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# A MODIFIED CG-DESCENT AND DPR ALGORITHM FOR UNCONSTRAINED OPTIMIZATION

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**Abstract:** In this paper, a modified CG-DESCENT and DPR conjugate gradient algorithm is presented, which produces sufficient descent search direction at every iteration. This property depends neither on the line search used, nor on the convexity of the objective function. Moreover, we establish the global convergence for general nonlinear functions under suitable conditions. Numerical results show that the proposed method is more efficient than the previous ones.

**Key words:** *Unconstrained optimization, conjugate gradient method, sufficient descent property, global convergence*

**Mathematics Subject Classification:** *90C30, 65K05, 49M37*

## 1 Introduction

Conjugate gradient methods comprise a class of unconstrained optimization algorithms which are characterized by low memory requirements and strong local and global convergence properties. For a general unconstrained optimization problem

$$\min f(x), \quad x \in R^n, \tag{1.1}$$

where $f$: $R^n \to R$ is a continuously differentiable function. The iterative formula of the conjugate gradient method is given by

$$x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, \cdots, \tag{1.2}$$

where the step-length $\alpha_k$ is obtained by carrying out some line search, and the direction $d_k$ is defined by

$$d_k = \begin{cases} -g_0, & \text{if } k = 0, \\ -g_k + \beta_k d_{k-1}, & \text{if } k > 0, \end{cases} \tag{1.3}$$

where $\beta_k$ is a parameter. Some well-known formulas for $\beta_k$ are the Fletcher-Reeves(FR) [2], Polak-Ribiere(PRP) [14], Liu-Storey(LS) [13], Dai-Yuan(DY) [4], the Conjugate Descent(CD) [8] and Hestenes-Stiefel(HS) [11] formulas, which are respectively given by

$$\beta_k^{DY} = \frac{\|g_k\|^2}{d_{k-1}^T y_{k-1}}, \quad \beta_k^{FR} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad \beta_k^{CD} = \frac{\|g_k\|^2}{-d_{k-1}^T g_{k-1}},$$

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$$\beta_k^{HS} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \beta_k^{PR} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \quad \beta_k^{LS} = \frac{g_k^T y_{k-1}}{-d_{k-1}^T g_{k-1}},$$

where $y_{k-1} = g_k - g_{k-1}$ and $\|\cdot\|$ stands for the Euclidean norm of vectors.

An important class of conjugate gradient algorithms is the hybrid conjugate gradient methods. Hu & Storey [12] and Dai & Yuan [5] proposed some hybrid methods which we call the H1 method and the H2 method, respectively, that is,

$$\beta_k^{H1} = \max\{0, \min\{\beta_k^{FR}, \beta_k^{PRP}\}\}, \tag{1.4}$$
$$\beta_k^{H2} = \max\{0, \min\{\beta_k^{DY}, \beta_k^{HS}\}\}. \tag{1.5}$$

Gilbert and Nocedal [9] extended H1 to the case that

$$\beta_k^{H1} = \max\{-\beta_k^{FR}, \min\{\beta_k^{FR}, \beta_k^{PRP}\}\}. \tag{1.6}$$

Numerical performances show that the H1 and the H2 methods are better than the PRP method [5,12].

Recently, there has been growing interest in the descent conjugate gradient methods. Hager and Zhang [10] proposed a new conjugate gradient method which was obtained by modifying the HS method and called CG-DESCENT method. The parameter $\beta_k$ in the CG-DESCENT method is given by

$$\beta_k^{N} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - 2\frac{\|y_{k-1}\|^2 g_k^T d_{k-1}}{(d_{k-1}^T y_{k-1})^2}, \tag{1.7}$$
$$\beta_k^{N+} = \max\{\beta_k^N, \eta_k\}, \eta_k = \frac{-1}{\|d_k\| \min\{\|g_k\|, \eta\}}, \tag{1.8}$$

where $\eta > 0$ is a constant. Later, Yu and Guan [15] motivated by their work proposed a PRP conjugate gradient method as following

$$\beta_k^{DPR} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2} - C\frac{\|y_{k-1}\|^2 g_k^T d_{k-1}}{\|g_{k-1}\|^4}, \tag{1.9}$$

