDM_1^2}{2T}. \\
ATC'_{2.3}(T) &= \frac{-A}{T^2} + \frac{DH}{2} + \frac{pI_c(1-r)(1-\rho)D}{2} + \frac{pI_c(1-r)(1-\rho)PM_1^2}{2T^2} + \frac{sI_dDM_1^2}{2T^2} \\
&= \frac{D(H + pI_c(1-r)(1-\rho))}{2} - \frac{2A - M_1^2\left(pI_c(1-r)(1-\rho)P + sI_dD\right)}{2T^2} \\
&= \frac{D(H + pI_c(1-r)(1-\rho))}{2} - \frac{\lambda_{2.3}}{2T^2} \\
ATC''_{2.3}(T) &= \frac{\lambda_{2.3}}{T^3}
\end{aligned}$$

and

$$\begin{aligned}
2A - k_1 - k_4 &= 2A - DM_1^2(H + sI_d) - \left(\frac{P^2}{D^2} - 1\right) DM_1^2(H + pI_c(1-r)) \\
&= 2A - \frac{P^2}{D^2} DM_1^2(H + pI_c(1-r)) - DM_1^2(sI_d - pI_c(1-r)) \\
&= 2A - DM_1^2 sI_d - \frac{P^2}{D^2} DM_1^2(H + pI_c(1-r)) + DM_1^2 pI_c(1-r) \\
&= \lambda_{2.3} + M_1^2 pI_c(1-r)(1-\rho)P - \frac{P^2}{D^2} DM_1^2(H + pI_c(1-r)) \\
&\quad + DM_1^2 pI_c(1-r) \\
&= \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H + M_1^2 pI_c(1-r)(1-\rho)P - \frac{P^2}{D^2} DM_1^2 pI_c(1-r) \\
&\quad + DM_1^2 pI_c(1-r) \\
&= \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H - DM_1^2 pI_c(1-r)\left(-(1-\rho)\frac{P}{D} + \frac{P^2}{D^2} - 1\right) \\
&= \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H - \frac{P^2}{D^2} DM_1^2 pI_c(1-r)\left(-(1-\rho)\rho + 1 - \rho^2\right) \\
&= \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H - \frac{P^2}{D^2} DM_1^2 pI_c(1-r)\left(1-\rho\right).
\end{aligned}$$

(1) Since $\lambda_{2.3} > 0$ and $T > 0$ implies $ATC''_{2.3}(T) > 0$, the function $ATC_{2.3}(T)$ is strictly convex on $\{T > 0\}$. The stationary point $\overline{T_{2.3}}$ of $ATC_{2.3}(T)$ can be obtained from the following computation

$$\begin{aligned}
ATC'_{2.3}(T) = 0 \quad &\Rightarrow \quad \frac{D(H + pI_c(1-r)(1-\rho))}{2} - \frac{\lambda_{2.3}}{2T^2} = 0 \\
&\Rightarrow \quad T = \sqrt{\frac{\lambda_{2.3}}{D(H + pI_c(1-r)(1-\rho))}}.
\end{aligned}$$

(2) If $\lambda_{2.3} = 0$, then $ATC'_{2.3}(T) = \dfrac{D(H + pI_c(1-r)(1-\rho))}{2} > 0$ which implies that the function $ATC_{2.3}(T)$ is strictly increasing.

(3) Since $\lambda_{2.3} < 0$ and $T > 0$ implies $ATC'_{2.3}(T) > 0$ and $ATC''_{2.3}(T) < 0$, the function $ATC_{2.3}(T)$ is strictly increasing and concave on $\{T > 0\}$.

(4) $2A < k_1 + k_4$ and $\lambda_{2.3} > 0$ imply $\overline{T_{2.3}} < \dfrac{PM_1}{D}$, since

$$2A < k_1 + k_4 \Rightarrow \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H - \frac{P^2}{D^2} DM_1^2 pI_c(1-r)(1-\rho) < 0 \Rightarrow \overline{T_{2.3}} < \frac{PM_1}{D}.$$

(5) $2A \geq k_1 + k_4$ implies $\lambda_{2.3} > 0$ and $\overline{T_{2.3}} \geq \dfrac{PM_1}{D}$, since

$$2A \geq k_1 + k_4 \Rightarrow \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H - \frac{P^2}{D^2} DM_1^2 pI_c(1-r)(1-\rho) \geq 0$$

$$\Rightarrow \lambda_{2.3} \geq \frac{P^2}{D^2} DM_1^2 H - \frac{P^2}{D^2} DM_1^2 pI_c(1-r)(1-\rho) > 0$$

and

$$2A \geq k_1 + k_4 \Rightarrow \lambda_{2.3} - \frac{P^2}{D^2} DM_1^2 H - \frac{P^2}{D^2} DM_1^2 pI_c(1-r)(1-\rho) \geq 0 \Rightarrow \overline{T_{2.3}} \geq \frac{PM_1}{D}.$$

(6) Suppose $2A < k_1 + k_4$ and $\lambda_{2.3} < 0$. Then $ATC_{2.3}(T)$ is strictly increasing. Hence $ATC_{2.3}\left(\frac{PM_1}{D}\right) = \min_{T \in \left[\frac{PM_1}{D}, \infty\right)} ATC_{2.3}(T)$,

Suppose $2A < k_1 + k_4$ and $\lambda_{2.3} = 0$. Then $ATC_{2.3}(T)$ is strictly increasing. Hence $ATC_{2.3}\left(\frac{PM_1}{D}\right) = \min_{T \in \left[\frac{PM_1}{D}, \infty\right)} ATC_{2.3}(T)$.

Suppose $2A < k_1 + k_4$ and $\lambda_{2.3} > 0$. Then $ATC_{2.3}(T)$ is strictly convex and $\overline{T_{2.3}} < \frac{PM_1}{D}$. Hence $ATC_{2.3}\left(\frac{PM_1}{D}\right) = \min_{T \in \left[\frac{PM_1}{D}, \infty\right)} ATC_{2.3}(T)$.

Suppose $2A \geq k_1 + k_4$. Then $\lambda_{2.3} > 0$, $ATC_{2.3}(T)$ is strictly convex and $\overline{T_{2.3}} \geq \frac{PM_1}{D}$. Hence $ATC_{2.3}\left(\overline{T_{2.3}}\right) = \min_{T \in \left[\frac{PM_1}{D}, \infty\right)} ATC_{2.3}(T)$.

