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ANALYSIS OF A FLUID MODEL DRIVEN BY AN M/M/C VACATION QUEUE*

Xiuli Xu, Xiaoyan Liu, Wuyi Yue and Ning Mei

Abstract: A fluid queue is an input-output system where a continuous fluid enters and leaves a storage device, called a buffer, according to a randomly varying rate influenced by an underling stochastic environment or background. This paper considers a fluid flow model driven by a multi-server M/M/c queue with classical vacation. We obtain the sets of differential equations satisfied by the stationary joint distribution of the buffer content, by which we gain the simple structure of the Laplace transform (LT) for the stationary distribution of the buffer content. Furthermore, we give the probability of an empty buffer content and the expected buffer content based on the relationship between the LT and the Laplace-Stieltjes transform (LST) of the stationary distribution.

Key words: *fluid model, buffer content, multiple vacations, the Laplace transform (LT), the Laplace-Stieltjes transform (LST)*

Mathematics Subject Classification: *60K25, 90B22*

1 Introduction

Owing to the application of fluid queueing models in the field of wireless communications, transport, storage and computer systems, or others fields, the model has recently attracted interest from probability researchers as a research subject.

As it is well known, the distribution function of any buffer content satisfies a set of differential equations. Spectral analysis method has been the most traditional and commonly used method to find the solution to these equations. Kulkarni [2] proposed using the spectral method to deal with a fluid model driven by a Markov process with limited state.

As well as the spectral analysis method, Doorn and Scheinhardt [1] provided using orthogonal polynomials to express the stationary distribution of the buffer content, which is driven by an infinite-state birth-death process. Ramaswami [9] advanced a matrix analytic method, Neuts [7] extended geometric solution method into multi-dimensional matrix geometric solution method.

Lenin and Parthasarathy [3] studied the fluid model driven by an M/M/1/N queue, while Parthasarathy et al. [8] and many other researchers learned from indicators of fluid models driven by an M/M/1 queue with different methods.

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However, fluid models driven by queues with different vacation policies just begin research, such as the fluid model driven by an M/M/1/N queue with multiple exponential vacations in [4], the fluid model driven by an M/G/1 queue with multiple exponential vacations in [5] and the fluid model driven by the M/M/1 queue with working vacation and vacation interruption in [10].

In this paper, we mainly study some indexes relating to fluids model driven by an M/M/c queue with multiple vacations. Firstly, we discuss the drive system and obtain the stationary distribution of the drive system. Then, we introduce the Laplace transform (LT) and Laplace-Stieltjes transform (LST) of distribution functions. The LST of the stationary distribution of the buffer content is given on the basis of the relationship between the LT and the LST. Furthermore, we obtain the brief expressions of the mean of the buffer content, as well as the probability of buffer being empty.

The rest of the paper is organized as follows. In Section 2, an M/M/c queue with multiple vacations is presented and the steady-state distribution of queue length is derived under the stationary condition. In Section 3, we establish the fluid model driven by an underlying M/M/c vacation queue, and obtain the differential equations satisfied the stationary joint distribution of the fluid flow model. Then we gain a simple structure of the LT of the stationary distribution of the buffer content. Furthermore, we give the stationary probability of empty buffer content and the mean of the buffer content in steady state based on the relationship between the LT and the LST of the distribution. Conclusions are presented in Sections 4.

2 Description of an M/M/c Queue with Multiple Vacations

In this system, the inter-arrival times and service times follow an exponential distribution with parameters λ and μ, respectively. When there is no customer in the system after a service completion, the server will take a vacation of a random length which follows an exponential distribution with parameters θ. If there are customers in the system when a vacation comes to an end, the servers enter the busy period; Otherwise, the servers take another vacation.

This model is identified as the M/M/c queue with multiple vacations, abbreviated as the M/M/c/MV queue.

We assume that interarrival times, service times and vocation times are all independent, and the service discipline is First-Come, First-Served (FCFS).

Let $L(t)$ be the number of customers in the system at time t, and $J(t) = 0$ or 1, decided according to whether the system stays in a vacation period or a busy period at time t. Namely,

$$J(t) = \begin{cases} 0, & \text{the system stays in a vacation period at time } t, \\ 1, & \text{the system stays in a busy period at time } t. \end{cases}$$

Then, the stochastic process $\{(L(t), J(t)), t \geq 0\}$ is a Quasi-Birth-and-Death (QBD) process with the state spaces as follows:

$$\boldsymbol{\Omega} = \{(0,0)\} \bigcup \{(k,j), k \geq 1, j = 0,1\}.$$

Arranging the state spaces in lexicographic order, the infinitesimal generator $\boldsymbol{Q}$ for the QBD process $\{(L(t), J(t)), t \geq 0\}$ can be expressed as a block tridiagonal matrix form, that is

$$Q = \begin{pmatrix} A_0 & C_0 & & & & & & \\ B_1 & A_1 & C & & & & & \\ & B_2 & A_2 & C & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & B_{c-1} & A_{c-1} & C & & \\ & & & & B & A & C & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_0 = -\lambda, \quad B_1 = (0, \mu)^T, \quad C_0 = (\lambda, 0),$$

$$A_k = \begin{pmatrix} -(\lambda+\theta) & \theta \\ 0 & -(\lambda + k\mu) \end{pmatrix}, \quad k = 1, 2, ..., c-1,$$

$$B_k = \begin{pmatrix} 0 & 0 \\ 0 & k\mu \end{pmatrix}, \quad k = 2, 3, ..., c-1,$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & c\mu \end{pmatrix}, \quad A = \begin{pmatrix} -(\lambda+\theta) & \theta \\ 0 & -(\lambda + c\mu) \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

