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# HADAMARD-TYPE WELLl-POSEDNESS FOR A VECTOR EQUILIBRIUM PROBLEM WITH SET-VALUED MAPPINGS*

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**Abstract:** In this paper, we introduce a kind of Hadamard-type well-posedness for vector equilibrium problems with set-valued mappings. By virtue of a scalarization function, we obtain some relationships between solutions of vector equilibrium problems and a scalar optimization problem and also derive a scalarization theorem of P.K. convergence for sequences of set-valued mappings. Based on these results, we establish a sufficient condition of Hadamard-type well-posedness for vector equilibrium problem.

**Key words:** *vector equilibrium problem, P.K. convergence, Hadamard-type well-posedness, nonlinear scalarization function*

**Mathematics Subject Classification:** *49K40, 49J53, 90C31*

## 1 Introduction

The concept of Hadamard well-posedness is inspired by the classical idea of Hadamard, which goes back to the beginning of the last century. Hadamard well-posedness together with Tykhonov well-posedness are two main types of concepts for well-posed optimization problems. Recently, Tykhonov well-posedness has been studied and generalized in other more complicated situations, such as scalar optimization problems, vector optimization problems, nonlinear optimal control problems and so on, see [3, 6, 8, 9, 10, 11, 14] and references therein. However, as much as we know, there are few papers which consider Hadamard-type well-posedness for vector equilibrium problems. As we know, Hadamard well-posedness requires existence and uniqueness of the optimal solution together with continuous dependence of solutions on the data in problems. Therefore, it is closely related to the stability of vector optimization problems. In the present paper, we try to deal with Hadamard-type well-posedness for a vector equilibrium problem and establish a sufficient condition for it.

In this paper, we introduce a kind of Hadamard-type well-posedness for a vector equilibrium problem with set-valued mappings. Following the idea of [12, Section 2], we try to find out an appropriate scalarization function to characterize the vector equilibrium problem by a scalar optimization problem, and obtain some solution relationships between them.

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Meanwhile, taking advantage of the suitable scalarization function, we also establish a scalarization theorem of Painlevé-Kuratowski convergence for sequences of set-valued mappings. Based on the solution relationships and the scalarization theorem, we get a sufficient condition for Hadamard-type well-posedness of a vector equilibrium problem is obtained. This paper is organized as follows. In Section 2, we give some basic concepts and notations. In Section 3, according to the method which has just mentioned, we establish a sufficient condition for Hadamard-type well-posedness of a vector equilibrium problem. In Section 4, we give a conclusion.

## 2 Preliminaries and Notations

In this section, we introduce some basic concepts and notations, which will be used in the sequel. In addition, we introduce a kind of Hadamard-type well-posedness for a vector equilibrium problem.

Throughout this paper, unless other specified, we assume that $C \subset \mathbb{R}^h$ is a pointed closed convex cone, which has a nonempty interior denoted by int$C$. Let $E_n(n \in \mathcal{N}, \mathcal{N}$is an positive integer ) and $E$ be nonempty subsets of $\mathbb{R}^k$, $F : E \times E \to 2^{\mathbb{R}^h}$ be a set-valued mapping.

Let $F : E \times E \to 2^{\mathbb{R}^h}$ be a set-valued mapping with $0 \in F(x, x)$ for every $x \in E$. We consider the following kind of vector equilibrium problem with set-valued mappings:

(VEP$(F, E)$): to find $\bar{x} \in E$ such that

$$F(\bar{x}, y) \subset Y \setminus (-\text{int}C),\ \forall\, y \in E.$$

i.e. to find $\bar{x} \in E$ such that

$$F(\bar{x}, E) \subset Y \setminus (-\text{int}C),\ \ where\ F(\bar{x}, E) = \bigcup_{y \in E} F(\bar{x}, y).$$

**Definition 2.1.** Let $e \in \text{int}C$ and $\varepsilon \geq 0$. A point $\bar{x} \in E$ is called an $(\varepsilon, e)-$solution of the problem (VEP$(F, E)$), written as $\bar{x} \in (\varepsilon, e) - \mathrm{S}(F, E)$, if

$$F(\bar{x}, E) \cap (-\varepsilon e - \text{int}C) = \emptyset.$$

In the sequel, we will suppose that all the $(\varepsilon, e)-$solution sets of the vector equilibrium problems, which are involved, are nonempty.