□

A.7 The proof of Theorem 3.7

Proof. We have

$$\begin{aligned}
ATC_{2.5}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD - \frac{sI_d DM_2^2}{2T} + \frac{pI_c D (T - M_2)^2}{2T} \\
ATC'_{2.5}(T) &= -\frac{A}{T^2} + \frac{DH}{2} + \frac{sI_d DM_2^2}{2T^2} + \frac{pI_c D}{2} - \frac{pI_c DM_2^2}{2T^2} \\
&= \frac{D(H + pI_c)}{2} - \frac{2A - DM_2^2(sI_d - pI_c)}{2T^2} \\
&= \frac{D(H + pI_c)}{2} - \frac{\lambda_{2.5}}{2T^2} \\
ATC''_{2.5}(T) &= \frac{\lambda_{2.5}}{T^3},
\end{aligned}$$

$$\begin{aligned}
2A - k_2 &= 2A - DM_2^2(H + sI_d) \\
&= 2A - DM_2^2(sI_d - pI_c) - DM_2^2(H + pI_c) \\
&= \lambda_{2.5} - DM_2^2(H + pI_c).
\end{aligned}$$

and

$$\begin{aligned}
2A - k_2 - k_5 &= \lambda_{2.5} - DM_2^2(H + pI_c) - \left(\frac{P^2}{D^2} - 1\right) DM_2^2(H + pI_c) \\
&= \lambda_{2.5} - \frac{P^2}{D^2} DM_2^2(H + pI_c).
\end{aligned}$$

(1) Since $\lambda_{2.5} > 0$ and $T > 0$ implies $ATC''_{2.5}(T) > 0$, the function $ATC_{2.5}(T)$ is strictly convex on $\{T > 0\}$. The stationary point $\overline{T_{2.5}}$ of $ATC_{2.5}(T)$ can be obtained from the following computation

$$\begin{aligned}
ATC'_{2.5}(T) = 0 \quad &\Rightarrow \quad \frac{D(H + pI_c)}{2} - \frac{\lambda_{2.5}}{2T^2} = 0 \\
&\Rightarrow \quad T = \sqrt{\frac{\lambda_{2.5}}{D(H + pI_c)}}.
\end{aligned}$$

(2) If $\lambda_{2.5} = 0$, then $ATC'_{2.5}(T) = \dfrac{D(H+pI_c)}{2} > 0$ which implies that the function $ATC_{2.5}(T)$ is strictly increasing.

(3) Since $\lambda_{2.5} < 0$ and $T > 0$ implies $ATC'_{2.5}(T) > 0$ and $ATC''_{2.5}(T) < 0$, the function $ATC_{2.5}(T)$ is strictly increasing and concave on $\{T > 0\}$.

(4) $2A \le k_2$ and $\lambda_{2.5} > 0$ implies $\overline{T_{2.5}} \le M_2$ since

$$2A \le k_2 \Rightarrow \lambda_{2.5} - DM_2^2(H+pI_c) \le 0 \Rightarrow \overline{T_{2.5}} \le M_2.$$

(5) $2A > k_2$ implies $\lambda_{2.5} > 0$ and $\overline{T_{2.5}} > M_2$ since

$$2A > k_2 \Rightarrow \lambda_{2.5} - DM_2^2(H+pI_c) > 0 \Rightarrow \lambda_{2.5} > DM_2^2(H+pI_c)$$

and

$$2A > k_2 \Rightarrow \lambda_{2.5} - DM_2^2(H+pI_c) > 0 \Rightarrow \overline{T_{2.5}} > M_2.$$

(6) $2A \in (k_2, k_2+k_5)$ implies $\lambda_{2.5} > 0$, $\overline{T_{2.5}} > M_2$ and $\overline{T_{2.5}} < \frac{PM_2}{D}$ because $2A > k_2$ and

$$2A \in (k_2, k_2+k_5) \Rightarrow \lambda_{2.5} - \frac{P^2}{D^2} DM_2^2(H+pI_c) < 0 \Rightarrow \overline{T_{2.5}} < \frac{PM_2}{D}.$$

(7) $2A \ge k_2 + k_5$ implies $\lambda_{2.5} > 0$, $\overline{T_{2.5}} > M_2$ and $\overline{T_{2.5}} \ge \frac{PM_2}{D}$ since because $2A > k_2$ and

$$2A \ge k_2 + k_5 \Rightarrow \lambda_{2.5} - \frac{P^2}{D^2} DM_2^2(H+pI_c) \ge 0 \Rightarrow \overline{T_{2.5}} \ge \frac{PM_2}{D}.$$

(8) Suppose $2A \le k_2$ and $\lambda_{2.5} < 0$. Then $ATC_{2.5}(T)$ is strictly increasing. Hence $ATC_{2.5}(M_2) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{2.5}(T)$.

Suppose $2A \le k_2$ and $\lambda_{2.5} = 0$. Then $ATC_{2.5}(T)$ is strictly increasing. Hence $ATC_{2.5}(M_2) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{2.5}(T)$,

Suppose $2A \le k_2$ and $\lambda_{2.5} > 0$. Then $ATC_{2.5}(T)$ is strictly convex and $\overline{T_{2.5}} < M_2$. Hence $ATC_{2.5}(M_2) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{2.5}(T)$,

Suppose $2A \in (k_2, k_2+k_5)$. Then $\lambda_{2.5} > 0$, $ATC_{2.5}(T)$ is strictly convex, $\overline{T_{2.5}} > M_2$ and $\overline{T_{2.5}} < \frac{PM_2}{D}$. Hence $ATC_{2.5}(\overline{T_{2.5}}) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{2.5}(T)$,

Suppose $2A \ge k_2 + k_5$. Then $\lambda_{2.5} > 0$, $ATC_{2.5}(T)$ is strictly convex, $\overline{T_{2.5}} > \frac{PM_2}{D}$. Hence, $ATC_{2.5}(\frac{PM_2}{D}) = \min_{T \in [M_2, \frac{PM_2}{D}]} ATC_{2.5}(T)$.