In order to get the expression of the stationary distribution of the process $\{(L(t), J(t)), t \geq 0\}$, it is necessary to obtain the minimal non-negative solution of the matrix equation $\boldsymbol{R}^2\boldsymbol{B} + \boldsymbol{RA} + \boldsymbol{C} = \boldsymbol{0}$. This solution $\boldsymbol{R}$ is called the rate matrix, which plays an important role in the analysis of the QBD process.

Lemma 2.1. *If $\rho_c = \lambda/(c\mu) < 1$, the quadratic matrix equation $\boldsymbol{R}^2\boldsymbol{B} + \boldsymbol{RA} + \boldsymbol{C} = \boldsymbol{0}$ has the minimal non-negative solution as follows:*

$$R = \begin{pmatrix} \dfrac{\lambda}{\lambda+\theta} & \rho_c \\ 0 & \rho_c \end{pmatrix}. \tag{2.1}$$

It is well known that the stationary distribution of $\{(L(t), J(t)), t \geq 0\}$ exists if and only if the spectral radius $SP(\boldsymbol{R}) < 1$ of the rate matrix $\boldsymbol{R}$ and the homogeneous linear equation $\boldsymbol{Z} \times \boldsymbol{B}[\boldsymbol{R}] = \boldsymbol{0}$ has a positive solution, where $\boldsymbol{Z}$ is a $(2 \times c + 1)$-dimensional row vector, and

$$B[R] = \begin{pmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C & & & \\ & B_2 & A_2 & C & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{c-1} & A_{c-1} & C \\ & & & & B & RB + A \end{pmatrix}.$$

It is easy to conclude that the M/M/c/MV system is stable if and only if the system workload $\rho_c < 1$. If $\rho_c < 1$, let

$$\pi_{kj} = \lim_{t \to +\infty} P\{L(t) = k, J(t) = j\}, \quad (k, j) \in \Omega,$$

then $\{\pi_{kj}, k \geq 0, j = 0, 1\}$ is the stationary distribution of the process $\{(L(t), J(t)), t \geq 0\}$.

Denoted by

$$\boldsymbol{\pi}_k = (\pi_{k0}, \pi_{k1}),\ k \geq 1, \quad \boldsymbol{\Pi} = (\pi_{00}, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, ...).$$

Then we have

Theorem 2.2. *If $\rho_c < 1$, the stationary distributions of $\{(L(t), J(t)), t \geq 0\}$ are as follows:*

$$\pi_{k0} = H\Big(\frac{\lambda}{\lambda+\theta}\Big)^k, \quad k \geq 0,$$

$$\pi_{k1} = \begin{cases} H\left\{\frac{1}{k!}\Big(\frac{\lambda}{\mu}\Big)^k + \sum\limits_{i=1}^{k-1}\frac{i!}{k!}\Big(\frac{\lambda}{\mu}\Big)^{k-i}\Big[\Big(\frac{\lambda}{\lambda+\theta}\Big)^c + \frac{\theta}{\lambda}\sum\limits_{v=i+1}^{c}\Big(\frac{\lambda}{\lambda+\theta}\Big)^v\Big]\right\}, & 1 \leq k \leq c, \\ \pi_{c0}\sum\limits_{v=1}^{k-c}\Big(\frac{\lambda}{\lambda+\theta}\Big)^{k-c-v}{\rho_c}^v + \pi_{c1}{\rho_c}^{k-c}, \quad k > c \end{cases}$$

where the constant $H = \pi_{00}$ can be determined by the normalization condition:

$$\sum_{k=0}^{\infty}\pi_{k0} + \sum_{k=1}^{\infty}\pi_{k1} = 1.$$

Proof. Since $\mathbf{\Pi}$ is a solution of matrix equation $\boldsymbol{Z} \times \boldsymbol{B}[\boldsymbol{R}] = \mathbf{0}$, we can obtain the linear equations as following:

$$\begin{cases} -\lambda\pi_{00} + \mu\pi_{11} = 0, \\ \theta\pi_{10} - (\lambda+\mu)\pi_{11} + 2\mu\pi_{21} = 0, \\ \lambda\pi_{k-1,0} - (\lambda+\theta)\pi_{k0} = 0, \quad 1 \leq k \leq c, \\ \lambda\pi_{k-1,1} + \theta\pi_{k0} - (\lambda+k\mu)\pi_{k1} + (k+1)\mu\pi_{k+1,1} = 0, \quad 2 \leq k \leq c-1, \\ \lambda\pi_{c-1,1} + (\lambda+\theta)\pi_{c0} - c\mu\pi_{c1} = 0. \end{cases} \tag{2.2}$$