where parameter C essentially controls the relative weight between conjugant and descent. Zhang and Zhou [16] proposed two hybrid methods called NH1 and NH2 method as follows

$$NH1: d_k = -(1 + \beta_k^{\mathrm{H1}} \frac{d_{k-1}^T g_k}{\|g_k\|^2}) g_k + \beta_k^{\mathrm{H1}} d_{k-1}, \tag{1.10}$$
$$NH2: d_k = -(1 + \beta_k^{\mathrm{H2}} \frac{d_{k-1}^T g_k}{\|g_k\|^2}) g_k + \beta_k^{\mathrm{H2}} d_{k-1}. \tag{1.11}$$

Obviously, these two new hybrid methods satisfy

$$g_k^T d_k = -\|g_k\|^2, \tag{1.12}$$

which shows that they are descent and independent of any line search used. The global convergence of these two methods [16] are presented and numerical results also showed their efficiency in real computations.

In this paper, in order to obtain an efficiency method in real computations, based on the idea of the methods all above, we proposed a new method, which is a projection of the

CG-DESCENT and DPR conjugate algorithms. The search direction $d_k$ has the following form

$$d_k = \begin{cases} -g_0, & \text{if} \quad k = 0, \\ -(1 + \beta_k^{HZPR} \frac{d_{k-1}^T g_k}{\|g_k\|^2}) g_k + \beta_k^{HZPR} d_{k-1}, & \text{if} \quad k \geq 1, \end{cases} \tag{1.13}$$

where

$$\beta_k^{HZPR} = \max\{0, \min\{\beta_k^N, \beta_k^{DPR}\}\}. \tag{1.14}$$

From the method above, we can easily obtain (1.12). For convenience, we call the method above as the HZPR method.

The rest of this paper are organized as follows. In the next section, we prove the global convergence of the method (1.13) for general nonlinear functions with strong Wolfe line search. In section 3, we report some numerical results to test the proposed method.

## 2 Algorithm and Convergence Analysis

First, we make the following standard assumptions for the objective function, which have been used often in the literature to analyze the global convergence of conjugate methods with inexact line search.

**Assumption (A)**

(H1) The level set $\Omega = \{x \in R^n | f(x) \leq f(x_0)\}$ is bounded.

(H2) In some neighborhood $N$ of $\Omega$, $f$ is continuously differentiable and its gradient is Lipschitz continuous, namely, there exists a constant $L > 0$ such that

$$\|g(x) - g(y)\| \leq L\|x - y\|, \quad \forall x, y \in N.$$

Assumption (A) implies that there exists a positive constant $\widehat{\gamma}$ and $B$ such that

$$\|g(x)\| \leq \widehat{\gamma}, \quad \forall x \in \Omega, \tag{2.1}$$

and

$$\|x - y\| \leq B, \quad \forall x, y \in \Omega. \tag{2.2}$$

Now we introduce the steps of the HZPR algorithm as following.

**Algorithm 2.1** (HZPR Method)**.**

**Step 0:** Choose an initial point $x_0 \in R^n$. Let $k = 0$.

**Step 1:** Compute $d_k$ by Eq. (1.13), where $\beta_k^{HZPR}$ is computed by (1.14).

**Step 2:** Determine $\alpha_k$ by the strong Wolfe line search

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k g_k^T d_k, \tag{2.3}$$

$$|g(x_k + \alpha_k d_k)^T d_k| \leq \sigma |g_k^T d_k|, \tag{2.4}$$

where $0 < \delta < \sigma < 1$.

**Step 3:** Let $x_{k+1} = x_k + \alpha_k d_k$.

**Step 4:** Let $k := k+1$ and go to step 1.

The following well-known lemma, called the zoutendijk condition, which was originally given in [18].

**Lemma 2.2.** *Suppose Assumption (A) holds. Consider any method in the form (1.2), where $d_k$ is a descent direction and $\alpha_k$ satisfies the strong Wolfe condition (2.3) and (2.4). Then we have*

$$\sum_{k=0}^{\infty} \frac{(d_k^T g_k)^2}{\|d_k\|^2} < \infty. \tag{2.5}$$

We now establish the global convergence theorem for the HZPR algorithm in a similar way to Theorem 3.1 in [6]. First, we give a useful lemma about $\beta_k^{HZPR}$ in (1.14) which plays an important role in the global convergence analysis.