□

A.8 The proof of Theorem 3.8

Proof. We have

$$\begin{aligned}
ATC_{2.6}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD + \frac{pI_c(1-\rho)\left(DT^2 - PM_2^2\right)}{2T} - \frac{sI_dDM_2^2}{2T}.\\
ATC'_{2.6}(T) &= \frac{-A}{T^2} + \frac{DH}{2} + \frac{pI_c(1-\rho)D}{2} + \frac{pI_c(1-\rho)PM_2^2}{2T^2} + \frac{sI_dDM_2^2}{2T^2}\\
&= \frac{D(H+pI_c(1-\rho))}{2} - \frac{2A - M_2^2\left(pI_c(1-\rho)P + sI_dD\right)}{2T^2}\\
&= \frac{D(H+pI_c(1-\rho))}{2} - \frac{\lambda_{2.6}}{2T^2}\\
ATC''_{2.6}(T) &= \frac{\lambda_{2.6}}{T^3}.
\end{aligned}$$

and

$$\begin{aligned}
2A - k_2 - k_5 &= 2A - DM_2^2(H + sI_d) - \left(\frac{P^2}{D^2} - 1\right) DM_2^2(H + pI_c)\\
&= 2A - \frac{P^2}{D^2} DM_2^2(H+pI_c) - DM_2^2(sI_d - pI_c)\\
&= 2A - DM_2^2 sI_d - \frac{P^2}{D^2} DM_2^2(H+pI_c) + DM_2^2 pI_c)\\
&= \lambda_{2.6} + M_2^2 pI_c(1-\rho)P - \frac{P^2}{D^2} DM_2^2(H+pI_c) + DM_2^2 pI_c\\
&= \lambda_{2.6} - \frac{P^2}{D^2} DM_2^2 H + M_2^2 pI_c(1-\rho)P - \frac{P^2}{D^2} DM_2^2 pI_c + DM_2^2 pI_c\\
&= \lambda_{2.6} - \frac{P^2}{D^2} DM_2^2 H - DM_2^2 pI_c \left(-(1-\rho)\frac{P}{D} + \frac{P^2}{D^2} - 1\right)\\
&= \lambda_{2.6} - \frac{P^2}{D^2} DM_2^2 H - \frac{P^2}{D^2} DM_2^2 pI_c \left(-(1-\rho)\rho + 1 - \rho^2\right)\\
&= \lambda_{2.6} - \frac{P^2}{D^2} DM_2^2 H - \frac{P^2}{D^2} DM_2^2 pI_c \left(1-\rho\right)\\
&= \lambda_{2.6} - \frac{P^2}{D^2} DM_2^2 (H + pI_c\left(1-\rho\right)).
\end{aligned}$$

(1) Since $\lambda_{2.6} > 0$ and $T > 0$ implies $ATC''_{2.6}(T) > 0$, the function $ATC_{2.6}(T)$ is strictly convex on $\{T > 0\}$. The stationary point $\overline{T_{2.6}}$ of $ATC_{2.6}(T)$ can be obtained from the following computation

$$\begin{aligned}
ATC'_{2.6}(T) = 0 \quad &\Rightarrow \quad \frac{D(H+pI_c(1-\rho))}{2} - \frac{\lambda_{2.6}}{2T^2} = 0\\
&\Rightarrow \quad T = \sqrt{\frac{\lambda_{2.6}}{D(H+pI_c(1-\rho))}}.
\end{aligned}$$

(2) If $\lambda_{2.6} = 0$, then $ATC'_{2.6}(T) = \dfrac{D(H+pI_c(1-\rho))}{2} > 0$ which implies that the function $ATC_{2.6}(T)$ is strictly increasing.

(3) Since $\lambda_{2.6} < 0$ and $T > 0$ implies $ATC'_{2.6}(T) > 0$ and $ATC''_{2.6}(T) < 0$, the function $ATC_{2.6}(T)$ is strictly increasing and concave on $\{T > 0\}$.

(4) $2A < k_2 + k_5$ and $\lambda_{2.6} > 0$ imply $\overline{T_{2.6}} < \dfrac{PM_2}{D}$, since

$$2A < k_2 + k_5 \Rightarrow \lambda_{2.6} - \frac{P^2}{D^2} D M_2^2 (H + pI_c(1-\rho)) < 0$$

$$\Rightarrow \overline{T_{2.6}} = \sqrt{\frac{\lambda_{2.6}}{D(H + pI_c(1-\rho))}} < \frac{PM_2}{D}.$$

(5) $2A \geq k_2 + k_5$, implies $\lambda_{2.6} > 0$ and $\overline{T_{2.6}} \geq \dfrac{PM_2}{D}$, since

$$2A \geq k_2 + k_5 \Rightarrow \lambda_{2.6} - \frac{P^2}{D^2} D M_2^2 (H + pI_c(1-\rho)) \geq 0$$

$$\Rightarrow \lambda_{2.6} \geq \frac{P^2}{D^2} D M_2^2 (H + pI_c(1-\rho)) > 0.$$

and

$$2A \geq k_2 + k_5 \Rightarrow \lambda_{2.6} - \frac{P^2}{D^2} D M_2^2 (H + pI_c(1-\rho)) \geq 0$$

$$\Rightarrow \overline{T_{2.6}} = \sqrt{\frac{\lambda_{2.6}}{D(H + pI_c(1-\rho))}} \geq \frac{PM_2}{D}.$$

(6) (a) Suppose $2A < k_2 + k_5$ and $\lambda_{2.6} < 0$. Then $ATC_{2.6}(T)$ is strictly increasing. Hence $ATC_{2.6}\left(\frac{PM_2}{D}\right) = \min_{T \in \left[\frac{PM_2}{D}, \infty\right)} ATC_{2.6}(T)$.

(b) Suppose $2A < k_2 + k_5$ and $\lambda_{2.6} = 0$. Then $ATC_{2.6}(T)$ is strictly increasing. Hence $ATC_{2.6}\left(\frac{PM_2}{D}\right) = \min_{T \in \left[\frac{PM_2}{D}, \infty\right)} ATC_{2.6}(T)$,

(c) Suppose $2A < k_2 + k_5$ and $\lambda_{2.6} > 0$. Then $ATC_{2.6}(T)$ is strictly convex and $\overline{T_{2.6}} < \dfrac{PM_2}{D}$. Hence $ATC_{2.6}\left(\frac{PM_2}{D}\right) = \min_{T \in \left[\frac{PM_2}{D}, \infty\right)} ATC_{2.6}(T)$,

(d) Suppose $2A \geq k_2 + k_5$. Then $\lambda_{2.6} > 0$, $ATC_{2.6}(T)$ is strictly convex and $\overline{T_{2.6}} \geq \dfrac{PM_2}{D}$,
Hence, $ATC_{2.6}(\overline{T_{2.6}}) = \min_{T \in \left[\frac{PM_2}{D}, \infty\right)} ATC_{2.6}(T)$.