Denoted by $\pi_{00} = H$, from the third expression in Eq. (2.2), we can obtain

$$\pi_{k0} = H(\frac{\lambda}{\lambda+\theta})^k, \quad 0 \leq k \leq c. \tag{2.3}$$

Especially, we have

$$\pi_{c0} = H(\frac{\lambda}{\lambda+\theta})^c.$$

On the basis of the forth and fifth expressions in Eq. (2.2), we can get

$$k\mu\pi_{k1} = \lambda\pi_{k-1,1} + \lambda\pi_{c0} + \theta\sum_{v=k}^{c}\pi_{v0}, \quad 2 \leq k \leq c.$$

Using the first expression in Eq. (2.2) and Eq. (2.3), after computing iteratively, we get

$$\pi_{k1} = H\left\{\frac{1}{k!}\Big(\frac{\lambda}{\mu}\Big)^k + \sum_{i=1}^{k-1}\frac{i!}{k!}\Big(\frac{\lambda}{\mu}\Big)^{k-i}\Big[\Big(\frac{\lambda}{\lambda+\theta}\Big)^c + \frac{\theta}{\lambda}\sum_{v=i+1}^{c}\Big(\frac{\lambda}{\lambda+\theta}\Big)^v\Big]\right\}, \quad 1 \leq k \leq c,$$

where we assume that the empty sum is equal to zero.

On the other hand, using geometric-matrix method (refer to [7]), we can get

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_c\boldsymbol{R}^{k-c} = (\pi_{c0}, \pi_{c1})\boldsymbol{R}^{k-c}, \quad k \geq c.$$

From Eq. (2.1), we know that

$$\boldsymbol{R}^k = \begin{pmatrix} \Big(\frac{\lambda}{\lambda+\theta}\Big)^k & \sum\limits_{v=1}^{k}\Big(\frac{\lambda}{\lambda+\theta}\Big)^{k-v}{\rho_c}^v \\ 0 & {\rho_c}^k \end{pmatrix}, \quad k \geq 1,$$

then the expression of $\boldsymbol{\pi}_k (k \geq c)$ is obtained. Finally, the constant H can be determined by the normalization condition $\sum_{k=0}^{+\infty}\pi_{k0} + \sum_{k=1}^{+\infty}\pi_{k1} = 1$. □

3 Analysis for the Fluid Model

Let $X(t)$ be the content of the buffer at time t, which is a non-negative random variable. Assume that the net input rate of fluid (the input rate minus the output rate) to the buffer is the function of the process $\{(X(t), L(t), J(t)), t \geq 0\}$:

$$\eta[X(t), L(t), J(t)] = \frac{dX(t)}{dt} = \begin{cases} 0, & (L(t), J(t)) = (0,0), \quad X(t) = 0, \\ \sigma, & (L(t), J(t)) = (0,0), \quad X(t) > 0, \\ \sigma_0, & (L(t), J(t)) = (k,0), \quad k \geq 1, \\ \sigma_1, & (L(t), J(t)) = (k,1), \quad k \geq 1, \end{cases}$$

where $\sigma < 0$, $\sigma_1 > \sigma_0 > 0$. Now, the fluid model driven by the M/M/c queue with multiple vacations is a three-dimensional Markov process with state space $\Omega' = [0, +\infty) \times \Omega$. Let

$$d = \sigma\pi_{00} + \sigma_0 \sum_{k=1}^{\infty} \pi_{k0} + \sigma_1 \sum_{k=1}^{\infty} \pi_{k1},$$

then d is called the average drift of the fluid model. It is not difficult to prove that the fluid model is stable if and only if $d < 0$ and $\rho_c < 1$ when the buffer capacity is infinite (see Kulkarni [2]).

Let $F_{k0}(t,x) = P\{L(t) = k, J(t) = 0, X(t) \leq x\}$, $k \geq 0$ and $F_{k1}(t,x) = P\{L(t) = k, J(t) = 1, X(t) \leq x\}$, $k \geq 1$, which are called the instantaneous joint probability distribution functions of the three-dimensional Markov process. If the process achieves balance, $\{(X(t), L(t), J(t)), t \geq 0\}$ converges to the random vector (X, L, J), where X is the stationary distribution of the buffer content. The joint distribution of (X, L, J) is denoted by

$$F_{k0}(x) = \lim_{t\to\infty} F_{k0}(t,x), \quad F_{k1}(x) = \lim_{t\to\infty} F_{k1}(t,x).$$

Then, the buffer content in steady state has a distribution function:

$$F(x) = P\{X \leq x\} = F_{00}(x) + \sum_{k=1}^{\infty} F_{k0}(x) + \sum_{k=1}^{\infty} F_{k1}(x), \quad x \geq 0.$$

Denoted by

$$\boldsymbol{F}(x) = (F_{00}(x), \boldsymbol{F}_1(x), \boldsymbol{F}_2(x), ...),$$

where $\boldsymbol{F}_k(x) = (F_{k0}(x), F_{k1}(x)), k \geq 1$.

