Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 9, No. 4, pp. 635–650, October 2013

FEEDBACK EQUIVALENCE AND THE CONTRAST PROBLEM IN NUCLEAR MAGNETIC RESONANCE IMAGING

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Abstract: The theoretical analysis of the contrast problem in NMR imaging is mainly reduced, thanks to the Maximum Principle, to the analysis of the so-called singular trajectories of the control system modeling the problem: a coupling of two Bloch equations representing the evolution of the magnetization vector of each spin particle. They are solutions of a constrained Hamiltonian equation. In this article we describe feedback invariants related to the singular flow to distinguish the different cases occurring in physical experiments.

Key words: *Mayer problem, geometric optimal control, contrast imaging*

Mathematics Subject Classification: *49K15, 81Q93*

1 Introduction

The contrast problem in magnetic resonance imaging was recently analyzed from theoretical, numerical and experimental points of view, using techniques of optimal control, see the series of articles [3], [4], [5], [6], [11]. In particular geometric optimal control is used to compute the optimal control field which produces a maximum contrast between two substances, e.g. oxygenated and deoxygenated blood, by saturation of the first spin $1/2$ magnetization vector based on [10]. Details about the NMR imaging application can be found in [6] and [11].

The model is a single-input control system, and the Pontryagin Maximum Principle states that in this case the optimal solution is a concatenation of bang arcs, where the amplitude of the control is maximum, and singular arcs. Based on the aforementioned articles, the optimal problem can be mainly reduced to the analysis of the singular flow. This flow depends on the physical parameters and in this article our focus is the understanding of the role of those parameters in the behavior of the singular arcs, e.g. stability properties. Such an analysis uses the concept of feedback classification [2].

The mathematical model is given by coupling two Bloch equations [13]

$$
\begin{aligned}
\frac{dM_x^i}{dt} &= -\omega M_y^i + \omega_y M_z^i - M_x^i/T_2^i \\
\frac{dM_y^i}{dt} &= \omega M_x^i - \omega_x M_z^i - M_y^i/T_2^i \\
\frac{dM_z^i}{dt} &= -\omega_y M_x^i + \omega_x M_y^i - (M_z^i - M_0)/T_1^i
\end{aligned}
\tag{1.1}
$$

ISSN 1348-9151

where $M^i = (M^i_x, M^i_y, M^i_z)$, $i = 1, 2$, is the magnetization vector representing each spin particle, M_0 is the equilibrium, T^i_1 and T^i_2 are the longitudinal and transverse relaxation rates, ω is the resonance offset and (ω_x, ω_y) represent the components of the control magnetic field.

Assuming the spin 1/2 particles are in resonance, $\omega = 0$, and the applied control field only tuned in amplitude one has $\omega_y = 0$ and using appropriate normalizations the control system is reduced to a four-dimensional system

$$\frac{dq}{dt} = F(q) + uG(q), \quad |u| \leq 2\pi$$

where $q = (q_1, q_2)$ represents the magnetization vector of the couple and each q_i is described by the dynamics (up to a time reparametrization)

$$\begin{aligned} \frac{dy_i}{dt} &= -\Gamma_i y_i - u_x z_i \\ \frac{dz_i}{dt} &= \gamma_i(1 - z_i) + u_x y_i, \end{aligned} \tag{1.2}$$

$i = 1, 2$, where $|u_x| \leq 2\pi$ and the parameters $\Lambda_i = (\gamma_i, \Gamma_i)$ are related to the relaxation times and satisfy $2\Gamma_i \geq \gamma_i$ so the Bloch ball $|q_i| \leq 1$ is invariant for the dynamics.

The various contrast problems in magnetic resonance imaging which differ mainly by the boundary conditions are defined by a *Mayer problem*: given a transfer time T, minimize a cost function c at a final time, $\min_{u(\cdot)} c(q(T))$, subject to $\dot{q} = F(q) + uG(q)$ and the boundary conditions $q(0) = ((0, 1), (0, 1))$ (the equilibrium point of free motion) with the final constraint $g(q(T)) = 0$.

For instance, in the contrast by *saturation* problem we have

- $q_1(T) = 0$ (saturation of the first spin)
- $c(q(T)) = -|q_2(T)|^2$, and $|q_2(T)|$ is the final contrast.

The candidates as minimizers are parameterized by the Maximum Principle [15].

Proposition 1.1 (Maximum Principle)**.** *Let $u^*(\cdot)$ be an optimal control whose associated trajectory $q^*(\cdot)$ is optimal for the contrast problem. Denoting $H(q, p, u) = \langle p, F(q) + uG(q)\rangle$ as the pseudo-Hamiltonian, there exists $p^*(\cdot)$ and a constant p^*_0 such that for almost every $t \in [0, T]$,*

(1) $\frac{dq^*}{dt} = \frac{\partial H}{\partial p}(q^*, p^*, u^*)$, $\quad \frac{dp^*}{dt} = -\frac{\partial H}{\partial q}(q^*, p^*, u^*)$

(2) $H(q^*, p^*, u^*) = \max_{|u| \leq 2\pi} H(q^*, p^*, u)$ *(maximization condition)*

and additionally we have the following boundary conditions,

(3) $g(q^*(T)) = 0$

(4) $p^*(T) = p^*_0 \frac{\partial c}{\partial q}(q^*(T)) + \sum_{i=1}^k \sigma_i \frac{\partial g_i}{\partial q_i}(q^*(T))$, $\sigma = (\sigma_1, \ldots, \sigma_k)$, $p^*_0 \leq 0$ *(transversality condition)*

Definition 1.2. Triples (q, p, u) which are solutions of the first two conditions of Proposition 1.1 are called extremals and if they satisfy the boundary conditions they are called BC-extremals. An extremal subarc on $[0, T]$ is called regular if $u(t) = 2\pi \operatorname{sgn}\langle p(t), G(q(t))\rangle$ and singular if $\langle p(t), G(q(t))\rangle = 0$.