**Lemma 2.3.** *Suppose Assumption (A) holds. $\{x_k\}$ is generated by algorithm 2.1. If there exist a constant $\epsilon > 0$ such that $\|g_k\| > \epsilon$ for all $k > 0$, then there exist a positive constant $D$ such that*

$$|\beta_k^{HZPR}| \leq D\|s_{k-1}\|, \tag{2.6}$$

*where $s_{k-1} = x_k - x_{k-1}$.*

*Proof.* From (2.4) and (1.12), we have

$$\begin{aligned} |d_{k-1}^T y_{k-1}| &= |d_{k-1}^T g_k - d_{k-1}^T g_{k-1}| \geq |d_{k-1}^T g_{k-1}| - |d_{k-1}^T g_k| \\ &\geq |d_{k-1}^T g_{k-1}| - \sigma |g_{k-1}^T d_{k-1}| = (1-\sigma)\|g_{k-1}\|^2. \end{aligned}$$

The above inequality together with (1.12) and (2.4), we have

$$\begin{aligned} |\beta_k^{HZPR}| &= |\max\{0, \min\{\beta_k^{DPR}, \beta_k^N\}\}| \\ &\leq \max\{\frac{\|g_k\|\|y_{k-1}\|}{\|g_{k-1}\|^2} + \frac{C\|y_{k-1}\|^2\sigma|g_{k-1}^T d_{k-1}|}{\|g_{k-1}\|^4}, \\ &\quad \frac{\|g_k\|\|y_{k-1}\|}{|d_{k-1}^T y_{k-1}|} + \frac{2\|y_{k-1}\|^2\sigma|g_{k-1}^T d_{k-1}|}{(d_{k-1}^T y_{k-1})^2}\} \\ &\leq \max\{\frac{\|g_k\|\|y_{k-1}\|}{\|g_{k-1}\|^2} + \frac{C\|y_{k-1}\|^2\sigma\|g_{k-1}\|^2}{\|g_{k-1}\|^4}, \\ &\quad \frac{\|g_k\|\|y_{k-1}\|}{(1-\sigma)\|g_{k-1}\|^2} + \frac{2\|y_{k-1}\|^2\sigma\|g_{k-1}\|^2}{(1-\sigma)\|g_{k-1}\|^4}\} \\ &\leq \max\{\frac{L\widehat{\gamma} + C\sigma L^2 B}{\epsilon^2}, \frac{L\widehat{\gamma} + 2\sigma L^2 B}{(1-\sigma)\epsilon^2}\}\|s_{k-1}\|. \end{aligned}$$

Defining

$$D = \max\{\frac{L\widehat{\gamma} + C\sigma L^2 B}{\epsilon^2}, \frac{L\widehat{\gamma} + 2\sigma L^2 B}{(1-\sigma)\epsilon^2}\},$$

then we have the result (2.6). The proof is complete. □

The next lemma corresponds to Lemma 3.4 in [3] and Lemma 2.3 in [6].

**Lemma 2.4.** *Suppose Assumption (A) holds. $\{x_k\}$ is generated by algorithm 2.1. If there exist a constant $\epsilon > 0$ such that $\|g_k\| > \epsilon$ for all $k > 0$, then we have*

$$\sum_{k=0}^{\infty} \|u_k - u_{k-1}\|^2 < \infty, \tag{2.7}$$

where $u_k = \frac{d_k}{\|d_k\|}$.