Now we recall the well-known definition of Painlevé-Kuratowski set convergence.

**Definition 2.2.** A sequence of nonempty subsets $(D_n)_{n \in \mathcal{N}}$ of $\mathbb{R}^h$ converges to $D$ in the sense of Painlevé-Kuratowski (P.K. for short), written as $D_n \xrightarrow{P.K.} D$, if

$$\limsup_n D_n \subset D \subset \liminf_n D_n,$$

where $\liminf_n D_n$, the inner limit, consists of all possible limit points of sequences $(x_n)_{n \in \mathcal{N}}$ with $x_n \in D_n (n \in \mathcal{N})$ and $\limsup_n D_n$, the outer limit, consists of all possible cluster points of such sequences.

Based on the above definition, we introduce the following two definitions for sequences of set-valued mappings in [4].

**Definition 2.3.** A sequence of nonempty set-valued mappings $H_n : R^k \to 2^{\mathbb{R}^h}$ Painlevé-Kuratowski (P.K. for short) converges to a set-valued mapping $H : R^k \to 2^{\mathbb{R}^h}$, written as $H_n \stackrel{P.K.}{\longrightarrow} H$, if epi$H_n \stackrel{P.K.}{\longrightarrow}$ epi$H$, where epi$H_n = \{(x, z) : z \in H_n(x)+C\}$ and epi$H = \{(x, z) : z \in H(x) + C\}$.

**Definition 2.4** ([13])**.** Let $\{H_n : E_n \to 2^{\mathbb{R}^h},\ n = 1, 2, \ldots\}$ be a sequence of nonempty set-valued mappings and denote by $\{(E_n, H_n) : n = 1, 2, \ldots\}$ the corresponding sequence of (domain, mapping) pairs representing the perturbing problems. Let $H : E \to 2^{\mathbb{R}^h}$ be a set-valued mapping. We say $(E_n, H_n)$ $P.K.$ converges to $(E, H)$ if $\overline{H}_n \stackrel{P.K.}{\longrightarrow} \overline{H}$, where

$$\overline{H}_n(x) = \begin{cases} H_n(x), & \text{if } x \in E_n, \\ \{+\infty\}, & \text{if } x \in R^k \backslash E_n; \end{cases}$$

$$\overline{H}(x) = \begin{cases} H(x), & \text{if } x \in E, \\ \{+\infty\}, & \text{if } x \in R^k \backslash E. \end{cases}$$

Now based on the above definition, we introduce a kind of Hadamard-type well-posedness for vector equilibrium problems with set-valued mapping.

For convenience, we shall denote the set-valued mappings $G_n : E_n \to 2^{R^h}$ and $G : E \to 2^{R^h}$ by $G_n(x) = F_n(x, E_n) = \cup_{y \in E_n} F_n(x, y)$ and $G(x) = F(x, E) = \cup_{y \in E} F(x, y)$. We suppose that $0 \in F(x, x)$ for every $x \in E$ and $0 \in F_n(x, x)$ for every $x \in E_n$, where $n \in \mathcal{N}$.

**Definition 2.5.** Let $(E_n, G_n) \stackrel{P.K.}{\longrightarrow} (E, G)$. The vector equilibrium problem (VEP$(F, E)$) is called to be Hadamard-type well-posed with respect to $(E_n, F_n)_{n \in N}$, if there exists $\varepsilon_0 > 0$ such that

$$\limsup_n [(\varepsilon, e) - \mathrm{S}(F_n, E_n)\ ] \subset (\varepsilon, e) - \mathrm{S}(F, E),\ \forall\, \varepsilon \in [0, \varepsilon_0].$$

## 3 Hadamard-type Well-posedness for Vector Equilibrium Problem

In this section, firstly, we use the following nonlinear scalarization function to characterize the problem (VEP$(F, E)$) by a scalar optimization problem and establish some solution relationships between them.

**Definition 3.1** ([2])**.** , the nonlinear scalarization function $\xi_e(.) : \mathbb{R}^h \to R$ is defined by:

$$\xi_e(y) = \inf\{t \in R :\ y \in te - C\},\ y \in \mathbb{R}^h.$$

The nonlinear scalarization function $\xi_e(.)$ has the following salient properties.