We use the following notation. If X is a vector field, $H_X = \langle p, X\rangle$ is the Hamiltonian lift. The Lie bracket of two vector fields is computed by the convention $[X_1, X_2](q) = \frac{\partial X_1}{\partial q}(q)X_2(q) - \frac{\partial X_2}{\partial q}(q)X_1(q)$ and the Poisson bracket of two Hamiltonians is $\{H_1, H_2\} = dH_1(\overrightarrow{H}_2)$. Singular extremals can be easily computed. Deriving $H_G(z(t)) = 0$, $z = (q, p)$, one gets

$$H_G = \{H_G, H_F\} = 0$$
$$\{\{H_G, H_F\}, H_F\} + u_s\{\{H_G, H_F\}, H_G\} = 0.$$

Denoting $\Sigma : H_G = 0$, $\Sigma' : H_G = \{H_G, H_F\} = 0$, $S : \{\{H_G, H_F\}, H_G\} = 0$, and $H_s = H_F + u_s H_G$, one gets the following proposition.

Proposition 1.3. *Outside S, the singular extremals are solutions of the Hamiltonian vector field $\overrightarrow{H}_s$, restricted to the surface Σ'.*

Most of the properties of the optimal solutions of the contrast problem are coded in the pair $(\overrightarrow{H}_s, \Sigma')$. In particular in the contrast problem, the four sets of parameters given below are important in the classification.

Physical parameter relaxation times (in seconds) [9]

(1) $\mathbf{P}_1$: water: $T_1 = T_2 = 2.5$; cerebrospinal fluid: $T_1 = 2$, $T_2 = 0.3$.

(2) $\mathbf{P}_2$: deoxygenated blood: $T_1 = 1.35$, $T_2 = 0.05$; oxygenated blood: $T_1 = 1.35$, $T_2 = 0.2$.

(3) $\mathbf{P}_3$: gray cerebral matter: $T_1 = 0.92$, $T_2 = 0.1$; white cerebral matter: $T_1 = 0.780$, $T_2 = 0.09$.

(4) $\mathbf{P}_4$: water: $T_1 = T_2 = 2.5$; fat tissue: $T_1 = 0.2$, $T_2 = 0.1$.

The aim of this article is to present a brief analysis of the relations between the relaxation parameters and the classification of $(\overrightarrow{H}_s, \Sigma')$. The adapted concept is the one of feedback classification.

2 Singular trajectories and the feedback classification pairs

2.1 Feedback classification

First of all, we need some standard concepts from geometric invariant theory [7, 14].

Definition 2.1. Let E and F be two $\mathbb{R}$-vector spaces and let $\mathcal{G}$ be a group acting linearly on E and F. A homomorphism $\chi : \mathcal{G} \to \mathbb{R} \setminus \{0\}$ is called a character. A semi-invariant of weight χ is a map $\lambda : E \to \mathbb{R}$ such that for all $g \in \mathcal{G}$ and all $x \in E$, $\lambda(g.x) = \chi(g)\lambda(x)$; it is an invariant if $\chi = 1$. A map $\lambda : E \to F$ is a semi-covariant of weight χ if for all $g \in \mathcal{G}$ and for all $x \in E$, $\lambda(g.x) = \chi(g)g.\lambda(x)$; it is called a covariant if $\chi = 1$.

Next, we introduce the feedback group, reducing our presentation to the single-input case. We denote by $\mathcal{C}$ the set $\{F, G\}$ of such (smooth) systems on the state space $V \cong \mathbb{R}^n$.

Definition 2.2. Let (F, G), (F', G') be two elements in $\mathcal{C}$. They are called feedback equivalent if there exists a smooth diffeomorphism φ of $\mathbb{R}^n$ and a feedback $u = \alpha(q) + \beta(q)v$, where α and β are smooth, β invertible such that

(i) $F' = \varphi * F + \varphi * (G\alpha)$

(ii) $G' = \varphi * (G\beta)$

where $\varphi * Z$ is the vector field image defined by $\varphi * Z = \frac{\partial \varphi^{-1}}{\partial q}(Z \circ \varphi)$.

Definition 2.3. Let $(F, G) \in \mathcal{C}$ and λ_1 be the map which associates the constrained Hamiltonian vector field $(\overrightarrow{H}_s, \Sigma')$ (see the introduction for the definition). We define the action of $(\varphi, \alpha, \beta) \in \mathcal{G}$ on $(\overrightarrow{H}_s, \Sigma')$ to be the action of the sympletic change of coordinates (on Hamiltonian vector fields and surfaces)

$$\overrightarrow{\varphi} : q = \varphi(Q), \quad p = P\frac{\partial \varphi^{-1}}{\partial Q},$$

in particular the feedback acts trivially.

Our classification relies on the general theorem.

Theorem 2.4 (see [2])**.** *The mapping λ_1 is a covariant.*

In order to analyze our classification in the contrast problem, we shall restrict ourselves to a subset of singular trajectories.

$\boxed{2.2}$ Exceptional trajectories and the feedback classification

Definition 2.5. A singular extremal (q, p) is called *exceptional* if $H_F = \langle p, F(q) \rangle = 0$.

According to the Maximum Principle they are associated to an optimal control problem where the transfer time T is free.

Application

In the contrast problem, the state space $V \cong \mathbb{R}^4$. Using the additional constraint $H_F = 0$, in the exceptional case the singular control is given by the *feedback*

$$u_s^e = -\frac{D'(q)}{D(q)}$$

where

$$D = \det(F, G, [G, F], [[G, F], G])$$
$$D' = \det(F, G, [G, F], [[G, F], F])$$

which leads us to introduce the vector field X^e defined by

$$\frac{dq}{dt} = F - \frac{D'}{D}G$$

which can be desingularized using the reparameterization $ds = dt/D(q(t))$ and this gives the smooth vector field

$$X_r^e = DF - D'G.$$

This leads to the following reduced action of the feedback group.

Notation

Let φ be a diffeomorphism of V. Then φ acts on the mapping $f : V \to \mathbb{R}$ according to $\varphi.f = f \circ \varphi$ and on vector fields as $\varphi.X = \varphi * X$ (image of X by φ).

According to this action, we have the following lemma.

Lemma 2.6. *We have that*

- $D^{F+\alpha G,\beta G} = \beta^4 D^{F,G}$
- $D'^{F+\alpha G,\beta G} = \beta^3 (D'^{F,G} + \alpha D^{F,G})$
- $D^{\varphi * F,\varphi * G}(q) = \det\left(\frac{\partial\varphi}{\partial q}^{-1}\right) D^{F,G}(\varphi(q))$
- $D'^{\varphi * F,\varphi * G}(q) = \det\left(\frac{\partial\varphi}{\partial q}^{-1}\right) D'^{F,G}(\varphi(q))$.