Using the standard methods (see [6] or [8]), we can prove that $\boldsymbol{F}(x)$ satisfies the matrix differential equation as follows:

$$\frac{d}{dx}\boldsymbol{F}(x)\boldsymbol{\Lambda} = \boldsymbol{F}(x)\boldsymbol{Q} \tag{3.1}$$

and the following boundary condition:

$$\boldsymbol{F}(0) = (a, 0, 0, 0, ...), \tag{3.2}$$

where $\boldsymbol{\Lambda} = diag(\sigma, \Sigma, \Sigma, ...)$, $\boldsymbol{\Sigma} = diag(\sigma_0, \sigma_1)$. The probability $a = F_{00}(0) = P\{X = 0, L = 0, J = 0\}$ is called the stationary probability of the empty buffer content, which will be determined in the following analysis.

In order to solve the differential Eq. (3.1), we have to get help from the LT of the joint distribution:

$$\hat{F}_{kj}(s) = \int_0^{\infty} e^{-sx} F_{kj}(x)dx, \quad s > 0, \quad (k,j) \in \Omega.$$

Denoted by

$$\hat{\boldsymbol{F}}_k(s) = (\hat{F}_{k0}(s), \hat{F}_{k1}(s)),\ k \geq 1, \quad \hat{\boldsymbol{F}}(s) = (\hat{F}_{00}(s), \hat{\boldsymbol{F}}_1(s), ...).$$

Taking the LT on both sides of Eq. (3.1) and using Eq. (3.2), we can found that $\hat{\boldsymbol{F}}(s)$ satisfies the equation as in the following structure:

$$\hat{\boldsymbol{F}}(s)(\boldsymbol{Q} - s\boldsymbol{\Lambda}) = -\boldsymbol{F}(0)\boldsymbol{\Lambda} = (-a\sigma, 0, 0, 0, ...). \tag{3.3}$$

Next we introduce a crucial quadratic equation, whose roots play an important role in the following analysis.

Lemma 3.1. *If $\rho_c < 1$, for any $s \geq 0$, $c\mu z^2 - (\lambda + c\mu + s\sigma_1)z + \lambda = 0$ has two real roots $\gamma_0(s)$ and $\gamma_1(s)$, where*

$$\gamma_0(s)(\gamma_1(s)) = \frac{(\lambda + c\mu + s\sigma_1) - (+)\sqrt{(\lambda + c\mu + s\sigma_1)^2 - 4c\lambda\mu}}{2c\mu}.$$

It is easy to verify $0 < \gamma_0(s) < 1, \gamma_1(s) > 1$, and

$$\gamma_0(0) = \frac{\lambda}{c\mu} = \rho_c, \quad \gamma_1(0) = 1.$$

Lemma 3.2. *If $\rho_c < 1$, the quadratic matrix equation $(\boldsymbol{R}(s))^2\boldsymbol{B} + \boldsymbol{R}(s)(\boldsymbol{A} - s\boldsymbol{\Sigma}) + \boldsymbol{C} = \boldsymbol{0}$ has the minimal non-negative solution as follows:*

$$\boldsymbol{R}(s) = \begin{pmatrix} \dfrac{\lambda}{\lambda + \theta + s\sigma_0} & \dfrac{\theta\gamma_0(s)}{\lambda + \theta + s\sigma_0 - c\mu\gamma_0(s)} \\ 0 & \gamma_0(s) \end{pmatrix}.$$

Defined a series of functions by

$$\begin{cases} \phi_0(s) = 1, \\ \phi_k(s) = \dfrac{\lambda - c\mu\gamma_0(s)}{\lambda} + \dfrac{(c-k)\mu}{\lambda}\phi_{k-1}(s) + \dfrac{s\sigma_1}{\lambda}\displaystyle\sum_{v=0}^{k-1}\phi_v(s), \quad 1 \leq k \leq c-1, \end{cases} \tag{3.4}$$

and by

$$\begin{cases} \varphi_0(s) = 0, \\ \varphi_1(s) = \dfrac{(\lambda + \theta + s\sigma_0)\theta}{\lambda(\lambda + \theta + s\sigma_0 - c\mu\gamma_0(s))}, \\ \varphi_k(s) = \dfrac{\lambda + (c-k)\mu + s\sigma_1}{\lambda}\varphi_{k-1}(s) - \dfrac{(c-k+1)\mu}{\lambda}\varphi_{k-2}(s) + \dfrac{\theta}{\lambda}\Big(\dfrac{\lambda + \theta + s\sigma_0}{\lambda}\Big)^{k-1}, \\ \hfill 2 \leq k \leq c-1. \end{cases} \tag{3.5}$$