*Proof.* Since $d_k \neq 0$ follows from (1.12) and $\|g_k\| > \epsilon$, so $u_k$ is well-defined. We rewrite (1.13) as following

$$d_k = -(1 + \beta_k^{HZPR} \frac{d_{k-1}^T g_k}{\|g_k\|^2}) g_k + \beta_k^{HZPR} d_{k-1} = v_k + \beta_k^{HZPR} d_{k-1}. \tag{2.8}$$

By defining

$$r_k = \frac{v_k}{\|d_k\|}, \quad \delta_k = \frac{\beta_k^{HZPR} \|d_{k-1}\|}{\|d_k\|}.$$

So, we have

$$u_k = r_k + \delta_k u_{k-1}. \tag{2.9}$$

Then we have from the fact that $\|u_k\| = \|u_{k-1}\| = 1$

$$\|r_k\| = \|u_k - \delta_k u_{k-1}\| = \|u_{k-1} - \delta_k u_k\|. \tag{2.10}$$

Using the condition $\delta_k \geq 0$, the triangle inequality and (2.10), we have

$$\|u_k - u_{k-1}\| \leq \|(1+\delta_k)(u_k - u_{k-1})\| \leq \|u_k - \delta_k u_{k-1}\| + \|u_{k-1} - \delta_k u_k\| \leq 2\|r_k\|. \tag{2.11}$$

From (2.1), (2.4) and (2.6), we have

$$|\beta_k^{HZPR}| \frac{|g_k^T d_{k-1}|}{\|g_k\|^2} \leq D\|s_{k-1}\| \frac{\sigma |g_{k-1}^T d_{k-1}|}{\|g_k\|^2} \leq D \frac{\sigma \widehat{\gamma}^2}{\epsilon^2} \|s_{k-1}\| \doteq M \|s_{k-1}\|. \tag{2.12}$$

From (2.2), (2.12) and (2.8), there exist a constant $M_1 \geq 0$ such that

$$\|v_k\| \leq \|g_k\| + M\|s_{k-1}\| \|g_k\| \leq M_1. \tag{2.13}$$

From the definition of $r_k$, (2.6) and (2.13), we have

$$\sum_{k=0}^{\infty} \|r_k\|^2 = \sum_{k=0}^{\infty} \frac{\|v_k\|^2}{\|d_k\|^2} \leq \sum_{k=0}^{\infty} \frac{M_1^2}{\|d_k\|^2} = \sum_{k=0}^{\infty} \frac{M_1^2}{\|g_k\|^4} \frac{\|g_k\|^4}{\|d_k\|^2} \leq \frac{M_1^2}{\epsilon^2} \sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} < \infty. \tag{2.14}$$

Together with (2.11), we have (2.7). □

The next theorem establishes the global convergence of the HZPR method. The proof of it is similar to Theorem 4.3 in [9] and Theorem 3.1 in [6] .

**Theorem 2.5.** *Suppose Assumption (A) holds. $\{x_k\}$ is generated by algorithm 2.1. Then we have*

$$\liminf_{k \to \infty} \|g_k\| = 0. \tag{2.15}$$

*Proof.* Assume that the conclusion (2.15) is not true, then there exist a constant $\varepsilon > 0$ such that for all $k$, $\|g_k\| > \varepsilon$. From the definition of $u_k$, we observe that for any $l \geq k$,

$$x_l - x_k = \sum_{j=k}^{l-1}(x_{j+1} - x_j) = \sum_{j=k}^{l-1} \|s_j\| u_k + \sum_{j=k}^{l-1} \|s_j\|(u_j - u_k). \tag{2.16}$$

By the triangle inequality, from the fact $\|u_k\| = 1$, so we have

$$\sum_{j=k}^{l-1} \|s_j\| \leq \|x_l - x_k\| + \sum_{j=k}^{l-1} \|s_j\| \|u_j - u_k\| \leq B + \sum_{j=k}^{l-1} \|s_j\| \|u_j - u_k\|. \tag{2.17}$$

Let $\Delta$ be a positive integer, chosen large enough that

$$\Delta \geq 4BD, \tag{2.18}$$

where $B$ and $D$ appear in (2.2) and (2.6). By lemma 2.4, we can find a large enough $k_0$ that

$$\sum_{i \geq k_0} \|u_k - u_{k-1}\|^2 \leq \frac{1}{4\Delta}. \tag{2.19}$$

If $j > k > k_0$ and $j - k \leq \Delta$, then by (2.19) and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|u_j - u_k\| &\leq \sum_{i=k}^{j-1} \|u_{i+1} - u_i\|^2 \\ &\leq \sqrt{j-k}(\sum_{i=k}^{j-1} \|u_{i+1} - u_i\|^2)^{1/2} \\ &\leq \sqrt{\Delta}(\frac{1}{4\Delta})^{1/2} = \frac{1}{2}. \end{aligned}$$