From this, we deduce the following proposition, where the weights are associated to β and $\det\left(\frac{\partial\varphi^{-1}}{\partial q}\right)$.

Proposition 2.7. *We have the following.*

- $\lambda_2 : (F, G) \to X^e$ *is a covariant.*
- $\lambda_3 : (F, G) \to D$ *is a semi-covariant.*
- $\lambda_4 : (F, G) \to D = D'$ *is a semi-covariant.*
- $\lambda_5 : (F, G) \to X_r^e = DF - D'G$ *is a semi-covariant.*

Application and geometric interpretation

The action of diffeomorphisms on X^e can be used to classify the set of systems (F, G). In the contrast problem we can construct a set of invariants related to the problem and in particular relate the properties of the optimal solution to the experimental parameters. The geometric interpretation of the invariants in connection with the above covariants is the following.

- Invariants can be found in the dynamical properties of the dynamical system X^e: equilibrium points, periodic solutions, stability analysis, and integrability properties of the set of solutions.
- The surface $D = 0$ encodes the set of points where the singular control explodes and is preceded by the saturation of the constraint $|u| \leq 2\pi$, while the surface $D = D' = 0$ encodes the set of points where the singular control can cross the set $D = 0$. This describes the main singularities of X_r^e, but the analysis is intricate since it is related to the classification of the behavior of solutions near non-isolated singularities, see [16, 17] for such recent studies.

3 Classification results

We shall make a short description of the classification in relation with the experimental parameters.

3.1 Preliminary results

The physical relaxation parameters T_1^i, T_2^i corresponding to the spin signatures of the substances are given in the introduction. The introduction of the parameters $\Lambda = (\Lambda_1, \Lambda_2)$ in relation with the control bound in the experiments is

$$\gamma_i = \frac{2\pi}{\omega_{\max} T_1^i}, \quad \Gamma_i = \frac{2\pi}{\omega_{\max} T_2^i}$$

where $\omega_{\max} = 32.3$ Hz is taken in the experiments [11].

We shall identify the set of parameters in Λ up to a scalar, taking the projective space $\mathbb{P}^3 : \Lambda/\mathbb{R}$ and only rational invariants will be obtained. They are related to the one-dimensional foliation X_r^e obtained by reparameterizing the singular trajectories and the quartic surface $D = 0$.

Both X_r^e and D can be compactified by standard techniques, that is,

- the polynomial vector field X_r^e in $\mathbb{R}^4$ is extended to $\mathbb{R}^5$ using the Poincaré vector field compactification: $X_r^e \to {}^P X_r^e = {}^H X_r^e \frac{\partial}{\partial q} + 0 \frac{\partial}{\partial q_0}$, where ${}^H X_r^e$ is the quadratic homogeneous vector field such that ${}^H X_r^e = X_r^e$ for $q_0 = 1$;
- similarly the quartic D in $\mathbb{R}^4$ can be replaced by the homogeneous quartic in dimension five: $D \to {}^H D$, defined by ${}^H D_{|_{q_0=1}} = D$.

Clearly even the classification of D is a complicated problem.

To extract the invariants we shall use mainly two *physical invariants* of the problem which are described below.

- *The point* $O_1 = ((0,1),(0,1))$. Each point $(0,1)$ corresponds to the north pole of the Bloch ball of each spin and $(0,1)$ is the globally attractive, stable equilibrium point of the uncontrolled Bloch equation. In the imaging process the experiment is repeated many times, letting the system relax to the equilibrium point before restarting the next trial.
- *The point* $O_2 = ((0,0),(0,0))$. The point $(0,0)$ of each spin corresponds to the center of the Bloch ball and in image processing it corresponds to the saturation of the spin, giving the color black in imaging.

We have the following straightforward but important result.

Proposition 3.1. *Both points O_1 and O_2 are equilibrium points of X_r^e and moreover the z_i-axes are line solutions along which the singular control is zero, connecting the singularities $O_2 \to O_1$.*

This gives the first (topological) information about the flow.

Next we introduce the following, see [12].

3.2 Quadratic differential equations

Consider a quadratic differential equation in $\mathbb{R}^n$,

$$\frac{dx_i}{dt} = \sum_{j,k=1}^{n} a_{jk}^i x_j x_k, \quad i = 1, \ldots, n$$

in which $a_{jk}^i = a_{kj}^i$, written as $\dot{x} = Q(x)$. Such an equation is identified to a $(1,2)$ tensor whose linear classification is analyzed by introducing the following.

Definition 3.2. Let (e_i) be the canonical basis of $\mathbb{R}^n$ and endow $\mathbb{R}^n$ with the multiplication defined by $e_j.e_k = \sum_{i=1}^n a_{jk}^i e_i$. The associated commutative algebra (in general non-associative) is denoted by $\mathcal{A}$.

The relation with Lie brackets is given by the following lemma.

Lemma 3.3. *Let $v_1, v_2 \in \mathbb{R}^n$. Then $[Q, v_1](v_2) = [v_2, [v_1, Q]] = 2v_1.v_2$.*

Clearly two quadratic differential equations are (linearly) isomorphic if and only if their associated algebras are isomorphic and the classification relies on the following.

Proposition 3.4. (1) *$\mathcal{E}$ is a subalgebra of $\mathcal{A}$ if and only if $\mathcal{E}$ is an invariant vector space for the solutions of $\dot{x} = Q(x)$ and one can define the restriction of the equation to $\mathcal{E}$.*

(2) *A subalgebra I of $\mathcal{A}$ is an ideal if and only if $\dot{x} = Q(x)$ can be projected on the quotient $\mathcal{A}/I$.*

Ray solutions

One-dimensional subalgebras are called *ray solutions* of $\dot{x} = Q(x)$. They correspond to lines $\mathbb{R}v$ such that $Q(v) = \lambda v$ for some $\lambda \in \mathbb{R}$. Geometrically they correspond to

- lines of non-isolated equilibrium points of $\dot{x} = Q(x)$ if $\lambda = 0$, and
- true ray solutions if $\lambda \neq 0$ and $\mathbb{R}v$ is a solution on which the dynamics are reduced to $\dot{y}_1 = y_1^2$.