□

A.9 The proof of Theorem 3.9

Proof.

a) Since

$$\begin{aligned} ATC_{2.1}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD(1-r) - sI_d D\left(M_1 - \frac{1}{2}T\right) \\ ATC_{2.4}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD - sI_d D\left(M_2 - \frac{1}{2}T\right), \end{aligned}$$

we obtain

$$\begin{aligned} ATC_{2.1}(T) - ATC_{2.4}(T) &= -rpD - sI_dD\left(M_1 - \frac{1}{2}T\right) + sI_dD\left(M_2 - \frac{1}{2}T\right) \\ &= -D\left(rp - sI_d\left(M_2 - M_1\right)\right). \end{aligned}$$

Hence, combining it with $D > 0$, we obtain

$$\begin{cases} ATC_{2.1}(T) < ATC_{2.4}(T) & \Leftrightarrow \quad \Delta > 0 \\ ATC_{2.1}(T) = ATC_{2.4}(T) & \Leftrightarrow \quad \Delta = 0 \\ ATC_{2.1}(T) > ATC_{2.4}(T) & \Leftrightarrow \quad \Delta < 0 \end{cases}$$

b) Since

$$\begin{aligned} ATC_{1.1}(T) &= \frac{A}{T} + \frac{DTH}{2} + \frac{pD\left(T - rM_1\right)}{T} \\ &\quad -sI_dD\frac{\left(M_1(M_1-M_2)-T\left(\frac{1}{2}T-M_2\right)\right)}{T} \\ ATC_{2.4}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD\left(1 - r\right) - sI_dD\left(M_2 - \frac{1}{2}T\right), \end{aligned}$$

we obtain

$$\begin{aligned} ATC_{1.1}(T) - ATC_{2.4}(T) &= -\frac{rpDM_1}{T} - \frac{sI_dD(M_1(M_1-M_2))}{T} \\ &= \frac{-DM_1(rp-sI_d(M_2-M_1))}{T}. \end{aligned}$$

Hence, combining it with $\dfrac{DM_1}{T} > 0$, we obtain

$$\begin{cases} ATC_{1.1}(T) < ATC_{2.3}(T) & \Leftrightarrow \quad \Delta > 0 \\ ATC_{1.1}(T) = ATC_{2.3}(T) & \Leftrightarrow \quad \Delta = 0 \\ ATC_{1.1}(T) > ATC_{2.3}(T) & \Leftrightarrow \quad \Delta < 0 \end{cases}.$$

c) Since

$$\begin{aligned} ATC_{1.3}(T) &= \frac{A}{T} + \frac{DTH}{2} + \frac{pD\left(T - rM_1\right)}{T} - sI_dD\frac{\left(M_1\left(M_1 - M_2\right) + \frac{1}{2}M_2^2\right)}{T} \\ &\quad +pI_c(1-\rho)\frac{DT^2 - PM_2^2}{2T} \\ ATC_{1.4}(T) &= \frac{A}{T} + \frac{DTH}{2} + \frac{pD\left(T - rM_1\right)}{T} - \frac{sI_dD\left(M_1\left(M_1 - M_2\right) + \frac{1}{2}M_2^2\right)}{T} \\ &\quad +\frac{pI_cD\left(T - M_2\right)^2}{2T} \\ ATC_{2.5}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD - \frac{sI_dDM_2^2}{2T} + \frac{pI_cD\left(T - M_2\right)^2}{2T} \\ ATC_{2.6}(T) &= \frac{A}{T} + \frac{DTH}{2} + pD - \frac{sI_dDM_2^2}{2T} + \frac{pI_cD\left(T - M_2\right)^2}{2T}, \end{aligned}$$

we obtain

$$\begin{aligned} ATC_{1.4}(T) - ATC_{2.5}(T) &= -\frac{rpDM_1}{T} - \frac{sI_dD\left(M_1\left(M_1 - M_2\right)\right)}{T} \\ &= \frac{-DM_1\left(rp - sI_d\left(M_2 - M_1\right)\right)}{T}. \end{aligned}$$

and

$$ATC_{1.3}(T) - ATC_{2.6}(T) = -\frac{rpDM_1}{T} - \frac{sI_dD(M_1(M_1 - M_2))}{T} = \frac{-DM_1(rp - sI_d(M_2 - M_1))}{T}.$$

Hence, combining it with $\frac{DM_1}{T} > 0$, we obtain

$$\begin{cases} ATC_{1.4}(T) < ATC_{2.5}(T) & \Leftrightarrow \quad \Delta > 0 \\ ATC_{1.4}(T) = ATC_{2.5}(T) & \Leftrightarrow \quad \Delta = 0 \\ ATC_{1.4}(T) > ATC_{2.5}(T) & \Leftrightarrow \quad \Delta < 0 \end{cases}$$

and

$$\begin{cases} ATC_{1.3}(T) < ATC_{2.6}(T) & \Leftrightarrow \quad \Delta > 0 \\ ATC_{1.3}(T) = ATC_{2.6}(T) & \Leftrightarrow \quad \Delta = 0 \\ ATC_{1.3}(T) > ATC_{2.6}(T) & \Leftrightarrow \quad \Delta < 0 \end{cases}$$

□

A.10 The proof of Theorem 3.10

Proof. In order to compare the values $ATC_{1.1}(T^*_{1.1})$, $ATC_{1.3}(T^*_{1.3})$, $ATC_{1.4}(T^*_{1.4})$, $ATC_{2.1}(T^*_{2.1})$, $ATC_{2.2}(T^*_{2.2})$, $ATC_{2.3}(T^*_{2.3})$, we compute the following values

$$\begin{aligned} ATC_{1.1}(M_1) &= \frac{A}{M_1} + \frac{DM_1H}{2} + \frac{pD(M_1 - rM_1)}{M_1} \\ &\quad - \frac{sI_dD(M_1(M_1-M_2) - M_1(\frac{1}{2}M_1 - M_2))}{M_1} \\ &= \frac{A}{M_1} + \frac{DM_1H}{2} + pD(1-r) - sI_dD(M_1 - M_2 - \frac{1}{2}M_1 + M_2) \\ &= \frac{A}{M_1} + \frac{DM_1H}{2} + pD(1-r) - \frac{sI_dDM_1}{2} \\ ATC_{2.1}(M_1) &= \frac{A}{M_1} + \frac{DM_1H}{2} + pD(1-r) - sI_dD(M_1 - \frac{1}{2}M_1) \\ &= \frac{A}{M_1} + \frac{DM_1H}{2} + pD(1-r) - \frac{sI_dDM_1}{2} \\ ATC_{2.2}(M_1) &= \frac{A}{M_1} + \frac{DM_1H}{2} + pD(1-r) - \frac{sI_dDM_1^2}{2M_1} + \frac{p(1-r)I_cD(M_1 - M_1)^2}{2M_1} \\ &= \frac{A}{M_1} + \frac{DM_1H}{2} + pD(1-r) - \frac{sI_dDM_1}{2}. \end{aligned}$$