Combining this with (2.2) and (2.17) yields

$$\sum_{j=k}^{l-1} \|s_j\| \leq 2B. \tag{2.20}$$

From (2.6), (2.8) and (2.13) we have

$$\|d_l\|^2 \leq (\|v_k\| + |\beta_l^{HZPR}| \|d_{l-1}\|)^2 \leq 2M_1^2 + 2D^2 \|s_{l-1}\|^2 \|d_{l-1}\|^2. \tag{2.21}$$

Defining $S_i = 2D^2 \|s_i\|^2$, by induction, we obtain

$$\begin{aligned} \|d_l\|^2 &\leq 2M_1^2 + S_{l-1} \|d_{l-1}\|^2 \\ &\leq 2M_1^2(1 + S_{l-1} + S_{l-1}S_{l-2} + \cdots + S_{l-1}S_{l-2} \cdots S_{k_0+1}) \\ &\quad + \|d_{k_0}\|^2 S_{l-1}S_{l-2} \cdots S_{k_0}. \end{aligned}$$

Then we have

$$\|d_l\|^2 \leq \begin{cases} 2M_1^2 + S_{k_0} \|d_{k_0}\|^2, & \text{if } l = k_0 + 1, \\ 2M_1^2(1 + \sum_{i=k_0+1}^{l-1} \prod_{j=i}^{l-1} S_j) + \|d_{k_0}\|^2 \prod_{j=k_0}^{l-1} S_j, & \text{if } l > k_0 + 1. \end{cases} \tag{2.22}$$

Let us consider as follows a product of $\Delta$ consecutive $S_j$, where $k \geq k_0$,

$$\begin{aligned}\prod_{j=k}^{k+\Delta-1} S_j &= \prod_{j=k}^{k+\Delta-1} 2D^2\|s_j\|^2 = (\prod_{j=k}^{k+\Delta-1} \sqrt{2}D\|s_j\|)^2 \\ &\leq (\frac{\sum_{j=k}^{k+\Delta-1} \sqrt{2}D\|s_j\|}{\Delta})^{2\Delta} \leq (\frac{2\sqrt{2}BD}{\Delta})^{2\Delta} \leq \frac{1}{2^\Delta}.\end{aligned}$$

The product of $\Delta$ consecutive $S_j$ is bounded by $\frac{1}{2^\Delta}$, it follows that the sum in (2.22) is bounded, and the bound is independent of $l$. This bound for $\|d_l\|$, independent of $l > k_0$, contradicts (2.5), hence we have (2.15). The proof is complete. □

## 3 Numerical Experiments

In this section, we do some numerical experiments to test the performance of the HZPR method and compare it with some existing methods for solving large scale unconstrained optimization problems. All codes are written in Fortran and ran on IBM T60 PC with two 1.83 GHz CPU and 2.5GB RAM.

The test problems are the unconstrained problems from Neculai Andrei [1]. For each problem, the dimension $n$ is set to 1 000 and 10 000. The parameters in the strong wolfe conditions are as follows: $\sigma = 0.9$ and $\delta = 0.1$, and $C = 1$ in (1.14). We stop the iteration if the inequality $\|g_k\| \leq 10^{-6}$ is satisfied.

We compare the performances of the HZPR method with that of the CG-DESCENT method [10] and the MPRP method [17]. The CG-DESCENT codes can be obtained from Hager's page at http://www.math.ufl.edu/hager/papers/CG.

Table 1 lists the results of the HZPR method, the CG-DESCENT method and the MPRP method which gives the total number of iterations(iter), the total number of function evaluations(fn), the total number of gradient evaluations(gn) and the cpu time(time) in seconds.

We adopt the performance profiles by Dolan and More [7] to compare the performance among the tested methods. That is, for each method, we plot the fraction $P$ of problems for which the method is within a factor $\tau$ of the best time. The left side of the figure gives the percentage of the test problems for which a method is the fastest; the right side gives the percentage of the test problems that are successfully solved by each of the methods. The top curve is the method that solved the most problems in a time that are within a factor $\tau$ of the best time. Figure 1-4 are the performance profile measured by CPU time, the number of iterations, the number of function evaluations and the number of gradient evaluations, respectively.