The matrix Eq. (3.3) can be rewritten as

$$\begin{cases} -(\lambda + s\sigma)\hat{F}_{00}(s) + \mu\hat{F}_{11}(s) = -a\sigma, \\ \theta\hat{F}_{10}(s) - (\lambda + \mu + s\sigma_1)\hat{F}_{11}(s) + 2\mu\hat{F}_{21}(s) = 0, \\ \lambda\hat{F}_{k-1,0}(s) - (\lambda + \theta + s\sigma_0)\hat{F}_{k0}(s) = 0, \quad 1 \leq k \leq c-1, \\ \lambda\hat{F}_{k-1,1}(s) + \theta\hat{F}_{k0}(s) - (\lambda + k\mu + s\sigma_1)\hat{F}_{k1}(s) + (k+1)\mu\hat{F}_{k+1,1}(s) = 0, \quad 2 \leq k \leq c-1, \\ \hat{\boldsymbol{F}}_{k-1}(s)\boldsymbol{C} + \hat{\boldsymbol{F}}_k(s)(\boldsymbol{A} - s\boldsymbol{\Sigma}) + \hat{\boldsymbol{F}}_{k+1}(s)\boldsymbol{B} = \boldsymbol{0}, \quad k \geq c. \end{cases} \tag{3.6}$$

Using Eq. (3.4) and Eq. (3.5), we get the following Theorem 3.3.

Theorem 3.3. *If $d < 0$ and $\rho_c < 1$, $\hat{F}_{k0}(s)$ and $\hat{F}_{k1}(s)$ can be expressed as*

$$\begin{cases} \hat{F}_{k0}(s) = \Big(\dfrac{\lambda+\theta+s\sigma_0}{\lambda}\Big)^{c-1-k} K_0(s), \quad 0 \le k \le c-1, \\ \hat{F}_{k1}(s) = K_1(s)\phi_{c-1-k}(s) - K_0(s)\varphi_{c-1-k}(s), \quad 1 \le k \le c-1, \\ \hat{\boldsymbol{F}}_k(s) = (K_0(s), K_1(s))(\boldsymbol{R}(s))^{k-c+1}, \quad k \ge c, \end{cases} \tag{3.7}$$

where $(K_0(s), K_1(s)) = \hat{\boldsymbol{F}}_{c-1}(s) = (\hat{F}_{c-1,0}(s), \hat{F}_{c-1,1}(s))$, *and*

$$\begin{aligned} K_0(s) =& \frac{a\sigma\lambda^{c-1}\Big[2\mu\phi_{c-3}(s) - (\lambda+\mu+s\sigma_1)\phi_{c-2}(s)\Big]}{H(s)}, \\ K_1(s) =& \frac{a\sigma\Big[2\mu\lambda^{c-1}\varphi_{c-3}(s) - (\lambda+\mu+s\sigma_1)\lambda^{c-1}\phi_{c-2}(s)\varphi_{c-2}(s) - \lambda\theta(\lambda+\theta+s\sigma_0)^{c-2}\phi_{c-2}(s)\Big]}{\phi_{c-2}(s)H(s)}, \\ H(s) =& \lambda\mu\theta(\lambda+\theta+s\sigma_0)^{c-2}\phi_{c-2}(s) + 2\mu(\lambda+s\sigma)(\lambda+\theta+s\sigma_0)^{c-1}\phi_{c-3}(s) - 2\mu^2\lambda^{c-1}\varphi_{c-3}(s) \\ &+ 2\mu^2\lambda^{c-1}\phi_{c-3}(s)\varphi_{c-2}(s) - (\lambda+\mu+s\sigma_1)(\lambda+s\sigma)(\lambda+\theta+s\sigma_0)^{c-1}\phi_{c-2}(s). \end{aligned}$$

Proof. If $d < 0$ and $\rho_c < 1$, the 3-dimensional Markov process $\{(X(t), L(t), J(t)), t \ge 0\}$ has unique stationary probability distribution $\{F_{kj}(x), (k,j) \in \Omega\}$. Therefore, there exists unique solution to Eq. (3.6). On the other hand, we can verify that $\hat{F}_{kj}(s), (k,j) \in \Omega$ in Eq. (3.7) is satisfied with Eq. (3.6).