Hence we obtain the following equalities:

$$ATC_{1.1}(M_1) = ATC_{2.1}(M_1) = ATC_{2.2}(M_1). \tag{4.1}$$

Then we computing the values

$$\begin{aligned} ATC_{1.1}(M_2) &= \frac{A}{M_2} + \frac{DM_2H}{2} + \frac{pD(M_2 - rM_1)}{M_2} \\ &\quad - \frac{sI_dD(M_1(M_1-M_2) - M_2(\frac{1}{2}M_2 - M_2))}{M_2} \\ &= \frac{A}{M_2} + \frac{DM_2H}{2} + \frac{pD(M_2 - rM_1)}{M_2} - \frac{sI_dD(M_1(M_1 - M_2) + \frac{1}{2}M_2^2)}{M_2} \\ ATC_{1.4}(M_2) &= \frac{A}{M_2} + \frac{DM_2H}{2} + \frac{pD(M_2 - rM_1)}{M_2} - \frac{sI_dD(M_1(M_1 - M_2) + \frac{1}{2}M_2^2)}{M_2}. \end{aligned}$$

We also obtain the equalities:

$$ATC_{1.1}(M_2) = ATC_{1.4}(M_2). \tag{4.2}$$

and we computing the values

$$\begin{aligned}
ATC_{1.3}(\tfrac{PM_2}{D}) &= \frac{A}{\frac{PM_2}{D}} + \frac{D\frac{PM_2}{D}H}{2} + \frac{pD(\frac{PM_2}{D} - rM_1)}{\frac{PM_2}{D}} - \frac{sI_dD(M_1(M_1-M_2)+\frac{1}{2}M_2^2)}{\frac{PM_2}{D}} \\
&\quad + \frac{pI_c(1-\rho)(D(\frac{PM_2}{D})^2 - PM_2^2)}{2\frac{PM_2}{D}} \\
ATC_{1.4}(\tfrac{PM_2}{D}) &= \frac{A}{\frac{PM_2}{D}} + \frac{D\frac{PM_2}{D}H}{2} + \frac{pD(\frac{PM_2}{D} - rM_1)}{\frac{PM_2}{D}} - \frac{sI_dD(M_1(M_1-M_2)+\frac{1}{2}M_2^2)}{\frac{PM_2}{D}} \\
&\quad \frac{pI_cD(\frac{PM_2}{D} - M_2)^2}{2\frac{PM_2}{D}}
\end{aligned}$$

and

$$\begin{aligned}
(1-\rho)\left(D\frac{P^2M_2^2}{D^2} - PM_2^2\right) &= (1-\rho)\left(\frac{P^2M_2^2}{D} - PM_2^2\right) \\
&= (P-D)\left(\frac{PM_2^2}{D} - M_2^2\right) \\
&= \left(\frac{P^2M_2^2}{D} - PM_2^2 - PM_2^2 + DM_2^2\right) \\
&= D\left(\frac{P^2M_2^2}{D^2} - \frac{2PM_2^2}{D} + M_2^2\right) \\
&= D\left(\frac{PM_2}{D} - M_2\right)^2.
\end{aligned} \tag{4.3}$$

We obtain the equalities:

$$ATC_{1.3}\left(\frac{PM_2}{D}\right) = ATC_{1.4}\left(\frac{PM_2}{D}\right). \tag{4.4}$$

After computing the values

$$\begin{aligned}
ATC_{2.2}\left(\tfrac{PM_1}{D}\right) &= \frac{A}{\frac{PM_1}{D}} + \frac{D\frac{PM_1}{D}H}{2} + pD(1-r) - \frac{sI_dDM_1^2}{2\frac{PM_1}{D}} \\
&\quad + \frac{pI_c(1-r)D(\frac{PM_1}{D} - M_1)^2}{2\frac{PM_1}{D}} \\
ATC_{2.3}\left(\tfrac{PM_1}{D}\right) &= \frac{A}{\frac{PM_1}{D}} + \frac{D\frac{PM_1}{D}H}{2} + pD(1-r) - \frac{sI_dDM_1^2}{2\frac{PM_1}{D}} \\
&\quad + \frac{pI_c(1-r)(1-\rho)(D(\frac{PM_1}{D})^2 - PM_1^2)}{2\frac{PM_1}{D}}
\end{aligned}$$

and (4.3) we obtain

$$ATC_{2.2}\left(\frac{PM_1}{D}\right) = ATC_{2.3}\left(\frac{PM_1}{D}\right). \tag{4.5}$$

(a) $2A < k_1$ implies that $2A < k_1 + k_3$, $2A < k_2 + k_3 + k_5$, $2A < k_2 + k_3$, $2A < k_1 + k_4$. Hence

$2A < k_1 + k_3 \Rightarrow T^*_{1.1} = M_1$	$2A < k_2 + k_3 + k_5 \Rightarrow T^*_{1.3} = \frac{PM_2}{D}$,
$2A < k_2 + k_3 \Rightarrow T^*_{1.4} = M_2$	$2A < k_1 \Rightarrow T^*_{2.1} = \overline{T_{2.1}}$,
$2A < k_1 \Rightarrow T^*_{2.2} = M_1$	$2A < k_1 + k_4 \Rightarrow T^*_{2.3} = \frac{PM_1}{D}$.