**Lemma 3.2** ([7])**.** *For a fixed $e \in \mathrm{int}C$ and any $t \in R$, we have*

(i) $\xi_e(y) < t \Leftrightarrow y \in te - \mathrm{int}C$,

(ii) $\xi_e(y) \le t \Leftrightarrow y \in te - C$,

(iii) $\xi_e(te) = t$,

(iv) $\xi_e(y)$ *is a continuous convex function on* $\mathbb{R}^h$ *and strictly monotone.*

Now we consider the following scalar optimization problem induced by (VEP$(F, E)$),

$$(\Omega_x, \xi_e(.)) : \quad \min_{y \in \Omega_x} \xi_e(y)$$

where $x \in E$.

**Definition 3.3.** Let $\bar{x} \in E$. A point $\bar{y} \in \Omega_{\bar{x}}$ is called an $\varepsilon$-approximate solution for the problem $(\Omega_{\bar{x}}, \xi_e(.))$, written as $\bar{y} \in \varepsilon - \mathrm{Inf}(\Omega_{\bar{x}}, \xi_e(.))$, if

$$\xi_e(\bar{y}) - \varepsilon \leq \xi_e(y), \quad \forall y \in \Omega_{\bar{x}}.$$

(VEP$(F,E)$) and $(\Omega_x, \xi_e(.))$ have the following solution relationships.

**Proposition 3.4.** *Let $x_0 \in E$. $x_0 \in (\varepsilon, e) - \mathrm{S}(F,E)$ if and only if $0 \in \varepsilon - \mathrm{Inf}(\Omega_{x_0}, \xi_e(.))$.*

*Proof.*

$$\begin{aligned}
& x_0 \in (\varepsilon, e) - \mathrm{S}(F,E) \\
& \Leftrightarrow F(x_0, E) \cap (-\varepsilon e - \mathrm{int}C) = \emptyset \\
& \Leftrightarrow (F(x_0, E) + C) \cap (-\varepsilon e - \mathrm{int}C) = \emptyset \\
& \Leftrightarrow \forall y \in F(x_0, E) + C,\ y \notin -\varepsilon e - \mathrm{int}C \\
& \Leftrightarrow \forall y \in F(x_0, E) + C,\ \xi_e(y) \geq -\varepsilon \\
& \Leftrightarrow 0 \in \varepsilon - \mathrm{Inf}(\Omega_{x_0}, \xi_e(.))
\end{aligned}$$

□

Secondly, by virtue of the nonlinear scalarization function, we also establish an important scalarization theorem of P.K. convergence for sequences of set-valued mappings.

**Theorem 3.5.** *Suppose that*

(i) *$E_n, E \subset R^k$ be nonempty subsets with $E_n \xrightarrow{P.K.} E$,*

(ii) *$F_n : E_n \times E_n \to 2^{\mathbb{R}^h}$ and $F : E \times E \to 2^{\mathbb{R}^h}$ be set-valued mappings,*

(iii) *$(E_n, G_n) \xrightarrow{P.K.} (E, G)$,*

(iv) *$q_n \in E_n, q \in E$ with $q_n \to q$.*

*Then, $I_{q_n} \xrightarrow{P.K.} I_q$, where scalar functions $I_{q_n}, I_q : \mathbb{R}^h \to R$ are defined as follows:*

$$I_{q_n}(y) = \begin{cases} \xi_e(y), & \text{if} \quad y \in F_n(q_n, E_n) + C, \\ +\infty, & \text{if} \quad y \in \mathbb{R}^h \backslash (F_n(q_n, E_n) + C), \end{cases} \tag{3.1}$$

$$I_q(y) = \begin{cases} \xi_e(y), & \text{if} \quad y \in F(q, E) + C, \\ +\infty, & \text{if} \quad y \in \mathbb{R}^h \backslash (F(q, E) + C). \end{cases} \tag{3.2}$$

*Proof.* Assume that $M_n = \mathrm{epi}I_{q_n}$ and $M = \mathrm{epi}I_q$. Firstly, we show that $M \subset \liminf_n M_n$. For arbitrary $(y,t) \in M$, we have that