Table 1: The result of HZPR, MPRP and CG-DESCENT

| Problem | N | CG-DESCENT<br>iter/fn/gn/time | MPRP<br>iter/fn/gn/time | HZPR<br>iter/fn/gn/time |
|---|---|---|---|---|
| ROTH | 1000 | 14/31/22/0.00E+00 | 12/30/22/1.56E-02 | 12/29/23/1.56E-02 |
| ROTH | 10000 | 18/39/27/9.38E-02 | 15/36/27/9.38E-02 | 11/25/19/4.69E-02 |
| TRIGMETRIC | 1000 | 100/211/117/2.34E-01 | 73/151/81/1.56E-01 | 70/146/78/1.56E-01 |
| TRIGMETRIC | 10000 | 94/196/109/2.17E+00 | 79/164/89/1.81E+00 | 86/183/104/2.06E+00 |
| ROSENBROCK | 1000 | 36/120/94/1.56E-02 | 40/104/76/1.56E-02 | 40/116/90/3.12E-02 |
| ROSENBROCK | 10000 | 34/111/87/1.56E-01 | 38/98/70/1.56E-01 | 46/132/102/2.03E-01 |
| WHITEHOLST | 1000 | 39/117/87/1.56E-02 | 40/109/77/1.56E-02 | 43/132/101/1.56E-02 |
| WHITEHOLST | 10000 | 39/123/94/1.88E-01 | 41/110/76/1.88E-01 | 42/134/105/2.19E-01 |
| BEALEU63 | 1000 | 17/36/21/1.56E-02 | 12/25/14/1.56E-02 | 17/35/21/1.56E-02 |
| BEALEU63 | 10000 | 17/36/21/7.81E-02 | 12/25/14/4.69E-02 | 17/35/21/7.81E-02 |
| PENALTY | 1000 | 36/70/42/1.56E-02 | 36/69/41/1.56E-02 | 33/66/39/1.56E-02 |
| PENALTY | 10000 | 42/80/48/1.56E-01 | 38/76/43/1.41E-01 | 38/76/42/1.41E-01 |
| PQUADRATIC | 1000 | 188/377/189/7.81E-02 | 188/377/189/6.25E-02 | 188/377/189/6.25E-02 |
| PQUADRATIC | 10000 | 598/1197/599/2.20E+00 | 598/1197/599/ 2.28E+00 | 598/1197/599/2.31E+00 |
| RAYDAN1 | 1000 | 252/381/377/2.66E-01 | 242/365/363/2.34E-01 | 232/352/346/2.34E-01 |