Blowing-up of the equation

To analyze a quadratic differential equation $\dot{x} = Q(x)$ we can introduce the differential equation on the sphere S^{n-1} defined by

$$\dot{v} = Q(v) - \langle v, Q(v) \rangle v$$

and the associated vector field will be denoted Q^π. Ray solutions are in correspondence with equilibrium points of Q^π and eigenvalues and eigenspaces of the linearized system $\dot{y} = Ay$ of a singular point y_0 of $Q^\pi : A = \frac{\partial Q^\pi}{\partial y}(y_0)$ are obtained as follows.

Find linear coordinates $(y_0, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ such that $\dot{x} = Q(x)$ takes the form

$$\begin{aligned} \dot{y}_0 &= \lambda y_0^2 + \cdots \\ \dot{y} &= y_0 B y + o(y) \end{aligned}$$

where the ray is identified to $\mathbb{R}y_0$.

Denoting $\sigma(B) = \{\lambda_1, \ldots, \lambda_{n-1}\}$ as the spectrum of B, one can use a change of coordinates to get B in Jordan normal form. The following straightforward computation is crucial.

Using projective coordinates $u_1 = y_1/y_0, \ldots, u_{n-1} = y_{n-1}/y_0$, the ray solution $\mathbb{R}y_0$ corresponds to the singular point $u_1 = \cdots = u_{n-1} = 0$ of Q^π, giving the following proposition.

Proposition 3.5. *The eigenvalues of $A = \frac{\partial Q^\pi}{\partial y}(y_0)$ are given by $\sigma(A) = \{\lambda_1 - \lambda, \ldots, \lambda_{n-1} - \lambda\}$ and for the classification problem only the ratio of two eigenvalues have invariant meaning.*

This allows us to define the following two blowing-ups.

- *Blowing-up at O_1.* This is a singular point of X_r^e whose linear part, $A_1 q$, is $A_1 = \frac{\partial X_r^e}{\partial q}(O_1) = 0$ and whose quadratic part is the homogeneous quadratic vector field

$$H_2(q) = h_1(q)F(q) - h_2'(q)G'(q)$$

where identifying O_1 to zero we have

$$\begin{aligned} h_1(q) &= (\Gamma_1 - \Gamma_2)[\gamma_2(2\Gamma_1 - \gamma_1)z_2 - \gamma_1(2\Gamma_2 - \gamma_2)z_1] \\ h_2'(q) &= (\Gamma_1 - \Gamma_2)[\gamma_2(2\Gamma_1 - \gamma_1)(\Gamma_2 - (\gamma_1 - \Gamma_1))y_1 z_2 \\ &\qquad - \gamma_1(2\Gamma_2 - \gamma_2)(\Gamma_1 - (\gamma_2 - \Gamma_2))z_1 y_2] \end{aligned}$$

where G' denotes the constant vector field $-\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}$, approximating G near O_1. Observe also that at O_1 the quartic D is regular if $\Gamma_1 \neq \Gamma_2$ and $2\Gamma_1 - \gamma_1$ or $2\Gamma_2 - \gamma_2$ is nonzero, and is approximated by the linear mapping h_1.

- *Blowing-up at O_2.* Similarly at the equilibrium O_2 of X_r^e, we have a zero linear part $A_2 q$, $A_2 = \frac{\partial X_r^e}{\partial q}(O_2) = 0$, and whose quadratic part is the homogeneous quadratic vector field

$$\bar{H}_2(q) = \bar{h}_2(q)F'(q) - \bar{h}_1'(q)G(q)$$

where O_2 is zero and we have

$$\begin{aligned} \bar{h}_2 &= \gamma_2^2\mu_1 y_1^2 - \gamma_1\gamma_2(\mu_1 + \mu_2)y_1 y_2 + \gamma_1^2\mu_2 y_2^2 \\ &\quad + \gamma_2^2\eta_1 z_1^2 - \gamma_1\gamma_2(\eta_1 + \eta_2)z_1 z_2 + \gamma_1^2\eta_2 z_2^2, \end{aligned}$$

with $\eta_i = 2\Gamma_i - \gamma_i$ and $\mu_i = 2\gamma_i - \Gamma_i$, and $\eta_i \geq 0$ and μ_i can be positive, negative, or zero (it is zero in the case of fat tissue),

$$\bar{h}_1' = \gamma_1\gamma_2^2(\eta_2 - \eta_1)y_1 + \gamma_1^2\gamma_2(\eta_1 - \eta_2)y_2$$

and

$$F'(q) = \gamma_1\frac{\partial}{\partial z_1} + \gamma_2\frac{\partial}{\partial z_2}$$

is the constant vector field approximating $F(q)$ while

$$G = -z_1\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial y_2} + y_2\frac{\partial}{\partial z_2}$$

is the (unmodified) control vector field.

Observe that in the formulas corresponding to O_1 and O_2 the role of F and G are exchanged, but this will not lead to a similar analysis: in particular F is a stable vector field while G corresponds to a rotation. Also at O_1, D is approximated by a linear form $h_1(q)$ but at O_2 the approximation is the quadratic form $\bar{h}_2$. In particular we have the following lemma.

Lemma 3.6. *The quartic form D is singular at O_2 for every set of parameters.*

The remainder of this article is devoted to the analysis of the two blowing-ups, in relation with the classification problem.

3.3 Analysis of the quadratic vector field $H_2(q)$

First of all, we have the following lemma.

Lemma 3.7. *Assume $\Gamma_1 - \Gamma_2 \neq 0$ and $\eta_i \neq 0$, then the set of singular points $h_1 = h_2' = 0$ is the union of two planes E_1 and E_2 given by*

- $E_1 : z_1 = z_2 = 0$
- $E_2 : (\Gamma_1 - \delta_2)y_2 = (\Gamma_2 - \delta_1)y_1,\ \gamma_2\eta_1 z_2 = \gamma_1\eta_2 z_1$

with $\delta_i = \gamma_i - \Gamma_i$.