$$y \in F(q,E) + C, \tag{3.3}$$

and

$$I_q(y) \leq t. \tag{3.4}$$

By (3.3), we have $(q, y) \in \mathrm{epi}G$. Since $(E_n, G_n) \xrightarrow{P.K.} (E, G)$, there exists a sequence $(q_n, y_n) \in \mathrm{epi}G_n (n \in N)$ such that $(q_n, y_n) \to (q, y)$. It follows that $y_n \in F_n(q_n, E_n) + C$. According to (3.4), we obtain that $y \in te - C$. Then, there exists $k \in C$ such that $y = te - k$. Together with $y_n \to y$, we get $y_n \to te - k$. Let $l_n = y_n + k$. It implies that $l_n \to te$. Let the value of the non-linear scalarization function $\xi_e$ on $l_n$ be $\xi_e(l_n) = t_n$. By Lemma 3.2(iii), (iv) and $l_n \to te$, we obtain that $t_n \to t$. Moreover, from the definition of $\xi_e(\cdot)$, we have $l_n \in t_n e - C$, that is, $y_n + k \in t_n e - C$ and it deduces that $y_n \in t_n e - C$. Together with $y_n \in F_n(q_n, E_n) + C$, we have $I_{q_n}(y_n) \leq t_n$. In addition, $(y_n, t_n) \in M_n$ and $(y_n, t_n) \to (y, t)$. Therefore, we get $M \subset \liminf_n M_n$.

Secondly, we verify that $\limsup_n M_n \subset M$. For arbitrary $(y, t) \in \limsup_n M_n$, there exists a sequence $(y_n, t_n) \in M_n$ with subsequence $(y_{n_k}, t_{n_k}) \to (y, t)$. Since $(y_{n_k}, t_{n_k}) \in M_{n_k}$, we have that

$$y_{n_k} \in F_{n_k}(q_{n_k}, E_{n_k}) + C, \tag{3.5}$$

and

$$\xi_e(y_{n_k}) \leq t_{n_k}. \tag{3.6}$$

By (3.5), we obtain that $(q_{n_k}, y_{n_k}) \in \text{epi}G_{n_k}$. Since $(E_n, G_n) \xrightarrow{P.K.} (E, G)$ and $(q_{n_k}, y_{n_k}) \to (q, y)$, we have $(q, y) \in \text{epi}G$. It implies that

$$y \in F(q, E) + C. \tag{3.7}$$

According to Lemma 3.2($ii$) and (3.6), we have $y_{n_k} \in t_{n_k} e - C$, i.e., $y_{n_k} - t_{n_k} e \in -C$. Together with $(y_{n_k}, t_{n_k}) \to (y, t)$ and C is closed, we get that $y - te \in -C$,which means that

$$\xi_e(y) \leq t. \tag{3.8}$$

By (3.7) and (3.8), we get that $(y, t) \in M$, which implies that $\limsup_n M_n \subset M$. □

By the above definitions of scalar function $I_{q_n}, I_q : \mathbb{R}^h \to R$, we consider the following two scalar optimization problems $(\mathbb{R}^h, I_{q_n})$ and $(\mathbb{R}^h, I_q)$:

$$(\mathbb{R}^h, I_{q_n}) : \quad \min_{y \in \mathbb{R}^h} I_{q_n}(y),$$

and

$$(\mathbb{R}^h, I_q) : \quad \min_{y \in \mathbb{R}^h} I_q(y).$$

Then, we give the following definition of $\varepsilon$-approximate solution for them.

**Definition 3.6.** A point $\bar{y} \in \mathbb{R}^h$ is called an $\varepsilon$-approximate solution for the problem $(\mathbb{R}^h, I_{q_n})$, written as $\bar{y} \in \varepsilon - \text{Inf}(\mathbb{R}^h, I_{q_n})$, if

$$I_{q_n}(\bar{y}) - \varepsilon \leq I_{q_n}(y), \quad \forall y \in \mathbb{R}^h.$$

A point $\bar{y} \in \mathbb{R}^h$ is called an $\varepsilon$-approximate solution for the problem $(\mathbb{R}^h, I_q)$, written as $\bar{y} \in \varepsilon - \text{Inf}(\mathbb{R}^h, I_q)$, if

$$I_q(\bar{y}) - \varepsilon \leq I_q(y), \quad \forall y \in \mathbb{R}^h.$$

Obviously, we have

$$0 \in \varepsilon - \text{Inf}(\Omega_{x_0}, \xi_e(.)) \Leftrightarrow 0 \in \varepsilon - \text{Inf}(\mathbb{R}^h, I_{x_0}),$$

$$0 \in \varepsilon - \text{Inf}\,(F_{n_k}(x_{n_k}, E_{n_k}) + C, \xi_e(.)) \Leftrightarrow 0 \in \varepsilon - \text{Inf}(\mathbb{R}^h, I_{x_{n_k}}).$$

According to [1, Proposition 1.14] and [1, Theorem 1.39], $g_n : X \to \overline{R}$ ($n \in \mathcal{N}$) P.K. converges to $g : X \to \overline{R}$ if and only if $g_n : X \to \overline{R}$ ($n \in \mathcal{N}$) variationally converges to $g : X \to \overline{R}$, where $X$ is a first countable space. As we all know, Hausdorff topological linear space is first countable. Then from [3, Chap 4, Theorem 5], we have the following lemma.

