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SECOND-ORDER OPTIMALITY CONDITIONS OF GENERALIZED VECTOR EQUILIBRIUM PROBLEMS AND APPLICATIONS*

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Abstract: This paper deals with the second-order optimality conditions of generalized vector equilibrium problems with constraints. By virtue of second-order weak composed contingent epiderivatives, the necessary conditions and the sufficient conditions are established for the weakly efficient solution of generalized vector equilibrium problems with constraints. As interesting applications of the results, we obtain the optimality conditions for vector optimization problems, which can conclude the corresponding results in the literature.

Key words: *generalized vector equilibrium problems, vector optimization problems, second-order weak composed contingent epiderivatives, weakly efficient solutions, optimality conditions*

Mathematics Subject Classification: *90C46, 91B50*

1 Introduction

Vector equilibrium problems, which contain vector optimization problems, vector variational inequality problems and vector complementarity problems as special cases, have been studied(see [1, 2, 4, 7, 9, 10, 12, 13, 15, 16, 25–28]). But so far, most papers focused mainly on the existence of solutions and the properties of the solutions, there are a few papers which deal with the optimality conditions. Giannessi et al. [11] turned the vector variational inequalities with constraints into another vector variational inequalities without constraints. They gave the sufficient conditions for the efficient solution and the weakly efficient solution of the vector variational inequalities in finite dimensional spaces. Morgan and Romaniello [31] gave the scalarization and Kuhn-Tucker-like conditions for weak vector generalized quasivariational inequalities in Hilbert space by using the concept of subdifferential of the function. Gong [14] presented the necessary and sufficient conditions for the weakly efficient solution, the Henig efficient solution and the superefficient solution for vector equilibrium problems with constraints under the condition of cone-convexity. Qiu [32] presented the necessary and sufficient conditions for globally efficient solutions under generalized cone-subconvexlikeness. Under the nearly cone-subconvexlikeness, Long et al. [29] obtained the necessary and sufficient conditions for the Henig efficient solution and the superefficient solution to the vector

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equilibrium problems with constraints. By using the concept of Fré*chet* differentiability of mapping, Wei and Gong [35] obtained the Kuhn-Tucker optimality conditions for weakly efficient solutions, Henig efficient solutions, superefficient solutions and globally efficient solutions to the vector equilibrium problems with constraints. Ma and Gong [30] obtained the first-order necessary and sufficient conditions for the weakly efficient solution, the Henig efficient solution and the globally roper efficient solution to the vector equilibrium problems with constraints. Recently, by virtue of the higher-order derivatives or epiderivatives, the higher-order optimality conditions and duality have been established for set-valued optimization problems(see [3,5,18,19,21–24,33]). Moreover, as far as we know, the second-order optimality conditions of the solutions remain unstudied in generalized vector equilibrium problems.

Motivated by the work reported in [5,6,21–24,30,33], we first recall second-order weak composed contingent epiderivatives of set-valued maps and discuss some properties of the derivative. Then, by virtue of the second-order weak composed contingent epiderivatives, we establish necessary optimality conditions and sufficient optimality conditions for the weakly efficient solution of generalized vector equilibrium problems with constraints.

The rest of the paper is organized as follows. In Section 2, we recall some notions. In Section 3, we recall second-order weak composed contingent epiderivatives of set-valued maps and discuss some properties of the derivative. In Section 4, we establish second-order necessary and second-order sufficient optimality conditions for the weakly efficient solution of generalized vector equilibrium problems with constraints. As interesting applications of the results of Section 4, the optimality conditions for vector optimization problems with constraints are obtained in Section 5.

2 Preliminaries and Notations

Throughout this paper, let X, Y and Z be three real normed spaces, Y^* and Z^* be the topological dual spaces of Y and Z, respectively. $0_X, 0_Y$ and 0_Z denote the origins of X, Y and Z, respectively. Let $C \subset Y$ and $D \subset Z$ be closed convex pointed cones in Y and Z, respectively. Let C^* be the dual cone of cone C, defined by $C^* := \{y^* \in Y^* : y^*(c) \geq 0, \text{for all } c \in C\}$. Let M be a nonempty subset in Y. The linear hull of M is defined by $line(M) := \{ty | t \in R, y \in M\}$. The cone hull of M is defined by $cone(M) := \{ty | t \geq 0, y \in M\}$. We denote by

$$WMin_C M := \{y \in M : (M - y) \bigcap (-intC) = \emptyset\}$$

the set of all weakly C-minimal points of M.