By (4.2), $ATC_{1.1}(M_2) = ATC_{1.4}(M_2)$. Hence we obtain

$$\begin{aligned} ATC_{1.1}(M_1) &= \min_{T\in[M_1,M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_2) \\ &= ATC_{1.4}(M_2) \end{aligned}$$

which implies

$$ATC_{1.1}(M_1) < ATC_{1.4}(M_2). \tag{4.6}$$

On the other hand, by (4.1), $ATC_{2.1}(M_1) = ATC_{2.2}(M_1)$. Hence

$$ATC_{2.1}\left(\overline{T_{2.1}}\right) = \min_{T\in(0,M_1]} ATC_{2.1}(T) < ATC_{2.1}(M_1) = ATC_{2.2}(M_1)$$

which implies

$$ATC_{2.1}\left(\overline{T_{2.1}}\right) < ATC_{1.1}(M_1). \tag{4.7}$$

By (4.5) and $T^*_{2.2} = M_1$, we obtain

$$ATC_{2.2}(M_1) = \min_{T\in\left[M_1,\frac{PM_1}{D}\right]} ATC_{2.2}(T) < ATC_{2.2}(\frac{PM_1}{D}) = ATC_{2.3}(\frac{PM_1}{D}).$$

By (4.4) and $T^*_{1.4} = M_2$, we obtain

$$ATC_{1.4}(M_2) = \min_{T\in\left[M_2,\frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}(\frac{PM_2}{D}) = ATC_{1.3}(\frac{PM_2}{D}).$$

Hence

$$ATC^* = ATC_{2.1}(\overline{T_{2.1}}).$$

(b) $k_1 \le 2A < k_1 + k_3$ implies that $2A < k_2 + k_3 + k_5$, $2A < k_2 + k_3$, $2A \in [k_1, k_1 + k_4)$, $2A < k_1 + k_4$. Hence

$2A < k_1 + k_3 \Rightarrow T^*_{1.1} = M_1$	$2A < k_2 + k_3 + k_5 \Rightarrow T^*_{1.3} = \frac{PM_2}{D}$
$2A < k_2 + k_3 \Rightarrow T^*_{1.4} = M_2$	$2A \ge k_1 \Rightarrow T^*_{2.1} = M_1$
$2A \in [k_1, k_1 + k_4) \Rightarrow T^*_{2.2} = \overline{T_{2.2}}$	$2A < k_1 + k_4 \Rightarrow T^*_{2.3} = \frac{PM_1}{D}$

By (4.1) and $T^*_{2.2} = \overline{T_{2.2}}$, we obtain

$$ATC_{2.2}\left(\overline{T_{2.2}}\right) = \min_{T\in\left[M_1,\frac{PM_1}{D}\right]} ATC_{2.2}(T) < ATC_{2.2}(M_1) = ATC_{1.1}(M_1) = ATC_{2.1}(M_1).$$

By (4.2) and $T^*_{1.1} = M_1$, we obtain

$$ATC_{1.1}\left(M_1\right) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T) < ATC_{1.1}\left(M_2\right) = ATC_{1.4}\left(M_2\right).$$

By (4.4) and $T^*_{1.4} = M_2$, we obtain

$$ATC_{1.4}\left(M_2\right) = \min_{T\in\left[M_2,\frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}\left(\frac{PM_2}{D}\right) = ATC_{1.3}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T_{2.2}^* = \overline{T_{2.2}}$, we obtain

$$ATC_{2.2}\left(\overline{T_{2.2}}\right) = \min_{T\in\left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T) < ATC_{2.2}\left(\frac{PM_1}{D}\right) = ATC_{2.3}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^* = ATC_{2.2}(\overline{T_{2.2}}).$$

(c) $k_1 + k_3 \leq 2A < \min\{k_2 + k_3, k_1 + k_4\}$ implies that $2A \in [k_1 + k_3, k_2 + k_3)$, $2A < k_2 + k_3 + k_5$, $2A \in [k_1, k_1 + k_4)$. Hence

$2A \in [k_1 + k_3, k_2 + k_3) \Rightarrow T_{1.1}^* = \overline{T_{1.1}}$	$2A < k_2 + k_3 + k_5 \Rightarrow T_{1.3}^* = \frac{PM_2}{D}$
$2A < k_2 + k_3 \Rightarrow T_{1.4}^* = M_2$	$2A \geq k_1 \Rightarrow T_{2.1}^* = M_1$
$2A \in [k_1, k_1 + k_4) \Rightarrow T_{2.2}^* = \overline{T_{2.2}}$	$2A < k_1 + k_4 \Rightarrow T_{2.3}^* = \frac{PM_1}{D}$

By (4.1) and $T_{1.1}^* = \overline{T_{1.1}}$, we obtain

$$ATC_{1.1}(\overline{T_{1.1}}) = \min_{T\in[M_1.M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_1) = ATC_{2.1}(M_1).$$

By (4.2) and $T_{1.1}^* = \overline{T_{1.1}}$, we obtain

$$ATC_{1.1}(\overline{T_{1.1}}) = \min_{T\in[M_1.M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_2) = ATC_{1.4}(M_2).$$

By (4.4) and $T_{1.4}^* = M_2$, we obtain

$$ATC_{1.4}\left(M_2\right) = \min_{T\in\left[M_2, \frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}\left(\frac{PM_2}{D}\right) = ATC_{1.3}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T_{2.2}^* = \overline{T_{2.2}}$, we obtain

$$ATC_{2.2}\left(\overline{T_{2.2}}\right) = \min_{T\in\left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T) < ATC_{2.2}\left(\frac{PM_1}{D}\right) = ATC_{2.3}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^* = \min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.2}(\overline{T_{2.2}})\}.$$

(d) $k_1 + k_4 \leq 2A < k_2 + k_3$ implies that $2A \in [k_1 + k_3, k_2 + k_3)$, $2A < k_2 + k_3 + k_5$. Hence

$2A \in [k_1 + k_3, k_2 + k_3) \Rightarrow T_{1.1}^* = \overline{T_{1.1}}$	$2A < k_2 + k_3 + k_5 \Rightarrow T_{1.3}^* = \frac{PM_2}{D}$
$2A < k_2 + k_3 \Rightarrow T_{1.4}^* = M_2$	$2A \geq k_1 \Rightarrow T_{2.1}^* = M_1$
$2A \geq k_1 + k_4 \Rightarrow T_{2.2}^* = \frac{PM_1}{D}$	$2A \geq k_1 + k_4 \Rightarrow T_{2.3}^* = \overline{T_{2.3}}$

By (4.1) and $T_{1.1}^* = \overline{T_{1.1}}$, we obtain

$$ATC_{1.1}(\overline{T_{1.1}}) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_1) = ATC_{2.1}(M_1).$$