At any point of E_1, the eigenvalues of H_2^π are zero, H_2^π denoting the projection of H_2 on the sphere S^3, but the second set of equilibrium points is less degenerated. The analysis is described next.

Denoting $q = (y_1, z_1, y_2, z_2)^t$, $x = (x_1, x_2, x_3, x_4)^t$ is related by $q = Px$ where P is the transition matrix

$$P = \begin{bmatrix} \Gamma_1 - \delta_2 & 0 & 0 & 0 \\ 0 & \gamma_2\eta_1 & 0 & 0 \\ \Gamma_2 - \delta_1 & 0 & 1 & 0 \\ 0 & \gamma_1\eta_2 & 0 & 1 \end{bmatrix}$$

so that E_2 is identified to $x_3 = x_4 = 0$, one gets the system

$$\dot{x} = h_1(Px)(P^{-1}FP)x - h_2'(Px)P^{-1}G'$$

and computing

$$L_1 = \frac{\partial P^{-1}H_2 \circ P}{\partial x}_{|x_3=x_4=0}$$

which represents the linearized system at a point of E_2, we have

$$L_1 = (\Gamma_1 - \Gamma_2)\begin{bmatrix} 0 & 0 & -\gamma_1\gamma_2\eta_1\eta_2 x_2 & -\eta_1\gamma_2(\gamma_1 - \Gamma_2)x_1 \\ 0 & 0 & 0 & -\gamma_1\gamma_2\eta_1 x_2 \\ 0 & 0 & -\gamma_1\gamma_2\eta_1\eta_2(\gamma_1 - \gamma_2)x_2 & -\eta_1\gamma_2(\delta_1 - \Gamma_2)(\gamma_1 + \Gamma_1 - \gamma_2 - \Gamma_2)x_1 \\ 0 & 0 & 0 & \gamma_1\gamma_2\eta_1\eta_2(\gamma_1 - \gamma_2)x_2 \end{bmatrix}$$

and we have the following lemma.

Lemma 3.8. *If $\gamma_1 \neq \gamma_2$, at a point of E_2 the eigenvalues of L_1 are of the form $(0, 0, -\lambda, \lambda)$ with $-\lambda = x_2(\Gamma_1 - \Gamma_2)\gamma_1\gamma_2\eta_1\eta_2(\gamma_2 - \gamma_1)$ and the eigenspace associated to $-\lambda$ is $\mathbb{R}G'$, the vector field G' being tangent to $h_1 = 0$.*

Introducing the projective coordinates $u_1 = x_1/x_2, u_3 = x_3/x_2$, and $u_4 = x_4/x_2$ to represent the projection of the system on S^3 and making the time reparameterization $ds =$

$x_2 dt$ we get the system

$$\begin{aligned}
\dot{u}_1 &= (\Gamma_1 - \Gamma_2)\gamma_2\eta_1 u_4 \left[\frac{\Gamma_1(\Gamma_1 - \delta_2) - \Gamma_2(\Gamma_2 - \delta_1)}{\gamma_2 - \gamma_1} u_1 + \frac{(\Gamma_1 - \Gamma_2)}{\gamma_2 - \gamma_1} u_3\right] \\
&\quad + (\Gamma_1 - \Gamma_2)\gamma_1\gamma_2\eta_1 u_1 u_4 \\
\dot{u}_3 &= (\Gamma_1 - \Gamma_2)\gamma_2\eta_1 \left[\gamma_1\eta_2(\gamma_2 - \gamma_1)u_3 + u_3 u_4 \left[\frac{\Gamma_2(\Gamma_1 - \delta_2) - \Gamma_1(\Gamma_2 - \delta_1)}{\gamma_2 - \gamma_1} + (\Gamma_2 - \delta_1) + \gamma_1\right]\right. \\
&\quad \left. + u_1 u_4 \left[(\Gamma_1 - \delta_2)(\Gamma_2 - \delta_1)\left(\frac{\Gamma_2 - \Gamma_1 + \gamma_2 - \gamma_1}{\gamma_2 - \gamma_1}\right)\right]\right] \\
\dot{u}_4 &= (\Gamma_1 - \Gamma_2)\gamma_2\eta_1 u_4 \left[\gamma_1\eta_2(\gamma_1 - \gamma_2) + (\gamma_1 - \gamma_2)u_4\right].
\end{aligned}$$

The analysis of the system in those coordinates is intricate: the singularities are non-isolated and the results in [16] cannot be used to find an invariant two-dimensional foliation. We can observe that u_4 can be integrated from the equations above.

Nevertheless the blood case, with $\gamma_1 = \gamma_2$, deserves a specific analysis, especially from the integrability point of view.

Special case: $\gamma_1 = \gamma_2$

In this case, we have $\lambda = 0$. The analysis is as follows. Denoting the transition matrix by Q,

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \eta_1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & \eta_2 & 0 & 1 \end{bmatrix}$$

and in the x-coordinates $q = Qx$, introducing the projective coordinates $u_1 = x_1/x_2, u_3 = x_3/x_2$, and $u_4 = x_4/x_2$ and making the time reparameterization, we get

$$\begin{aligned}
\dot{u}_1 &= -\gamma_1(\Gamma_1 - \Gamma_2)(\Gamma_1 - \delta_2)\eta_1\eta_2 u_3 + (\Gamma_1 - \Gamma_2)\gamma_1\Gamma_2\eta_1 u_1 u_4 \\
\dot{u}_3 &= (\Gamma_1 - \Gamma_2)\gamma_1\eta_1[(\Gamma_1 - \Gamma_2)u_1 u_4 + (\gamma_1 - \Gamma_2)u_3 u_4] \\
\dot{u}_4 &= 0.
\end{aligned}$$

From the last equation u_4 is a constant and the first two equations form a linear system which can be easily solved, giving the following lemma.