Let E be a nonempty subset of X, $G : E \to 2^Z$ be a set-valued map. The domain, the graph and the epigraph of G are defined respectively by

$$\begin{aligned} &dom(G) := \{x \in E | G(x) \neq \emptyset\}, \\ &graph(G) := \{(x, z) \in X \times Z | x \in E, z \in G(x)\}, \\ &epi(G) := \{(x, z) \in X \times Z | x \in E, z \in G(x) + C\}. \end{aligned}$$

Denote

$$G(E) = \bigcup_{x \in E} G(x) \ and \ (G - z_0)(x) = G(x) - \{z_0\}.$$

Definition 2.1 (see [6])**.** A set-valued map $W : X \to 2^Y$ is said to be

(i) strictly positive homogeneous if

$$W(\alpha x) = \alpha W(x), \forall \alpha > 0, \forall x \in X;$$

(ii) subadditive if

$$W(x_1) + W(x_2) \subseteq W(x_1 + x_2) + C.$$

Definition 2.2 (see [8]). Let $E \subset X$ be convex and $G : E \to 2^Z$ be a set-valued map with $G(x) \neq \emptyset$, for all $x \in E$. G is said to be D-convex on E, if for any $x_1, x_2 \in E$ and $\lambda \in (0, 1)$,

$$\lambda G(x_1) + (1 - \lambda)G(x_2) \subseteq G(\lambda x_1 + (1 - \lambda)x_2) + D.$$

Let X be a normed space supplied with a distance d and K be a subset of X. We denote by $d(x, K) = inf_{y \in K} d(x, y)$ the distance from x to K, where we set $d(x, \emptyset) = +\infty$.

Definition 2.3. (see [3]) Let K be a nonempty subset of a X and $x \in K, u \in X$.

(i) The contingent cone of K at x is

$$T(K, x) := \{v \in X | \exists t_n \downarrow 0, \exists v_n \to v, \text{such that } x + t_n v_n \in K, \forall n \in N\}.$$

(ii) The second contingent set of K at x in the direction u is

$$T^2(K, x, u) := \{v \in X | \exists t_n \downarrow 0, \exists v_n \to v, \text{such that } x + t_n u + \frac{1}{2} t_n^2 v_n \in K, \forall n \in N\}.$$

Proposition 2.4 (see [3]). *Let $K \subseteq X$ and $x \in K$. Then $T(K, x)$ is a closed cone.*

Proposition 2.5 (see [20]). *Let $K \subseteq X$ be a convex set, $x \in K$ and $u \in T(K, x)$. Then*

$$T(T(K, x), u) = clcone(cone(K - x) - u)$$

and

$$T^2(K, x, u) \subseteq T(T(K, x), u).$$

Definition 2.6 (See [17]). Let Y be a topological linear space and be partially ordered by a convex cone $C \subset Y$ with apex at the origin.

(i) A sequence $\{y_n\} \subseteq Y$ is said to be C-decreasing iff $\forall i, j \in N, i \leq j$ implies $y_j \leq_C y_i$.

(ii) A subset $D \subset Y$ is said to be C-lower bounded iff there exists a $y \in Y$ such that $D \subset \{y\} + C$.

(iii) The cone C is called Daniell iff every C-decreasing and C-lower bounded sequence in Y converges to its infimum.

(iv) The weak domination property is said to hold for a subset M of Y iff $M \subset WMin_C M + intC \bigcup \{0_Y\}$.

Let E be a nonempty subset of X, $F : E \times E \to 2^Y$ be a set-valued bifunction, $F(x_1, x_2) \neq \emptyset$, for all $x_1, x_2 \in E$. We suppose that $0_Y \in F(x, x)$, for all $x \in E$.

Let $x_0 \in E$ be given. $F_{x_0} : E \to 2^Y$ is the set-valued map defined by

$$graph(F_{x_0}) := \{(x, y) \in E \times Y : y \in F(x_0, x)\}.$$

The set

$$epi(F_{x_0}) := \{(x, y) \in E \times Y : y \in F(x_0, x) + C\}.$$

is called the epigraph of F_{x_0}. Denote

$$F_{x_0}(E) := F(x_0, E) = \{y \in F(x_0, x) : x \in E\}.$$

In this paper, we consider the generalized vector equilibrium problem with constraints (GVEP): find $x_0 \in K$ such that

$$F(x_0, x) \cap (-A_0) = \emptyset, \text{for all } x \in K,$$

where $A_0 := A \setminus \{0_Y\}$, A is a convex cone in Y and $K := \{x \in E : G(x) \cap (-D) \neq \emptyset\}$.

Definition 2.7. Let $intC \neq \emptyset$. A vector $x_0 \in K$ is called a weakly efficient solution of (GVEP) if

$$F(x_0, K) \cap (-intC) = \emptyset.$$

3 Second-Order Weak Composed Contingent Epiderivatives

In this section, we recall second-order weak composed contingent epiderivatives of set-valued maps, and then investigate some of their properties.