Denoted by $\hat{\boldsymbol{F}}_{c-1}(s) = (K_0(s), K_1(s))$. If $k \ge c$, substituting the third expression of Eq. (3.7) into the fifth line of Eq. (3.6), using lemma 3.2 we obtain

$$\begin{aligned} &\hat{F}_{k-1}(s)\boldsymbol{C} + \hat{F}_k(s)(\boldsymbol{A} - s\boldsymbol{\Sigma}) + \hat{F}_{k+1}(s)\boldsymbol{B} \\ &=(K_0(s), K_1(s))(\boldsymbol{R}(s))^{k-c}\boldsymbol{C} + (K_0(s), K_1(s))(\boldsymbol{R}(s))^{k-c+1}(\boldsymbol{A} - s\boldsymbol{\Sigma}) \\ &\quad + (K_0(s), K_1(s))(\boldsymbol{R}(s))^{k-c+2}\boldsymbol{B} \\ &=(K_0(s), K_1(s))(\boldsymbol{R}(s))^{k-c}[\boldsymbol{C} + \boldsymbol{R}(s)(\boldsymbol{A} - s\boldsymbol{\Sigma}) + (\boldsymbol{R}(s))^2\boldsymbol{B}] \\ &=\boldsymbol{0}. \end{aligned}$$

From the third line of Eq. (3.6), we know that

$$\hat{F}_{k0}(s) = \Big(\frac{\lambda+\theta+s\sigma_0}{\lambda}\Big)^{c-1-k} K_0(s), \quad 0 \le k \le c-1.$$

Noting that Eq. (3.4) and Eq. (3.5), substituting the second expression of Eq. (3.7) into the fourth line of Eq. (3.6), we obtain

$$\begin{aligned} &\lambda\hat{F}_{k-1,1}(s) + \theta\hat{F}_{k0}(s) - (\lambda+k\mu+s\sigma_1)\hat{F}_{k1}(s) + (k+1)\mu\hat{F}_{k+1,1}(s) \\ &=\lambda\big[K_1(s)\phi_{c-k}(s) - K_0(s)\varphi_{c-k}(s)\big] + \theta\Big(\frac{\lambda+\theta+s\sigma_0}{\lambda}\Big)^{c-1-k}K_0(s) \\ &\quad - (\lambda+k\mu+s\sigma_1)\big[K_1(s)\phi_{c-k-1}(s) - K_0(s)\varphi_{c-k-1}(s)\big] \\ &\quad + (k+1)\mu\big[K_1(s)\phi_{c-k-2}(s) - K_0(s)\varphi_{c-k-2}(s)\big] \\ &=K_1(s)\big[\lambda\phi_{c-k}(s) - (\lambda+k\mu+s\sigma_1)\phi_{c-k-1}(s) + (k+1)\mu\phi_{c-k-2}(s)\big] \\ &\quad - K_0(s)\Big[\lambda\varphi_{c-k}(s) - \theta\Big(\frac{\lambda+\theta+s\sigma_0}{\lambda}\Big)^{c-1-k} - (\lambda+k\mu+s\sigma_1)\varphi_{c-k-1}(s) + (k+1)\mu\varphi_{c-k-2}(s)\Big] \\ &=0. \end{aligned}$$

Furthermore, the expressions of $K_0(s)$ and $K_1(s)$ can be determined by the first two expressions of Eq. (3.6), then theorem 3.3 is proved. □

Theorem 3.4. *If $d < 0$ and $\rho_c < 1$, the stable buffer content X has the LST as*

$$\begin{aligned} F^*(s) &= \frac{a\sigma}{\sigma_1}\Big\{1 + \frac{\lambda(\sigma_1-\sigma_0)+(\theta+s\sigma_0)(\sigma_1-\sigma)}{(\theta+s\sigma_0)} \times \frac{s}{H(s)} \\ &\quad \times(\lambda+\theta+s\sigma_0)^{c-1}\big[2\mu\phi_{c-3}(s)-(\lambda+\mu+s\sigma_1)\phi_{c-2}(s)\big]\Big\}. \end{aligned}$$

Proof. Taking sum on both sides of the forth line of Eq. (3.6) from 2 to c-1, we have

$$\theta\sum_{k=2}^{c-1}\hat{F}_{k0}(s) - s\sigma_1\sum_{k=2}^{c-1}\hat{F}_{k1}(s) + \lambda\hat{F}_{11}(s) - \lambda\hat{F}_{c-1,1}(s) - 2\mu\hat{F}_{21}(s) + c\mu\hat{F}_{c1}(s) = 0. \tag{3.8}$$

Summing up Eq. (11) and the second line of Eq. (3.6), we obtain

$$s\sigma_1\sum_{k=1}^{c-1}\hat{F}_{k1}(s) = \theta\sum_{k=1}^{c-1}\hat{F}_{k0}(s) - \mu\hat{F}_{11}(s) - \lambda\hat{F}_{c-1,1}(s) + c\mu\hat{F}_{c1}(s).$$

Using Eq. (3.7) and the first line of Eq. (3.6), then we have

$$\begin{aligned} \sum_{k=1}^{c-1}\hat{F}_{k1}(s) &= \frac{\theta}{s\sigma_1}\sum_{k=0}^{c-1}\hat{F}_{k0}(s) - \frac{\lambda+\theta+s\sigma}{s\sigma_1}\hat{F}_{00}(s) + \frac{a\sigma}{s\sigma_1} + \frac{c\mu\gamma_0(s)-\lambda}{s\sigma_1}K_1(s) \\ &\quad + \frac{c\mu\theta\gamma_0(s)}{s\sigma_1(\lambda+\theta+s\sigma_0-c\mu\gamma_0(s))}K_0(s). \end{aligned}$$