By (4.2) and $T_{1.1}^* = \overline{T_{1.1}}$, we obtain

$$ATC_{1.1}(\overline{T_{1.1}}) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_2) = ATC_{1.4}(M_2).$$

By (4.4) and $T_{1.4}^{*} = M_2$, we obtain

$$ATC_{1.4}(M_2) = \min_{T\in\left[M_2, \frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}\left(\frac{PM_2}{D}\right) = ATC_{1.3}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T_{2.3}^{*} = \overline{T_{2.3}}$, we obtain

$$ATC_{2.3}\left(\overline{T_{2.3}}\right) = \min_{T\in\left[\frac{PM_1}{D}, \infty\right)} ATC_{2.3}(T) < ATC_{2.3}\left(\frac{PM_1}{D}\right) = ATC_{2.2}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^{*} = \min\{ATC_{1.1}(\overline{T_{1.1}}), ATC_{2.3}(\overline{T_{2.3}})\}.$$

(e) $k_2 + k_3 \leq 2A < k_1 + k_4$ implies that $2A \in [k_2 + k_3, k_2 + k_3 + k_5)$, $2A < k_2 + k_3 + k_5$, $2A \in [k_1, k_1 + k_4)$. Hence

$2A \geq k_2 + k_3 \Rightarrow T_{1.1}^{*} = M_2$	$2A < k_2 + k_3 + k_5 \Rightarrow T_{1.3}^{*} = \frac{PM_2}{D}$
$2A \in [k_2 + k_3, k_2 + k_3 + k_5) \Rightarrow T_{1.4}^{*} = \overline{T_{1.4}}$	$2A \geq k_1 \Rightarrow T_{2.1}^{*} = M_1$
$2A \in [k_1, k_1 + k_4) \Rightarrow T_{2.2}^{*} = \overline{T_{2.2}}$	$2A < k_1 + k_4 \Rightarrow T_{2.3}^{*} = \frac{PM_1}{D}$

By (4.1) and $T_{1.1}^{*} = M_2$, we obtain

$$ATC_{1.1}(M_2) = \min_{T\in[M_1, M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_1) = ATC_{2.1}(M_1).$$

By (4.2) and $T_{1.4}^{*} = \overline{T_{1.4}}$, we obtain

$$ATC_{1.4}(\overline{T_{1.4}}) = \min_{T\in[M_2, \frac{PM_2}{D}]} ATC_{1.4}(T) < ATC_{1.4}(M_2) = ATC_{1.1}(M_2).$$

By (4.4) and $T_{1.4}^{*} = \overline{T_{1.4}}$, we obtain

$$ATC_{1.4}(\overline{T_{1.4}}) = \min_{T\in\left[M_2, \frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}\left(\frac{PM_2}{D}\right) = ATC_{1.3}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T_{2.2}^{*} = \overline{T_{2.2}}$, we obtain

$$ATC_{2.2}\left(\overline{T_{2.2}}\right) = \min_{T\in\left[M_1, \frac{PM_1}{D}\right]} ATC_{2.2}(T) < ATC_{2.2}\left(\frac{PM_1}{D}\right) = ATC_{2.3}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^{*} = \min\{ATC_{1.4}(\overline{T_{1.4}}), ATC_{2.2}(\overline{T_{2.2}})\}.$$

(f) $k_1 + k_4 \leq 2A < k_1 + k_3$ implies that $2A < k_2 + k_3$, $2A < k_2 + k_3 + k_5$. Hence

$2A < k_1 + k_3 \Rightarrow T_{1.1}^{*} = M_1$	$2A < k_2 + k_3 + k_5 \Rightarrow T_{1.3}^{*} = \frac{PM_2}{D}$
$2A < k_2 + k_3 \Rightarrow T_{1.4}^{*} = M_2$	$2A \geq k_1 \Rightarrow T_{2.1}^{*} = M_1$
$2A \geq k_1 + k_4 \Rightarrow T_{2.2}^{*} = \frac{PM_1}{D}$	$2A \geq k_1 + k_4 \Rightarrow T_{2.3}^{*} = \overline{T_{2.3}}$

By (4.1) and $T^*_{2.2} = \frac{PM_1}{D}$, we obtain

$$\begin{aligned} ATC_{2.2}\left(\frac{PM_1}{D}\right) &= \min_{T\in\left[M_1,\frac{PM_1}{D}\right]} ATC_{2.2}(T) < ATC_{2.2}(M_1) \\ &= ATC_{1.1}(M_1) = ATC_{2.1}(M_1). \end{aligned}$$

By (4.2) and $T^*_{1.1} = M_1$, we obtain

$$ATC_{1.1}(M_1) = \min_{T\in[M_1.M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_2) = ATC_{1.4}(M_2).$$

By (4.4) and $T^*_{1.4} = M_2$, we obtain

$$ATC_{1.4}(M_2) = \min_{T\in\left[M_2,\frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}\left(\frac{PM_2}{D}\right) = ATC_{1.3}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T^*_{2.3} = \overline{T_{2.3}}$, we obtain

$$ATC_{2.3}\left(\overline{T_{2.3}}\right) = \min_{T\in\left[\frac{PM_1}{D},\infty\right)} ATC_{2.3}(T) < ATC_{2.3}\left(\frac{PM_1}{D}\right) = ATC_{2.2}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^* = ATC_{2.3}(\overline{T_{2.3}}).$$

(g) $\max\{k_2+k_3, k_1+k_4\} \le 2A < k_2+k_3+k_5$ implies that $2A \in [k_2+k_3, k_2+k_3+k_5)$. Hence

$2A \ge k_2+k_3 \Rightarrow T^*_{1.1} = M_2$	$2A < k_2+k_3+k_5 \Rightarrow T^*_{1.3} = \dfrac{PM_2}{D}$
$2A \in [k_2+k_3, k_2+k_3+k_5) \Rightarrow T^*_{1.4} = \overline{T_{1.4}}$	$2A \ge k_1 \Rightarrow T^*_{2.1} = M_1$
$2A \ge k_1+k_4 \Rightarrow T^*_{2.2} = \dfrac{PM_1}{D}$	$2A \ge k_1+k_4 \Rightarrow T^*_{2.3} = \overline{T_{2.3}}$

By (4.1) and $T^*_{1.1} = M_2$, we obtain

$$ATC_{1.1}(M_2) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_1) = ATC_{2.1}(M_1).$$