We first recall two definitions in [24, 34]. Let H be a set-valued map from E to Y, $(x_0, y_0) \in graph(H)$, $(u, v) \in X \times Y$. Let $\mathcal{N}$ be the set of natural numbers

Definition 3.1 (see [24])**.** The generalized second-order composed contingent epiderivative $D''_g H(x_0, y_0, u, v)$ of H at (x_0, y_0) in the directive (u, v) is the set-valued map from X to Y defined by

$$D''_g H(x_0, y_0, u, v)(x) := Min_C\{y \in Y | (x, y) \in T(T(epi(H), (x_0, y_0)), (u, v))\}.$$

Definition 3.2 (see [34])**.** Let $(x_0, y_0) \in graph(H)$, $(u, v) \in X \times Y$. The second-order weak composed contingent epiderivative $D''_w H(x_0, y_0, u, v)$ of H at (x_0, y_0) in the directive (u, v) is the set-valued map from X to Y defined by

$$D''_w H(x_0, y_0, u, v)(x) := WMin_C\{y \in Y | (x, y) \in T(T(epi(H), (x_0, y_0)), (u, v))\}.$$

Now we discuss some crucial propositions of the second-order composed contingent epiderivatives.

Proposition 3.3. *Let $(x_0, y_0) \in graph(H)$, $(u, v) \in T(epi(H), (x_0, y_0))$ with $v \in C$ and E be convex. If H is C-convex on E and the set $P(x - x_0 - u) := \{y \in Y : (x - x_0 - u, y) \in T(T(epi(F), (x_0, y_0)), (u, v))$ fulfills the weak domination property for all $x \in E$, then,*

$$H(x) - \{y_0\} + C \subset D''_w H(x_0, y_0, u, v)(x - x_0 - u) + C, \forall x \in E.$$

Proof. It follows from Proposition 2.5 that

$$T(T(epi(H), (x_0, y_0)), (u, v)) = clcone(cone(epi(H) - \{(x_0, y_0)\}) - \{(u, v)\}). \tag{3.1}$$

Since for every $c \in C$, $x \in E$ and $y \in H(x)$, one has

$$(x - x_0 - u, y - y_0 + c) = (x - x_0 - u, y + c + v - y_0 - v) \in \{x\} \times (H(x) + C) - \{(x_0, y_0)\} - \{(u, v)\},$$

it follows from (3.1) that

$$(x - x_0 - u, y - y_0 + c) \in T(T(epi(H), (x_0, y_0)), (u, v)).$$

Thus, by the weak domination property of $P(x-x_0-u)$ and the definition of the second-order weak composed contingent epiderivatives, we have

$$y - y_0 + c \in D''_w H(x_0, y_0, u, v)(x - x_0 - u) + C,$$

and then,

$$H(x) - \{y_0\} + C \subset D''_w H(x_0, y_0, u, v)(x - x_0 - u) + C, \forall x \in E.$$

The proof is complete. □

Remark 3.1. If we use generalized second-order composed contingent epiderivatives instead of second-order weak composed contingent epiderivatives in Proposition 3.3, then corresponding result may not hold. The following example shows the case.

Example 3.4. Let $C = R^2_+$, $H : R \to 2^{R^2}$ be a set-valued map with

$$H(x) = \{y \in R^2 | y_1 \geq x^2, y_2 \geq x^2\}, x \in R.$$

Take $(x_0, y_0) = (0, (1, 0)) \in graph(H)$ and $(u, v) = (0, (1, 0))$. By directly calculating, we have

$$T(T(epi(H), (x_0, y_0)), (u, v)) = \{(x, y) \in R \times R^2 | x \in R, y_1 \in R, y_2 \geq 0\}.$$

We have checked that the assumptions of Proposition 3.3 are satisfied and

$$H(x) - \{y_0\} + C \subset D''_w H(x_0, y_0, u, v)(x - x_0 - u) + C, \forall x \in E.$$

However, for all $x \in E$, $D''_g H(x_0, y_0, u, v)(x - x_0 - u) = \emptyset$. Thus,

$$H(x) - \{y_0\} + C \not\subset D''_g H(x_0, y_0, u, v)(x - x_0 - u) + C, \forall x \in E.$$

From the proof of Proposition 3.3, we can obtain the following result.