Lemma 3.9. *In the case $\gamma_1 = \gamma_2$ the dynamics of the quadratic approximation of the system reduces to a linear system as follows. Introducing $u_4 = c$ and $\alpha = \gamma_1(\Gamma_1 - \Gamma_2)\eta_1$, we have*

$$\begin{aligned}
\dot{u}_1 &= \alpha[\Gamma_2 c u_1 - (\Gamma_1 - \delta_2)\eta_2 u_3] \\
\dot{u}_3 &= \alpha[(\Gamma_1 - \Gamma_2)c u_1 + (\gamma_1 - \Gamma_2)c u_3].
\end{aligned}$$

Physical application: Blood case

In this case, we have $\gamma_1 = \gamma_2$, and the numerical values of the linear system are

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_3 \end{pmatrix} = \begin{pmatrix} 0.00200c & -0.00279 \\ 0.00601c & -0.00171c \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \end{pmatrix} \tag{3.1}$$

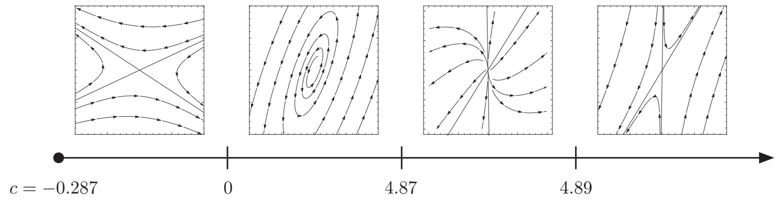

Figure 1: Classification of the flow of H_2 (north pole) in the blood case, given in Lemma 3.10

and the eigenvalues are given by $\lambda_\pm = \alpha \pm \sqrt{\beta}$ where $\alpha = 0.000148c$ and $\beta = (-16.7c + 3.44c^2) \cdot 10^{-6}$. Since z_1, z_2 belong to the translated Bloch ball it imposes constraints on the constant $c = \frac{z_2}{z_1}\eta_1 - \eta_2$, namely $c \in [-0.287, +\infty)$. We obtain a foliation of the phase portraits by linear planes for the linear system above with respect to c, and the behavior is completely described in the following lemma, and is illustrated in Figure 1.

Lemma 3.10. *We conclude that:*

(1) *For* $c \in [-0.287, 0)$*,* λ_+ *and* λ_- *have opposite sign. It is a saddle.*

(2) *For* $c \in (0, 4.87)$*, the eigenvalues are complex with a positive real part. It is an unstable focus.*

(3) *For* $c \in [4.87, 4.89)$*, both eingenvalues are positive* $\lambda_+ > \lambda_- > 0$*. It is an unstable node.*

(4) *For* $c > 4.89$*,* λ_+ *and* λ_- *have opposite sign. It is a saddle.*

More invariants are found along the z_i-axis which are true ray solutions contained in $h_1 \neq 0$. The analysis goes as follows.

Eigenvalues corresponding to the true ray solution z_2-axis: $y_1 = y_2 = z_1 = 0$

Let us introduce the coordinates $x = (x_1, x_2, x_3, x_4) = (z_2, y_1, z_1, y_2)$ and the system becomes

$$\begin{aligned} \dot{x}_1 &= \lambda x_1^2 + \cdots \\ \dot{y} &= x_1 A y + \cdots \end{aligned}$$

with $y = (x_2, x_3, x_4)$, $\lambda = -\gamma_2^2 \eta_1 (\Gamma_1 - \Gamma_2)$ and

$$A = (\Gamma_1 - \Gamma_2) \begin{bmatrix} \gamma_2\eta_1(\Gamma_2 - \gamma_1) & 0 & 0 \\ 0 & -\gamma_1\gamma_2\eta_1 & 0 \\ \gamma_2\eta_1(\Gamma_2 - \delta_1) & 0 & -\gamma_2\Gamma_2\eta_1 \end{bmatrix}$$

and the eigenvalues of A are

$$\lambda_1 = \gamma_2\eta_1(\Gamma_2 - \gamma_1)(\Gamma_1 - \Gamma_2), \quad \lambda_2 = -\gamma_1\gamma_2\eta_1(\Gamma_1 - \Gamma_2), \quad \lambda_3 = -\gamma_2\Gamma_2\eta_1(\Gamma_1 - \Gamma_2).$$

Denoting $\sigma_i = \lambda_i - \lambda$, $i = 1, 2, 3$, and $I_i = \sigma_{i+1}/\sigma_1$, $i = 1, 2$, we have

$$\sigma_1 = \gamma_2\eta_1(\gamma_2 + \Gamma_2 - \gamma_1)(\Gamma_1 - \Gamma_2), \quad \sigma_2 = \gamma_2\eta_1(\gamma_2 - \gamma_1)(\Gamma_1 - \Gamma_2), \quad \sigma_3 = \gamma_2\eta_1\delta_2(\Gamma_1 - \Gamma_2),$$

and

$$I_1 = \frac{\gamma_2 - \gamma_1}{\gamma_2 + \Gamma_2 - \gamma_1}, \quad I_2 = \frac{\delta_2}{\gamma_2 + \Gamma_2 - \gamma_1}.$$

One can check that the numerical values are different for the four sets of experimental parameters, which gives the following classification theorem.

Theorem 3.11. (1) *The eigenvalues of the linearized system correspond to the z_2-axis true ray solution projected on the sphere S^3 are σ_1, σ_2, and σ_3.*

(2) *I_1 and I_2 are two independent rational invariants.*

(3) *Computing along the z_1 axis amounts to exchanging the indices in the computation, and allows us to define four rational invariants and a choice of three of them separates the generic orbits in the feedback classification problem.*

Concerning the analysis of the integrability of the flow, we have the following partial result.

Introducing the projective coordinates $u_1 = x_2/x_1$, $u_2 = x_3/x_1$, and $u_3 = x_4/x_1$ and making a time reparameterization, one gets the system

$$\begin{aligned}
\dot{u}_1 &= (\Gamma_1 - \Gamma_2)[\gamma_2\eta_1(\Gamma_2 - \gamma_1 + \gamma_2)u_1 + \gamma_1\eta_2(\Gamma_1 - \gamma_2)u_1u_2 - \gamma_1\eta_2(\Gamma_1 - \delta_2)u_2u_3] \\
\dot{u}_2 &= (\Gamma_1 - \Gamma_2)(\gamma_2 - \gamma_1)[\gamma_2\eta_1u_2 - \gamma_1\eta_2u_2^2] \\
\dot{u}_3 &= (\Gamma_1 - \Gamma_2)[\gamma_2\eta_1\delta_2u_3 + \gamma_2\eta_1(\Gamma_2 - \delta_1)u_1 - \gamma_1\eta_2\Gamma_1u_2u_3]
\end{aligned}$$

where the second equation is integrable. Hence the analysis in the general case amounts to integrating a time-dependent, two-dimensional system.