Similarly, from the third expression in Eq. (3.6), we get

$$\sum_{k=0}^{c-1}\hat{F}_{k0}(s) = \frac{\lambda+\theta+s\sigma_0}{\theta+s\sigma_0}\hat{F}_{00}(s) - \frac{\lambda}{\lambda+\theta+s\sigma_0}K_0(s).$$

Hence, the LT of the stationary distribution $\boldsymbol{F}(x)$ of the buffer content can be given by

$$\begin{aligned} \hat{F}(s) &= \int_0^{+\infty} e^{-sx}F(x)dx \\ &= \sum_{k=0}^{c-1}\hat{F}_{k0}(s) + \sum_{k=1}^{c-1}\hat{F}_{k1}(s) + \sum_{k=c}^{\infty}\hat{\boldsymbol{F}}_k(s)\boldsymbol{e} \\ &= \left[\frac{\lambda(\sigma_1-\sigma_0)+(\theta+s\sigma_0)(\sigma_1-\sigma)}{\sigma_1(\theta+s\sigma_0)}\Big(\frac{\lambda+\theta+s\sigma_0}{\lambda}\Big)^{c-1} - \frac{\lambda(\theta+s\sigma_1)}{s\sigma_1(\theta+s\sigma_0)}\right]K_0(s) \\ &\quad + \frac{c\mu\theta\gamma_0(s)}{s\sigma_1(\lambda+\theta+s\sigma_0-c\mu\gamma_0(s))}K_0(s) + \frac{a\sigma}{s\sigma_1} + \frac{c\mu\gamma_0(s)-\lambda}{s\sigma_1}K_1(s) \\ &\quad + (K_0(s),K_1(s))\boldsymbol{R}(s)(\boldsymbol{I}-\boldsymbol{R}(s))^{-1}\boldsymbol{e}, \end{aligned}$$

where $\boldsymbol{e} = (1,1)^T$.

For the spectral radius

$$SP[\boldsymbol{R}(s)] = \max\left(\gamma_0(s), \frac{\lambda}{\lambda+\theta+s\sigma_0}\right) < 1,$$

so $\boldsymbol{I}-\boldsymbol{R}(s)$ is invertible, and

$$(\boldsymbol{I}-\boldsymbol{R}(s))^{-1}=\begin{bmatrix} \dfrac{\lambda+\theta+s\sigma_0}{\theta+s\sigma_0} & \dfrac{\theta\gamma_0(s)(\lambda+\theta+s\sigma_0)}{(\lambda+\theta+s\sigma_0-c\mu\gamma_0(s))(\theta+s\sigma_0)\,(1-\gamma_0(s))} \\ 0 & \dfrac{1}{1-\gamma_0(s)} \end{bmatrix}.$$

After calculating, we have

$$\hat{F}(s) \quad = \quad \frac{a\sigma}{s\sigma_1}+\frac{\lambda(\sigma_1-\sigma_0)+(\theta+s\sigma_0)(\sigma_1-\sigma)}{\sigma_1(\theta+s\sigma_0)}\Big(\frac{\lambda+\theta+s\sigma_0}{\lambda}\Big)^{c-1}K_0(s).$$

Next, we define the LST of the stationary joint distribution for the fluid model and the stationary distribution of the buffer content as

$$F^*_{kj}(s)=\int_0^{+\infty}e^{-sx}dF_{kj}(x),\quad (k,j)\in\Omega,$$

$$F^*(s)=\int_0^{+\infty}e^{-sx}dF(x).$$

It is easy to prove

$$F^*_{00}(s)=-a+s\hat{F}_{00}(s),$$

$$\boldsymbol{F}^*_k(s)=s\hat{\boldsymbol{F}}_k(s),\ k\geq 1.$$

Substituting the expression of $\hat{F}_{kj}(s)$ into the above expressions and after calculation and arrangement, we can obtain the LST of the stationary distribution of the buffer content as

$$\begin{aligned} F^*(s) \quad = \quad & \frac{a\sigma}{\sigma_1}\Big\{1+\frac{\lambda(\sigma_1-\sigma_0)+(\theta+s\sigma_0)(\sigma_1-\sigma)}{(\theta+s\sigma_0)}\times\frac{s}{H(s)} \\ & \times(\lambda+\theta+s\sigma_0)^{c-1}\big[2\mu\phi_{c-3}(s)-(\lambda+\mu+s\sigma_1)\phi_{c-2}(s)\big]\Big\}. \end{aligned} \tag{3.9}$$

Then theorem 3.4 is proved. □

With the normalization condition $\lim_{s\to 0}F^*(s)=1$ and Hopital's rule, we can acquire the expression of the probability a as

$$a=\frac{\sigma_1}{\sigma}\frac{\theta H'(0)}{\theta H'(0)+\big[\lambda(\sigma_1-\sigma_0)+\theta(\sigma_1-\sigma)\big](\lambda+\theta)^{c-1}\big[2\mu\phi_{c-3}(0)-(\lambda+\mu)\phi_{c-2}(0)\big]}.$$