Proposition 3.5. *(see [34]) Let $(x_0, y_0) \in graph(H)$, $(u, v) \in T(epi(H), (x_0, y_0))$ with $v \in -C$ and E be convex. If H is C-convex on E and the set $P(x - x_0 - u) := \{y \in Y : (x - x_0 - u, y) \in T(T(epi(F), (x_0, y_0)), (u, v))$ fulfills the weak domination property for all $x \in E$, then,*

$$H(x) - \{y_0\} - \{v\} \subset D''_w H(x_0, y_0, u, v)(x - x_0 - u) + C, \forall x \in E.$$

Proposition 3.6 (see [34]). *Let $(x_0, y_0) \in graph(H)$, $(u, v) \in T(epi(H), (x_0, y_0))$, $M = dom[D''_w H(x_0, y_0, u, v)]$. Then*

(i) *$D''_w H(x_0, y_0, u, v)$ is strictly positive homogeneous on M.*

Moreover, if H is C-convex on convex set E and $P(x) := \{y \in Y : (x, y) \in T(T(epi(H), (x_0, y_0)), (u, v))$ fulfills the weak domination property for all $x \in M$, then

(ii) *$D''_w H(x_0, y_0, u, v)$ is subadditive on M.*

Proposition 3.7 (see [34]). *Let $(x_0, y_0) \in graph(H)$, $(u, v) \in T(epi(H), (x_0, y_0))$, $M = dom[D''_w H(x_0, y_0, u, v)]$. If H is C-convex on E, then $D''_w H(x_0, y_0, u, v)(M) + C$ is a convex cone.*

$\boxed{4}$ Second-Order Optimality Conditions

Throughout this section, let $x_0 \in K$, $y_0 = 0_Y \in F_{x_0}(x_0)$, $intC \neq \emptyset$ and $intD \neq \emptyset$. Firstly, we recall a result in [19]. Let $K \subset X$ and $x_0 \in K$. The interior tangent cone of K at x_0 defined as

$$IT(K, x_0) := \{u \in X | \exists \delta > 0 \text{ such that } x_0 + tu' \in K, \forall t \in (0, \delta], \forall u' \in B_X(u, \delta)\},$$

where $B_X(u, \delta)$ stands for the closed ball centered at $u \in X$ and of radius δ.

Lemma 4.1 (See [19])**.** *If $K \subset X$ is convex, $x_0 \in K$ and $intK \neq \emptyset$, then*

$$IT(intK, x_0) = intcone(K - x_0).$$

Theorem 4.2. *Let $(u, v, w) \in X \times (-C) \times (-D)$. If x_0 is a weakly efficient solution of (GVEP), then for any $z_0 \in G(x_0) \bigcap (-D)$,*

$$\begin{aligned} & D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x) \\ & \bigcap [-cone(int(C \times D) + line\{(v, w)\}) \setminus \{(0_Y, 0_Z)\}] = \emptyset, \end{aligned} \tag{4.1}$$

for all $x \in \Omega := dom[D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)]$.

Proof. Suppose that there exists an $x \in \Omega$ such that (4.1) does not hold, that is, there exists a $(y, z) \in Y \times Z$ such that

$$(y, z) \in D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x)$$

and

$$(y, z) \in -cone(int(C \times D) + line\{(v, w)\}) \setminus \{(0_Y, 0_Z)\}. \tag{4.2}$$

Then, by the definition of second-order weak composed contingent epiderivatives, there exist sequences $\lambda_n \to +\infty$ and $(u_n, v_n, w_n) \in T(epi(F_{x_0}, G), (x_0, y_0, z_0))$ such that $(u_n, v_n, w_n) \to (u, v, w)$ and

$$\lambda_n((u_n, v_n, w_n) - (u, v, w)) \to (x, y, z), \text{as } n \to +\infty. \tag{4.3}$$

It follows from (4.2) that there exist $\mu > 0, \nu \in R$, $c \in intC$ and $d \in intD$ such that

$$y = -\mu(c + \nu v), \ z = -\mu(d + \nu w). \tag{4.4}$$

Let us consider two possible cases for ν.

Case 1: If $\nu \leq 0$, then, from (4.4), $v \in -C$, and $w, \ z_0 \in -D$, we have $y \in -intC$ and $z \in -intD$. Thus, by (4.3), there exists $N_1 \in \mathcal{N}$ such that

$$\lambda_n(v_n - v) \in -intC, \ \lambda_n(w_n - w) \in -intD, \forall n > N_1.$$

Thus, it follows from $v \in -C$ and $w \in -D$ that

$$v_n \in -intC, \ w_n \in -intD, \ \forall n > N_1. \tag{4.5}$$

Since $(u_n, v_n, w_n) \in T(epi(F_{x_0}, G), (x_0, y_0, z_0))$, for every $n \in \mathcal{N}$, there exist a sequence $\{\lambda_n^k\}$ with $\lambda_n^k \to +\infty$ as $k \to +\infty$ and a sequence $(x_n^k, y_n^k, z_n^k) \in epi(F_{x_0}, G)$, such that $(x_n^k, y_n^k, z_n^k) \to (x_0, y_0, z_0)$ and

$$\lambda_n^k((x_n^k, y_n^k, z_n^k) - (x_0, y_0, z_0)) \to (u_n, v_n, w_n), \text{as } k \to +\infty. \tag{4.6}$$