Continued on next page

**Table 1 – continued from previous page**

| Problem | N | CG-DESCENT iter/fn/gn/time | MPRP iter/fn/gn/time | HZPR iter/fn/gn/time |
|---|---|---|---|---|
| RAYDAN1 | 10000 | 829/1101/1388/7.98E+00 | 839/1101/1418/8.17E+00 | 808/1061/1365/7.89E+00 |
| RAYDAN2 | 1000 | 5/11/7/0.00E+00 | 6/13/9/0.00E+00 | 6/12/8/1.56E-02 |
| RAYDAN2 | 10000 | 6/13/9/6.25E-02 | 6/13/9/6.25E-02 | 6/13/9/7.81E-02 |
| DIAGONAL1 | 1000 | 307/430/493/3.59E-01 | 330/463/529/3.91E-01 | 296/416/474/3.59E-01 |
| DIAGONAL1 | 10000 | 989/1218/1753/1.15E+01 | 998/1235/1761/1.18E+01 | 991/1216/1761/1.17E+01 |
| DIAGONAL2 | 1000 | 201/382/247/2.50E-01 | 233/436/307/2.97E-01 | 207/386/273/2.50E-01 |
| DIAGONAL2 | 10000 | 615/1175/809/7.88E+00 | 641/1161/873/8.17E+00 | 616/1174/841/8.03E+00 |
| DIAGONAL3 | 1000 | 262/374/414/4.22E-01 | 270/385/427/4.38E-01 | 261/373/412/4.22E-01 |
| DIAGONAL3 | 10000 | 889/1095/1574/1.42E+01 | 881/1095/1550/1.42E+01 | 885/1095/1562/1.42E+01 |
| HAGER | 1000 | 50/85/77/7.81E-02 | 50/85/77/7.81E-02 | 50/84/78/7.81E-02 |
| HAGER | 10000 | 93/144/151/1.33E+00 | 92/143/149/1.31E+00 | 93/143/152/1.33E+00 |
| GTRIDIAG1 | 1000 | 26/45/37/0.00E+00 | 26/44/36/1.56E-02 | 26/44/36/0.00E+00 |
| GTRIDIAG1 | 10000 | 26/43/37/1.25E-01 | 26/42/38/1.41E-01 | 26/42/38/1.25E-01 |
| TRIDIAG1 | 1000 | 18/37/21/0.00E+00 | 21/43/29/1.56E-02 | 22/45/27/1.56E-02 |
| TRIDIAG1 | 10000 | 18/37/21/7.81E-02 | 21/43/29/9.38E-02 | 24/49/27/9.38E-02 |
| TETERMS | 1000 | 9/19/13/1.56E-02 | 10/20/14/1.56E-02 | 10/20/14/1.56E-02 |
| TETERMS | 10000 | 9/19/13/1.88E-01 | 12/26/18/2.66E-01 | 9/20/14/2.03E-01 |
| GTRIDIAG2 | 1000 | 50/92/60/1.56E-02 | 47/85/58/3.12E-02 | 46/84/56/1.56E-02 |
| GTRIDIAG2 | 10000 | 52/95/63/2.97E-01 | 54/99/65/3.12E-01 | 53/100/61/2.97E-01 |
| DIAGONAL4 | 1000 | 4/9/6/0.00E+00 | 4/9/6/0.00E+00 | 4/9/6/0.00E+00 |
| DIAGONAL4 | 10000 | 4/9/6/0.00E+00 | 4/9/6/1.56E-02 | 4/9/6/1.56E-02 |
| DIAGONAL5 | 1000 | 3/8/5/1.56E-02 | 3/8/5/1.56E-02 | 3/8/5/0.00E+00 |
| DIAGONAL5 | 10000 | 3/8/5/1.09E-01 | 3/8/5/1.09E-01 | 3/8/5/1.09E-01 |
| HIMMELB | 1000 | 9/22/14/0.00E+00 | 9/22/14/0.00E+00 | 9/22/14/0.00E+00 |
| HIMMELB | 10000 | 9/22/14/4.69E-02 | 9/22/14/4.69E-02 | 9/22/14/3.12E-02 |
| GPSC1 | 1000 | 726/1152/1471/1.06E+00 | 488/845/854/ 7.34E-01 | 564/898/1124/8.28E-01 |
| GPSC1 | 10000 | 840/1234/1797/1.17E+01 | 1057/1639/2193/1.53E+01 | 729/1139/1504/1.06E+01 |
| PSC1 | 1000 | 15/29/18/3.12E-02 | 12/24/15/1.56E-02 | 13/26/17/3.12E-02 |
| PSC1 | 10000 | 12/26/15/1.88E-01 | 11/23/13/1.56E-01 | 12/24/15/1.56E-01 |
| POWELL | 1000 | 116/236/135/4.69E-02 | 216/434/235/6.25E-02 | 75/153/93/3.12E-02 |
| POWELL | 10000 | 581/1186/659/2.00E+00 | 178/358/197/6.41E-01 | 328/660/393/1.19E+00 |