Denoted by

$$\begin{aligned} f(s)&=\frac{\lambda(\sigma_1-\sigma_0)+(\theta+s\sigma_0)(\sigma_1-\sigma)}{(\theta+s\sigma_0)}, \\ g(s)&=\frac{s}{H(s)}, \\ h(s)&=(\lambda+\theta+s\sigma_0)^{c-1}\big[2\mu\phi_{c-3}(s)-(\lambda+\mu+s\sigma_1)\phi_{c-2}(s)\big]. \end{aligned}$$

Now, taking the derivatives on both sides of Eq. (3.9) with respect to s, then let $s\to 0$, we get the mean of the buffer content:

$$E(X)=-\frac{a\sigma}{\sigma_1}\Big(f'(0)g(0)h(0)+f(0)g'(0)h(0)+f(0)g(0)h'(0)\Big),$$

where

$$\begin{aligned}
f(0) = &\ \frac{\lambda(\sigma_1 - \sigma_0) + \theta(\sigma_1 - \sigma)}{\theta}, \\
g(0) = &\ \frac{1}{H'(0)}, \\
h(0) = &\ (\lambda + \theta)^{c-1}\big[2\mu\phi_{c-3}(0) - (\lambda + \mu)\phi_{c-2}(0)\big], \\
f'(0) = &\ \frac{\sigma_0[(\sigma_1 - \sigma_0)(1 - \theta) - \lambda(\sigma_1 - \sigma_0)]}{\theta^2}, \\
g'(0) = &\ \frac{-H''(0)}{2(H'(0))^2}, \\
h'(0) = &\ (c-1)(\lambda + \theta)^{c-2}\sigma_0\big[2\mu\phi_{c-3}(0) - (\lambda + \mu)\phi_{c-2}(0)\big] \\
&\ +(\lambda + \theta)^{c-1}\big[2\mu\phi'_{c-3}(0) - (\lambda + \mu)\phi'_{c-2}(0) - \sigma_1\phi_{c-2}(0)\big].
\end{aligned}$$

4 Conclusions

In this paper, we discussed the fluid model driven by an M/M/c multiple vocations queue, where the input rate and output rate are determined by the drive system. That is the queue length of the M/M/c multiple vacations queue. Using a QBD process and a matrix-geometric solution method, the steady state distribution of the queue length was derived. Furthermore, we obtained the main expressions for the LST of the stationary distribution, the probability of empty buffer content and the mean of the buffer content.

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References

[1] E.A. van Doorn and W.R.W. Scheinhardt, A fluid queue driven by an infinite-state birth-death process, in *Proc. 15th International Teletraffic Congress*, 1997, pp. 465–475.

[2] V.G. Kulkarni, Fluid models for single buffer systems, in *Frontiers in Queueing: Models and Applications in Science and Engineering*, J. Dhashaloe (ed.), CRC Press, 1997, pp. 321–338.

[3] R.B. Lenin and P.R. Parthasarathy, Fluid queues driven by an M/M/1/N queue, *Mathematical Problems in Engineering* (2000) 439–460.

[4] B. Mao, F. Wang and N. Tian, Fluid model driven by an M/M/1/N queue with multiple exponential vacations, *Journal of Computational Information System* 6 (2010) 1809–1816.

[5] B. Mao, F. Wang and N. Tian, Fluid model driven by an M/G/1 queue with multiple exponential vacations, *Applied Mathematics and Computation* 218 (2011) 4041–4048.

[6] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer, *IEEE Advances in Applied Probability* 20 (1998) 646–676.

[7] M. Neuts, Matrix-geometric solution in stochastic models, *Advances in Mathematics* 55 (1981) 62–64.

[8] P.R. Parthasarathy, K.V. Vijayashree and R.B. Lenin, An M/M/1 driven fluid queue-continued fraction approach, *Queueing Systems* 42 (2002) 189–199.

[9] V. Ramaswami, Matrix analytic methods for stochastic fluid flows, in *Proc. Stochastic Analysis of Computer and Communication Systems*, 1999, pp. 1019–1030.

[10] X. Xu, H. Guo, Y. Zhao and J. Geng, The fluid model driven by the M/M/1 queue with working vacation and vacation interruption, *Journal of Computational Information Systems* 8 (2012) 7643–7651.

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Xiuli Xu
College of Science, Yanshan University
Qinhuangdao 066004 China
E-mail address: `xxl-ysu@163.com`

Xiaoyan Liu
College of Science, Yanshan University
Qinhuangdao 066004 China
E-mail address: `329878595@qq.com`

Wuyi Yue
Department of Intelligence and Informatics
Konan University, Kobe 658-8501 Japan
E-mail address: `yue@konan-u.ac.jp`

Ning Mei
College of Science, Yanshan University
Qinhuangdao 066004 China
E-mail address: `570673913@qq.com`