It follows from (4.5) and (4.6) that there exists $N_1(n) \in \mathcal{N}$ such that

$$\lambda_n^k(y_n^k - y_0) \in -intC,\ \lambda_n^k(z_n^k - z_0) \in -intD, \forall k > N_1(n), \forall n > N_1,$$

which implies

$$y_n^k - y_0 \in -intC,\ z_n^k - z_0 \in -intD, \forall k > N_1(n), \forall n > N_1. \tag{4.7}$$

Since $(x_n^k, y_n^k, z_n^k) \in epi(F_{x_0}, G)$, there exist $\bar{y}_n^k \in F_{x_0}(x_n^k)$, $\bar{z}_n^k \in G(x_n^k)$, $c \in C$ and $d \in D$ such that $y_n^k = \bar{y}_n^k + c$ and $z_n^k = \bar{z}_n^k + d$. Then, by (4.7), we have

$$\bar{y}_n^k - y_0 \in -intC, \forall k > N_1(n), \bar{z}_n^k \in -intD \forall n > N_1.$$

So

$$F_{x_0}(x_n^k) \bigcap -intC \neq \emptyset, x_n^k \in K,\ \forall k > N_1(n), \forall n > N_1,$$

which contradicts that x_0 is a weakly efficient solution of (GVEP).

Case 2: If $\nu > 0$, then, from (4.4), we get $y = -\mu\nu(\frac{1}{\nu}c + v)$ and $z = -\mu\nu(\frac{1}{\nu}d + w)$. So it follows from $c \in intC$ and $d \in intD$ that

$$y \in -intcone(C + \{v\}),\ z = -intcone(D + \{w\}). \tag{4.8}$$

Then, by Lemma 4.1 and (4.8), we get that $y \in IT(-intC, v)$ and $z = IT(-intD, w)$. Therefore, there exists $\delta > 0$ such that

$$v + \delta y' \in -intC, \forall y' \in B_Y(y, \delta). \tag{4.9}$$

$$w + \delta z' \in -intD, \forall z' \in B_Z(z, \delta). \tag{4.10}$$

For this δ, it follows from (4.3) that there exists $N_2 \in \mathcal{N}$ such that $\delta\lambda_n > 1$ and

$$\lambda_n(v_n - v) \in B_Y(y, \delta),\ \lambda_n(w_n - w) \in B_Z(z, \delta), \forall n > N_2.$$

Then, by (4.9) and (4.10), we have

$$v_n - (1 - \frac{1}{\delta\lambda_n})v \in -intC,\ w_n - (1 - \frac{1}{\delta\lambda_n})w \in -intD, \forall n > N_2.$$

Thus, from $v \in -C$, $w, z_0 \in -D$ and $\delta\lambda_n > 1, \forall n > N_2$, we have

$$v_n \in -intC,\ w_n \in -intD, \forall n > N_2.$$

By the similar proof method of case 1, there exists $N_2(n) \in \mathcal{N}$ such that

$$F_{x_0}(x_n^k) \bigcap -intC \neq \emptyset, x_n^k \in K, \forall k > N_2(n), \forall n > N_2,$$

which contradicts that x_0 is a weakly efficient solution of (GVEP). Thus (4.1) holds and the proof is complete. □

Corollary 4.3. *Let $(u, v, w) \in X \times (-C) \times (-D)$. If x_0 is a weakly efficient solution of (GVEP), then for any $z_0 \in G(x_0) \bigcap (-D)$,*

$$[D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x) + \{(v, w)\}] \bigcap -int(C \times D) = \emptyset,$$

for all $x \in \Omega := dom[D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)]$.

Now we give a Fritz John type necessary condition for the weakly efficient solution to (GVEP).

Theorem 4.4. *Let* $(u, v, w) \in X \times (-C) \times (-D)$ *with* $(u, v, w) \in T(epi(F_{x_0}, G), (x_0, y_0, z_0))$, $z_0 \in G(x_0) \bigcap (-D)$ *and* $E \subset X$ *be a nonempty convex set. Suppose that the following conditions are satisfied:*

(i) (F_{x_0}, G) *is* $C \times D$*-convex on* E.

(ii) x_0 *is a weakly efficient solution of (GVEP).*

Then there exist $\phi \in C^*$ *and* $\psi \in D^*$*, not both zero functionals, such that*

$$min_{(y,z) \in A} \phi(y) + \psi(z) = 0$$

and

$$\phi(v) = \psi(w) = \psi(z_0) = 0,$$

where $A := \bigcup_{x \in \Omega} D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w+z_0)(x)$ *and* $\Omega := dom[D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w+z_0)]$.