**Lemma 3.7.** *Assume that* $g_n : S \to \overline{R}$, $g : S \to \overline{R}$ *and* $g_n \stackrel{P.K.}{\longrightarrow} g$, *where* $n \in \mathcal{N}$. *Then there exists* $\varepsilon_0 > 0$ *such that*

$$\limsup_n [\varepsilon - \text{Inf } (S, g_n)] \subset \varepsilon - \text{Inf } (S, g),$$

*for all* $0 \leq \varepsilon \leq \varepsilon_0$.

Finally, we apply the above results to obtain sufficient condition for Hadamard-type well-posedness of vector equilibrium problem with set-valued mappings.

**Theorem 3.8.** *Suppose that*

(i) $E_n, E \subset R^k$ *be nonempty subsets with* $E_n \stackrel{P.K.}{\longrightarrow} E$,

(ii) $F_n : E_n \times E_n \to 2^{\mathbb{R}^h}$ *and* $F : E \times E \to 2^{\mathbb{R}^h}$ *be set-valued mappings,*

(iii) $(E_n, G_n) \stackrel{P.K.}{\longrightarrow} (E, G)$.

*Then* $(VEP(F, E))$ *is Hadamard-type well-posed with respect to* $(E_n, F_n)_{n \in N}$.

*Proof.* We need to prove that there exists $\varepsilon_0 > 0$ such that

$$\limsup_n [(\varepsilon, e) - \text{S}(F_n, E_n)\ ] \subset (\varepsilon, e) - \text{S}(F, E),$$

for all $0 \leq \varepsilon \leq \varepsilon_0$. Let $\bar{x} \in \limsup_n [(\varepsilon, e) - \text{S}(F_n, E_n)\ ]$. Then, there exists a sequence $x_n \in (\varepsilon, e) - \text{S}(F_n, E_n)$ with subsequence $x_{n_k} \in (\varepsilon, e) - \text{S}(F_{n_k}, E_{n_k})$ such that $x_{n_k} \to \bar{x}$. Since $E_n \stackrel{P.K.}{\longrightarrow} E$, we have $\bar{x} \in E$. From Proposition 3.4, we have $0 \in \varepsilon - \text{Inf } (F_{n_k}(x_{n_k}, E_{n_k}) + C, \xi_e(.))$. Obviously, $0 \in \varepsilon - \text{Inf } (\mathbb{R}^h, I_{x_{n_k}}(.))$. Thus, $0 \in \limsup_{n_k} [\varepsilon - \text{Inf } (R^h, I_{x_{n_k}}(.))]$. According to conditions (i), (ii) and (iii), by Theorem 3.5, we have $I_{x_{n_k}} \stackrel{P.K.}{\longrightarrow} I_{\bar{x}}$. Then by lemma 3.7, there exists $\varepsilon_0 > 0$ such that

$$\limsup_{n_k} [\varepsilon - \text{Inf } (R^h, I_{x_{n_k}}(.))] \subset \varepsilon - \text{Inf } (R^h, I_{\bar{x}}(.)),$$

for all $0 \leq \varepsilon \leq \varepsilon_0$. It follows that $0 \in \varepsilon - \text{Inf } (R^h, I_{\bar{x}}(.))$, i.e., $0 \in \varepsilon - \text{Inf } (\Omega_{\bar{x}}, I_{\bar{x}}(.))$. By Proposition 3.4 and $\bar{x} \in E$, we obtain $\bar{x} \in (\varepsilon, e) - \text{S}(F, E)$. ☐

## 4 Conclusion

This paper considered a kind of Hadamard-type well-posedness for a vector equilibrium problem with set-valued mappings, which based on the concepts of Painlevé-Kuratowski convergence of sets. By virtue of the suitable scalarization function, we established the sufficient condition for Hadamard-type well-posedness of the vector equilibrium problem.

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