Remark 3.1. In this case, the hyperbolic singularity is isolated and the Hartman-Grobman theorem applies [8]. The singularity is unstable. More results about integrability can be obtained using the work of Poincaré-Siegel [1].

3.4 Analysis of the quadratic vector field $\bar{H}_2(q)$

3.4.1 Reduction of $\bar{h}_2$

The only invariants are the index of $\bar{h}_2$ and next we compute the Gauss normal form. Let us write $\bar{h}_2 = q_1(y) + q_2(z)$, where

$$\begin{aligned}
q_1(y) &= \gamma_2\mu_1y_1^2 - \gamma_1\gamma_2(\mu_1 + \mu_2)y_1y_2 + \gamma_1^2\mu_2y_2^2 \\
q_2(z) &= \gamma_2\eta_1z_1^2 - \gamma_1\gamma_2(\eta_1 + \eta_2)z_1z_2 + \gamma_1^2\eta_2z_2^2.
\end{aligned}$$

Introducing the linear forms

$$\begin{aligned}
x_1 &= \gamma_2y_1 - \gamma_1y_2 \\
x_2 &= \mu_1\gamma_2y_1 - \mu_2\gamma_1y_2 \\
x_3 &= \gamma_2z_1 - \gamma_1z_2 \\
x_4 &= \eta_1\gamma_2z_1 - \eta_2\gamma_1z_2,
\end{aligned} \tag{3.2}$$

Proposition 3.12. *In the new coordinates, we have*

(1) $\bar{h}_1' = \gamma_1\gamma_2(\eta_2 - \eta_1)x_1$

(2) $\bar{h}_2 = x_1x_2 + x_3x_4$

$\boxed{3.4.2}$ Analysis of the projection of the system on the sphere S^3

We express $\bar{H}_2(q)$ in the coordinates $x = (x_1, x_2, x_3, x_4)$, $q' = Px$ where $q' = (y_1, y_2, z_1, z_2)$ and the transition matrix is defined by (3.2).

This implies that

$$F'(x) = P^{-1}F' = \gamma_1\gamma_2(\eta_1 - \eta_2)\frac{\partial}{\partial x_4}$$

and moreover $G(x) = P^{-1}APx$ where

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & \frac{\mu_1\eta_2-\mu_2\eta_1}{\eta_1-\eta_2} & \frac{\mu_2-\mu_1}{\eta_1-\eta_2} \\ 1 & 0 & 0 & 0 \\ \frac{\mu_1\eta_2-\mu_2\eta_1}{\mu_1-\mu_2} & \frac{\eta_1-\eta_2}{\mu_1-\mu_2} & 0 & 0 \end{bmatrix}.$$

Lemma 3.13. *We have the following,*

(1) *There are no ray solutions for $\bar{H}_2(q)$.*

(2) *The singular lines are contained in the union of planes $\{x : x_1 = 0$ and $(x_3 = 0$ or $x_4 = 0)\}$.*

Proof. The singular points belong to $\bar{h}'_2 = h'_1 = 0$. Solving in Gauss coordinates, one gets if $\eta_1 - \eta_2 \neq 0$ that $x_1 = 0$ and either $x_2 = 0$ or $x_3 = 0$. We therefore have the following two planes $\bar{E}_1 : x_1 = x_2 = 0$, $\bar{E}_2 : x_1 = x_3 = 0$.

Let $\mathbb{R}\zeta$ be a ray, then there exists $\lambda \neq 0$ such that $\bar{H}_2(\zeta) = \lambda\zeta$. This is written as

$$\lambda\zeta_1 = \bar{h}'_1\zeta_2 \tag{3.3}$$
$$\lambda\zeta_2 = \gamma_1\bar{h}_2 - \bar{h}'_1\zeta_1 \tag{3.4}$$
$$\lambda\zeta_3 = \bar{h}'_1\zeta_4 \tag{3.5}$$
$$\lambda\zeta_4 = \gamma_2\bar{h}_2 - \bar{h}'_1\zeta_3 \tag{3.6}$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$.

Assume ζ_1 is nonzero, then $\bar{h}'_1 \neq 0$, $\zeta_2 \neq 0$ and from (3.3) we deduce that

$$\lambda = \frac{\bar{h}'_1\zeta_2}{\zeta_1}.$$

Plugging back into (3.5) one gets with $\bar{h}'_1 \neq 0$ the relation

$$\zeta_1\zeta_4 = \zeta_2\zeta_3. \tag{3.7}$$

Substituting λ into (3.4) and (3.6), we obtain

$$\frac{\bar{h}'_1\zeta_2}{\zeta_1}\zeta_2 = \gamma_1\bar{h}_2 - \bar{h}'_1\zeta_1$$
$$\frac{\bar{h}'_1\zeta_2}{\zeta_1}\zeta_4 = \gamma_2\bar{h}_2 - \bar{h}'_1\zeta_3$$

and this leads to the equations

$$\bar{h}'_1\zeta_2^2 = \gamma_1\bar{h}_2\zeta_1 - \bar{h}'_1\zeta_1^2 \tag{3.8}$$
$$\bar{h}'_1\zeta_2\zeta_4 = \gamma_2\bar{h}_2\zeta_1 - \bar{h}'_1\zeta_3\zeta_1. \tag{3.9}$$

From (3.8) and since $\zeta_2 \neq 0$, we can write $\bar{h}_1'\zeta_2 = \frac{\gamma_1\bar{h}_2\zeta_1 - \bar{h}_1'\zeta_1^2}{\zeta_2}$ and substitute in (3.9) to obtain

$$\gamma_1\bar{h}_2\zeta_1\zeta_4 - \bar{h}_1'\zeta_1^2\zeta_4 = \gamma_2\bar{h}_2\zeta_1\zeta_2 - \bar{h}_1'\zeta_3\zeta_1\zeta_2$$

which using relation (3.7) is equivalent to $\gamma_1\zeta_4 = \gamma_2\zeta_2$ since $\bar{h}_2 = 0$ would imply $\zeta_2^2 = -\zeta_1^2$. Both relations $\zeta_1\zeta_4 = \zeta_2\zeta_3$ and $\gamma_1\zeta_4 = \gamma_2\zeta_2$ imply that $\bar{h}_1' = 0$ which is a contradiction.