Proof. Define $M = A + \{(v, w + z_0)\} + C \times D$. By Proposition 3.7, we obtain that M is a convex set. By Corollary 4.3, we get

$$M \bigcap (-int(C \times D)) = \emptyset.$$

By the separation theorem of convex sets, there exist $\phi \in Y^*$ and $\psi \in Z^*$, not both zero functionals, such that

$$\phi(y) + \psi(z) \geq \phi(\bar{y}) + \psi(\bar{z}),\ for\ all\ (y, z) \in M, (\bar{y}, \bar{z}) \in -int(C \times D). \tag{4.11}$$

Since $intC \bigcup \{0_Y\}$ and $intD \bigcup \{0_Z\}$ are cones, by (4.11), we have

$$\phi(\bar{y}) \leq \psi(\bar{z}), for\ all\ (\bar{y}, \bar{z}) \in (-intC) \times intD, \tag{4.12}$$

and

$$\phi(y) + \psi(z) \geq 0, for\ all\ (y, z) \in M. \tag{4.13}$$

From (4.12), we can conclude that ψ is bounded below on $intD$. Then, $\psi(z) \geq 0$, for all $z \in intD$. Naturally $\psi \in D^*$.

By the similar line of the proof for $\psi \in D^*$, we can obtain $\phi \in C^*$.

It follows from Proposition 3.7 that $(0_Y, 0_Z) \in A$, and then, by (4.13), we have

$$\phi(v) + \psi(w + z_0) \geq 0.$$

Since $v \in -D, w, z_0 \in -D$, $\phi \in C^*$ and $\psi \in D^*$,

$$\phi(v) = \psi(w) = \psi(z_0) = 0.$$

Combine with (4.13) and $(0_Y, 0_Z) \in A$, we can conclude that

$$min_{(y,z) \in A} \phi(y) + \psi(z) = 0.$$

The proof is complete. □

Now we give an example to illustrate Theorems 4.2 and 4.4.

Example 4.5. Suppose that $X = Z = R, Y = R^2, C = R^2_+, D = R_+, E = R$. Let $F : E \times E \to 2^Y$ be a set-valued bifunction with

$$F(x_1, x_2) = \{(y_1, y_2) \in Y : y_1 \in R, y_2 \geq x_2^2 - x_1^2\}, \forall x_1, x_2 \in E,$$

and $G : E \to Z$ be a set-valued map with

$$G(x) = \{z \in R : z \geq 0\}, x \in E.$$

We consider the generalized vector equilibrium problem with constraints $(GVEP1)$: find $x_0 \in K$ such that

$$F(x_0, x) \cap (-intC) = \emptyset, \text{for all } x \in K,$$

where $K := \{x \in E : G(x) \cap (-D) \neq \emptyset\}$.

Take $(x_0, y_0) = (0, (0, 0)) \in graph(F)$ and $(x_0, z_0) = (0, 0) \in graph(G)$. Then $F_{x_0}(x) = \{(y_1, y_2) \in Y : y_1 \in R, y_2 \geq x^2\}, \forall x \in E$, x_0 is a weakly efficient solution of $(GVEP1)$, and (F_{x_0}, G) is $C \times D$-convex on E. By directly calculating, we have

$$T(epi(F_{x_0}, G), (x_0, y_0, z_0)) = \{(x, y, z) \in R \times R^2 \times R | x \in R, y_1 \in R, y_2 \geq 0, z \geq 0\}.$$

Take $(u, v, w) = (1, (-1, 0), 0) \in T(epi(F_{x_0}, G), (x_0, y_0, z_0))$. Then

$$\begin{aligned} B &:= cone(int(C \times D) + line\{v, w\}) \setminus \{(0_Y, 0_Z)\} \\ &= \{(y, z) \in R^2 \times R | y_1 \in R, y_2 > 0, z > 0\}, \end{aligned}$$

and

$$\begin{aligned} &T(T(epi(F_{x_0}, G), (x_0, y_0, z_0)), (u, v, w)) \\ &= \{(x, y, z) \in R \times R^2 \times R | x \in R, y_1 \in R, y_2 \geq 0, z \geq 0\}. \end{aligned}$$

So

$$A = (R \times R_+) \times \{0\} \bigcup (R \times \{0\}) \times R_+ \text{ and } A \bigcap (-B) = \emptyset,$$

which shows that Theorem 4.2 holds here.

Take $\phi = (0, 1) \in C^*$ and $\psi = 1 \in D^*$. Clearly,

$$inf\{\bigcup_{(y,z) \in A} \phi(y) + \psi(z)\} = 0 \text{ and } \phi(v) = \psi(w) = \psi(z_0) = 0.$$

which shows that Theorem 4.4 holds here.