By (4.2) and $T^*_{1.4} = \overline{T_{1.4}}$, we obtain

$$ATC_{1.4}(\overline{T_{1.4}}) = \min_{T\in[M_2,\frac{PM_2}{D}]} ATC_{1.4}(T) < ATC_{1.4}(M_2) = ATC_{1.1}(M_2).$$

By (4.4) and $T^*_{1.4} = \overline{T_{1.4}}$, we obtain

$$ATC_{1.4}(\overline{T_{1.4}}) = \min_{T\in\left[M_2,\frac{PM_2}{D}\right]} ATC_{1.4}(T) < ATC_{1.4}\left(\frac{PM_2}{D}\right) = ATC_{1.3}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T^*_{2.3} = \overline{T_{2.3}}$, we obtain

$$ATC_{2.3}\left(\overline{T_{2.3}}\right) = \min_{T\in\left[M_1,\frac{PM_1}{D}\right]} ATC_{2.3}(T) < ATC_{2.3}\left(\frac{PM_1}{D}\right) = ATC_{2.2}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^* = \min\{ATC_{1.4}(\overline{T_{1.4}}), ATC_{2.3}(\overline{T_{2.3}})\}.$$

(h) $k_2+k_3+k_5 \le 2A$ implies that $2A \ge k_2+k_3$, $2A \ge k_2+k_3+k_5$, $2A \ge k_1+k_4$. Hence

$2A \ge k_2+k_3 \Rightarrow T^*_{1.1} = M_2$	$2A \ge k_2+k_3+k_5 \Rightarrow T^*_{1.3} = \overline{T_{1.3}}$
$2A \ge k_2+k_3+k_5 \Rightarrow T^*_{1.4} = \dfrac{PM_2}{D}$	$2A \ge k_1 \Rightarrow T^*_{2.1} = M_1$
$2A \ge k_1+k_4 \Rightarrow T^*_{2.2} = \dfrac{PM_1}{D}$	$2A \ge k_1+k_4 \Rightarrow T^*_{2.3} = \overline{T_{2.3}}$

By (4.1) and $T^*_{1.1} = M_2$, we obtain

$$ATC_{1.1}(M_2) = \min_{T\in[M_1,M_2]} ATC_{1.1}(T) < ATC_{1.1}(M_1) = ATC_{2.1}(M_1).$$

By (4.2) and $T^*_{1.4} = \frac{PM_2}{D}$, we obtain

$$ATC_{1.4}\left(\frac{PM_2}{D}\right) = \min_{T\in[M_2,\frac{PM_2}{D}]} ATC_{1.4}(T) < ATC_{1.4}(M_2) = ATC_{1.1}(M_2).$$

By (4.4) and $T^*_{1.3} = \overline{T_{1.3}}$, we obtain

$$ATC_{1.3}(\overline{T_{1.3}}) = \min_{T\in[\frac{PM_2}{D},\infty)} ATC_{1.3}(T) < ATC_{1.3}\left(\frac{PM_2}{D}\right) = ATC_{1.4}\left(\frac{PM_2}{D}\right).$$

By (4.5) and $T^*_{2.3} = \overline{T_{2.3}}$, we obtain

$$ATC_{2.3}\left(\overline{T_{2.3}}\right) = \min_{T\in[M_1,\frac{PM_1}{D}]} ATC_{2.3}(T) < ATC_{2.3}\left(\frac{PM_1}{D}\right) = ATC_{2.2}\left(\frac{PM_1}{D}\right).$$

Hence

$$ATC^* = \min\{ATC_{1.3}(\overline{T_{1.3}}), ATC_{2.3}(\overline{T_{2.3}})\}.$$

□

A.11 The proof of Theorem 3.11

Proof. As in [3], we define the values Δ_1, Δ_2, Δ_3, Δ_4 as follows

$$\begin{aligned}
\Delta_1 &= -2A + \frac{M_1^2}{D}[pI_c(1-r)(P^2-D^2) + sI_dD^2 + hP(P-D)] \\
\Delta_2 &= -2A + DM_1^2(H+SI_d) \\
\Delta_3 &= -2A + \frac{M_2^2}{D}[pI_c(P^2-D^2) + sI_dD^2 + hP(P-D)] \\
\Delta_4 &= -2A + DM_2^2(H+SI_d).
\end{aligned}$$

Then

$$\Delta_1 = -2A+k_1+k_4, \Delta_2 = -2A+k_1, \Delta_3 = -2A+k_2+k_5, \Delta_4 = -2A+k_2.$$

(a) $2A \le k_1$ implies that $\Delta_2 \ge 0$. Hence, by Theorem 1(A) of [3], we obtain

$$ATC^* = ATC_{2.4}(\overline{T_{2.4}}).$$

(b) $k_1 < 2A \leq \min\{k_2, k_1+k_4\}$ implies that $\Delta_1 \geq 0$, $\Delta_2 < 0$, $\Delta_4 \geq 0$. Hence, by Theorem 1(B) of [3], we obtain

$$ATC^* = \min\{ATC_{2.3}(\overline{T_{2.3}}), ATC_{2.4}(\overline{T_{2.4}})\}.$$

(c) $k_1 + k_4 < 2A \leq k_2$ implies that $\Delta_1 < 0$, $\Delta_4 \geq 0$. Hence, by Theorem 1(D) of [3], we obtain

$$ATC^* = \min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.5}(\overline{T_{2.5}})\}.$$

(d) $k_2 < 2A \leq k_1 + k_4$ implies that $\Delta_1 \geq 0$, $\Delta_2 < 0$, $\Delta_4 < 0$. Hence, by Theorem 1(C) of [3], we obtain

$$ATC^* = \min\{ATC_{2.2}(\overline{T_{2.2}}), ATC_{2.5}(\overline{T_{2.5}})\}.$$

(e) $\max\{k_2, k_1 + k_4\} < 2A < k_2 + k_5$ implies that $\Delta_1 < 0$, $\Delta_3 > 0$ $\Delta_4 < 0$. Hence, by Theorem 1(E) of [3], we obtain

$$ATC^* = \min\{ATC_{2.3}(\overline{T_{2.3}}), ATC_{2.5}(\overline{T_{2.5}})\}.$$

(f) $k_2 + k_5 \leq 2A$ implies that $\Delta_3 \leq 0$. Hence, by Theorem 1(F) of [3], we obtain

$$ATC^* = \min\{ATC_{2.3}(\overline{T_{2.3}}), ATC_{2.6}(\overline{T_{2.6}})\}.$$

□

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