Assume now that $\zeta_1 = 0$. In that case we have from (3.3) that either $\bar{h}_1' = 0$ or $\zeta_2 = 0$. If $\bar{h}_1' = 0$, it implies from (3.4) that $\lambda = \frac{\gamma_1\bar{h}_2}{\zeta_2}$. By plugging into (3.6) we obtain $\gamma_1\zeta_4\bar{h}_2 = \gamma_2\zeta_2\bar{h}_2$ which forces $\bar{h}_2$ to be zero (indeed, $\gamma_1\zeta_4 = \gamma_2\zeta_2$ and $\bar{h}_1' = 0$ imply $\bar{h}_2 = 0$) which provides a contradiction. If $\zeta_2 = 0$, it again forces $\bar{h}_2 = 0$ and is a contradiction. □

Theorem 3.14. *The eigenvalues of the linearized system are given by:*

(1) *At $\bar{E}_1 = \{x_1 = x_2 = 0\}$ the eigenvalues are given by $\{0, 0, \lambda, -\lambda)$ where $\lambda = \gamma_1\gamma_2(\eta_1 - \eta_2)x_3$.*

(2) *At $\bar{E}_2 = \{x_1 = x_3 = 0\}$ the eigenvalues are zero.*

Proof. We have

$$\bar{H}_2(x) = \gamma_1\gamma_2(\eta_1-\eta_2)\begin{bmatrix} -x_1x_3 \\ \frac{(\mu_1\eta_2-\mu_2\eta_1)x_1x_3+(\mu_2-\mu_1)x_1x_4}{\eta_1-\eta_2} \\ x_1^2 \\ x_1x_2 + x_3x_4 + \frac{(\mu_1\eta_2-\mu_2\eta_1)x_1^2+(\eta_1-\eta_2)x_1x_2}{\mu_1-\mu_2} \end{bmatrix}$$

and

$$\frac{\partial \bar{H}_2}{\partial x} = \begin{bmatrix} -x_3 & 0 & -x_1 & 0 \\ \frac{(\mu_1\eta_2-\mu_2\eta_1)x_3+(\mu_2-\mu_1)x_4}{\eta_1-\eta_2} & 0 & \frac{\mu_1\eta_2-\mu_2\eta_1}{\eta_1-\eta_2}x_1 & \frac{\mu_2-\mu_1}{\eta_1-\eta_2}x_1 \\ 2x_1 & 0 & 0 & 0 \\ x_2 + \frac{2(\mu_1\eta_2-\mu_2\eta_1)x_1+(\eta_1-\eta_2)x_2}{\mu_1-\mu_2} & x_1 + \frac{\eta_1-\eta_2}{\mu_1-\eta_2}x_1 & x_4 & x_3 \end{bmatrix},$$

modulo the constant $\gamma_1\gamma_2(\eta_1 - \eta_2)$. Then

$$\frac{\partial \bar{H}_2}{\partial x}|_{x_1=x_2=0} = \gamma_1\gamma_2(\eta_1-\eta_2)\begin{bmatrix} -x_3 & 0 & 0 & 0 \\ \frac{(\mu_1\eta_2-\mu_2\eta_1)x_3+(\mu_2-\mu_1)x_4}{\eta_1-\eta_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_4 & x_3 \end{bmatrix}$$

which has eigenvalues $\{0, 0, \gamma_1\gamma_2(\eta_1-\eta_2)x_3, -\gamma_1\gamma_2(\eta_1-\eta_2)x_3\}$.

Also

$$\frac{\partial \bar{H}_2}{\partial x}|_{x_1=x_3=0} = \gamma_1\gamma_2(\eta_1-\eta_2)\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{\mu_2-\mu_1}{\eta_1-\eta_2}x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x_2 + \frac{\eta_1-\eta_2}{\mu_1-\mu_2}x_2 & 0 & x_4 & 0 \end{bmatrix}$$

which has eigenvalues $\{0, 0, 0, 0\}$. □

Remark 3.2. Again [16] cannot be used to find an invariant integral, but the case $\Gamma_1 - \Gamma_2 = \gamma_2 - \gamma_1$ is integrable by quadratures.

4 Conclusion

As a conclusion we present in Fig. 2 partial results about the phase portrait of the vector field X_r^e. Those results are valid only for the quadratic approximation near the north pole and the origin. A first step in the analysis of the equilibrium points of X_r^e is presented in [3] where the authors analyze the difficult problem of parametrizing this set using Gröbner basis. Also additional feedback invariants related to optimality properties of the singular arcs are described in [3]. Connected work is ongoing on the analysis of the singular flow in the bi-input case where both the amplitude and the phase of the radio-frequency field are controlled.

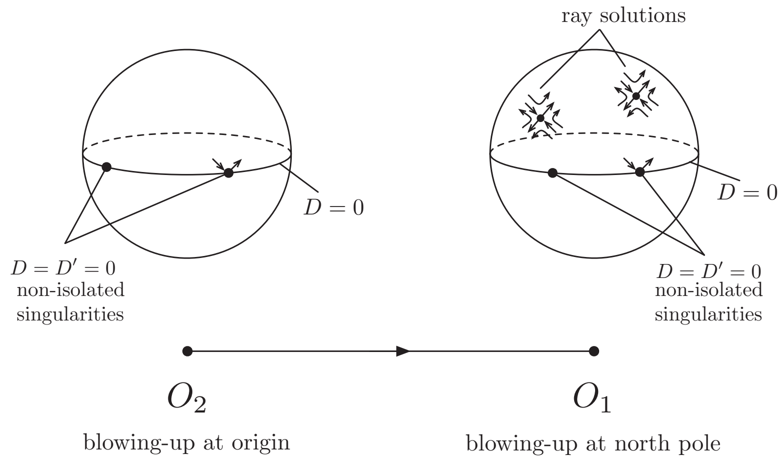

Figure 2: Partial phase portrait of the vector field X_r^e.

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Manuscript received 30 January 2013
revised 12 June 2013
accepted for publication 18 September 2013

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