Theorem 4.6. *Let $z_0 \in G(x_0) \bigcap (-D)$, $(u, v, w) \in X \times (-C) \times (-D)$ with $(u, v, w) \in T(epi(F_{x_0}, G), (x_0, y_0, z_0))$, and $K \subset X$ be a nonempty convex set. Suppose that the following conditions are satisfied:*

(i) *(F_{x_0}, G) is $C \times D$-convex on K.*

(ii) *The set $\tilde{G}(x - x_0 - u) := \{y \in Y : (x - x_0 - u, y) \in T(T(epi(F), (x_0, y_0)), (u, v))\}$ fulfills the weak domination property for all $x \in K$.*

(iii)

$$\begin{aligned} &D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x - x_0 - u) \\ &\bigcap((-intC - \{v\}) \times (-D - \{x_0\} - \{w\})) = \emptyset, \forall x \in K. \end{aligned} \quad (4.14)$$

Then (x_0, y_0) is a weak minimizer of problem $(GVEP)$.

Proof. It follows from condition (4.14) that

$$(D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x - x_0 - u) + C \times D) \bigcap$$
$$((-intC - \{v\}) \times (-D - \{x_0\} - \{w\})) = \emptyset, \forall x \in K.$$

And then, from Proposition 3.5, for all $x \in K$,

$$((F_{x_0}, G)(x) - \{(y_0, z_0)\} - \{(v, w)\}) \bigcap ((-intC - \{v\}) \times (-D - \{x_0\} - \{w\})) = \emptyset.$$

Since $x \in K$ and $y_0 = 0_Y$,

$$F_{x_0}(K) \bigcap (-intC) = \emptyset,$$

which implies that (x_0, y_0) is a weak minimizer of problem $(GVEP)$. The proof is complete. □

Theorem 4.7. *Let $z_0 \in G(x_0) \bigcap (-D)$, $(u, v, w) \in X \times C \times D$ with $(u, v, w) \in T(epi(F_{x_0}, G), (x_0, y_0, z_0))$ and $E \subset X$ be a nonempty convex set. Suppose that the following conditions are satisfied:*

(i) *(F_{x_0}, G) is $C \times D$-convex on E.*

(ii) *There exist $\phi \in C^* \setminus \{0\}$ and $\psi \in D^*$ such that*

$$inf\{ \bigcup_{(y,z)\in V} \phi(y) + \psi(z)\} = 0 \text{ and } \psi(z_0) = 0,$$

where $V := \bigcup_{x\in\Delta} D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x)$ and $\Delta := dom[D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)]$.
Then x_0 is a weakly efficient solution of (GVEP).

Proof. Assume that x_0 is not a weakly efficient solution of (GVEP). Then there exist $x' \in K$ and $y' \in F(x_0, x')$ such that $y' \in -intC$. Since $x' \in K$, there exists $z' \in G(x') \bigcap (-D)$. By Proposition 3.3, we have $(y' - y_0, z' - z_0) \in V + C \times D$, and then, from assumption (ii), we obtain

$$\phi(y' - y_0) + \psi(z' - z_0) \geq 0. \tag{4.15}$$

Since $y' - y_0 = y' \in -intC$, $\phi \in C^* \setminus \{0\}$, $\phi(y' - y_0) < 0$. By $z' \in G(x') \bigcap (-D)$, $\psi \in D^*$ and $\psi(z_0) = 0$, we get $\psi(z' - z_0) \leq 0$, thus

$$\phi(y' - y_0) + \psi(z' - z_0) < 0,$$

which contradicts (4.15). So x_0 is a weakly efficient solution of (GVEP), and this completes the proof. □

5 Applications

In this section, we use the results of Section 4 to get the optimality conditions for weakly efficient solutions to the set-valued vector optimization problems with constraints.

Consider the set-valued vector optimization problems with constraints:

$$(SVOP)\begin{cases} min & H(x), \\ s.t. & G(x)\bigcap(-D)\neq\emptyset, x\in E, \end{cases}$$

i.e., to find all $(x_0, y^{'}) \in K \times H(K)$ satisfying $y^{'} \in H(x_0)$ and $y^{'} \in WMin_C H(K)$, where $K := \{x \in E | G(x)\bigcap(-D) \neq \emptyset\}$. x_0 is said to be a weakly efficient solution of $(SVOP)$ and $(x_0, y^{'})$ is said to be a weakly efficient element of $(SVOP)$.

Theorem 5.1. *Let $(u, v, w) \in X \times (-C) \times (-D)$. If $(x_0, y^{'})$ is a weakly efficient element of $(SVOP)$. Then for any $z_0 \in G(x_0)\bigcap(-D)$,*

$$\begin{aligned} &D''_w(H, G)(x_0, y^{'}, z_0, u, v, w)(x) \\ &\bigcap -cone(int(C \times D) + line\{(v, w)\}) \setminus \{(0_Y, 0_Z)\} = \emptyset, \end{aligned} \tag{5.1}$$

for all $x \in \Omega := dom[D''_w(H, G)(x_0, y^{'}, z_0, u, v, w)]$.

Proof. Let

$$F(x, y) = H(y) - h(x), \forall x, y \in E,$$

where $h : X \to Y$ is a vector-valued map with $h(x) \in H(x)$ and $h(x_0) = y^{'}$.

Obviously, for any $x \in E$, $0_Y \in F(x, x)$. Take $y_0 = 0_Y$. By Definition 3.2, we know that

$$D''_w(F_{x_0}, G)(x_0, y_0, z_0, u, v, w)(x) = D''_w(H, G)(x_0, y^{'}, z_0, u, v, w)(x). \tag{5.2}$$

By assumptions, the conditions of Theorem 4.2 are satisfied. Combined with (5.2) and (4.1), we can conclude that (5.1) holds. This completes the proof. □

Remark 5.1. When $G(x) \equiv Z$, for any $x \in E$, Theorem 5.1 can conclude [24, Theorem 4.1] and [34, Theorem 4.1].

From Theorems 4.4 and 5.1, we can conclude that Theorem 5.2 holds.

Theorem 5.2. *Let $(u, v, w) \in X \times (-C) \times (-D)$ with $(u, v, w) \in T(epi(H, G), (x_0, y_0, z_0))$, $z_0 \in G(x_0)\bigcap(-D)$ and $E \subset X$ be a nonempty convex set. Suppose that the following conditions are satisfied:*

(i) *(H, G) is $C \times D$-convex on E.*

(ii) *$(x_0, y^{'})$ is a weakly efficient element of $(SVOP)$.*

Then there exist $\phi \in C^*$ and $\psi \in D^*$, not both zero functionals, such that

$$inf\{\bigcup_{(y,z)\in A'} \phi(y) + \psi(z)\} = 0$$

and

$$\phi(v) = \psi(w) = \psi(z_0) = 0,$$

where $A' := \bigcup_{x\in\Omega'} D''_w(H, G)(x_0, y', z_0, u, v, w)(x)$ and $\Omega' := dom[D''_w(H, G)(x_0, y', z_0, u, v, w)]$.

From Theorem 4.6, we can conclude that the following Theorem 5.3 holds.

Theorem 5.3. *Let $z_0 \in G(x_0) \bigcap (-D)$, $(u, v, w) \in X \times (-C) \times (-D)$ with $(u, v, w) \in T(epi(H, G), (x_0, y', z_0))$, and $K \subset X$ be a nonempty convex set. Suppose that the following conditions are satisfied:*

(i) *(H, G) is $C \times D$-convex on K.*

(ii) *The set $\{(y, z) \in Y \times Z : (x - x_0 - u, y, z) \in T(T(epi(H, G), (x_0, y', z_0)), (u, v, w))\}$ fulfills the weak domination property for all $x \in K$.*

(iii) *For all $x \in K$,*

$$D''_w(H, G)(x_0, y', z_0, u, v, w)(x - x_0 - u) \bigcap ((-intC - \{v\}) \times (-D - \{x_0\} - \{w\})) = \emptyset.$$

Then (x_0, y') is a weak minimizer of problem $(SVOP)$.

Remark 5.2. When $G(x) \equiv Z$, for any $x \in E$, from Theorems 5.3, we can conclude [34, Theorem 4.2].

From Theorem 4.7, we can conclude that the following Theorem holds.

Theorem 5.4. *Let $z_0 \in G(x_0) \bigcap (-D)$, $(u, v, w) \in X \times C \times D$ with $(u, v, w) \in T(epi(H, G), (x_0, y', z_0))$, and $E \subset X$ be a nonempty convex set. Suppose that the following conditions are satisfied:*

(i) *(H, G) is $C \times D$-convex on E.*

(ii) *There exist $\phi \in C^* \setminus \{0\}$ and $\psi \in D^*$ such that*

$$inf\{ \bigcup_{(y,z) \in V'} \phi(y) + \psi(z)\} = 0 \text{ and } \psi(z_0) = 0,$$

where $V' := \bigcup_{x \in \Delta'} D''_w(H, G)(x_0, y', z_0, u, v, w)(x)$ and $\Delta' := dom[D''_w(H, G)(x_0, y', z_0, u, v, w)]$.
Then $(x_0, y^{'})$ is a weakly efficient element of $(SVOP)$.

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