| BD1 | 1000 | 25/64/56/3.12E-02 | 17/47/42/1.56E-02 | 24/64/53/3.12E-02 |
| BD1 | 10000 | 25/64/56/2.97E-01 | 17/47/42/2.34E-01 | 24/64/53/2.97E-01 |
| MARATOS | 1000 | 52/159/127/1.56E-02 | 70/230/186/3.12E-02 | 70/233/193/3.12E-02 |
| MARATOS | 10000 | 51/168/134/2.34E-01 | 67/198/158/2.97E-01 | 74/235/193/3.44E-01 |
| QDP | 1000 | 135/271/162/4.69E-02 | 137/275/165/6.25E-02 | 136/273/163/6.25E-02 |
| QDP | 10000 | 442/885/544/1.91E+00 | 430/861/523/1.89E+00 | 430/861/525/1.91E+00 |
| WOOD | 1000 | 189/416/240/7.81E-02 | 269/594/339/1.09E-01 | 146/344/219/6.25E-02 |
| WOOD | 10000 | 183/421/256/7.50E-01 | 248/528/308/1.00E+00 | 142/338/218/6.09E-01 |
| HIEBERT | 1000 | 79/257/197/4.69E-02 | 76/248/198/3.12E-02 | 100/368/306/6.25E-02 |
| HIEBERT | 10000 | 71/236/190/3.28E-01 | 75/251/197/3.59E-01 | 98/327/264/4.69E-01 |
| QF1 | 1000 | 189/379/190/6.25E-02 | 189/379/190/7.81E-02 | 189/379/190/6.25E-02 |
| QF1 | 10000 | 600/1201/601/1.95E+00 | 600/1201/601/2.03E+00 | 600/1201/601/2.03E+00 |
| QP1 | 1000 | 15/29/18/0.00E+00 | 14/29/17/0.00E+00 | 14/30/17/0.00E+00 |
| QP1 | 10000 | 17/35/21/7.81E-02 | 16/35/20/7.81E-02 | 17/36/21/7.81E-02 |
| QP2 | 1000 | 42/132/100/9.38E-02 | 55/163/124/1.09E-01 | 58/188/153/1.25E-01 |
| QP2 | 10000 | 39/133/103/9.06E-01 | 50/152/116/1.05E+00 | 43/136/105/9.38E-01 |
| QF2 | 1000 | 393/687/501/1.25E-01 | 394/688/503/1.41E-01 | 390/683/496/1.41E-01 |
| QF2 | 10000 | 1253/2167/1601/4.39E+00 | 1276/2191/1646/4.62E+00 | 1251/2164/1598/4.55E+00 |
| EP1 | 1000 | 3/7/5/0.00E+00 | 3/7/5/0.00E+00 | 3/7/5/0.00E+00 |
| EP1 | 10000 | 3/7/5/3.12E-02 | 3/7/5/3.12E-02 | 3/7/5/1.56E-02 |
| TRIDIAG2 | 1000 | 39/63/56/1.56E-02 | 37/61/52/1.56E-02 | 38/62/54/1.56E-02 |
| TRIDIAG2 | 10000 | 42/68/65/1.72E-01 | 39/61/62/1.72E-01 | 37/60/60/1.72E-01 |
| BDQRTIC | 1000 | 1006/1877/1947/6.72E-01 | 710/1264/1607/5.16E-01 | 539/1126/724/3.44E-01 |
| TRIDIA | 1000 | 356/713/357/1.41E-01 | 358/717/359/1.41E-01 | 358/717/359/1.41E-01 |
| TRIDIA | 10000 | 1175/2351/1176/4.45E+00 | 1176/2353/1177/4.59E+00 | 1177/2355/1178/4.61E+00 |
| ARWHEAD | 1000 | 12/28/19/1.56E-02 | 14/37/27/0.00E+00 | 9/22/16/1.56E-02 |
| ARWHEAD | 10000 | 10/22/15/4.69E-02 | 8/19/13/4.69E-02 | 9/24/18/4.69E-02 |
| NONDIA | 1000 | 13/29/21/1.56E-02 | 10/22/15/0.00E+00 | 12/26/17/1.56E-02 |
| NONDIA | 10000 | 11/37/32/6.25E-02 | 9/29/22/4.69E-02 | 11/30/23/4.69E-02 |
| NONDQUAR | 1000 | 16252/32520/17640/7.30E+00 | 12041/24093/12529/5.47E+00 | 6501/13096/8345/3.17E+00 |
| DQDRTIC | 1000 | 7/15/8/0.00E+00 | 7/15/8/0.00E+00 | 7/15/8/0.00E+00 |
| DQDRTIC | 10000 | 10/21/11/4.69E-02 | 7/15/8/1.56E-02 | 7/15/8/1.56E-02 |
| EG2 | 1000 | 125/285/229/1.72E-01 | 124/257/238/1.56E-01 | 35/99/96/6.25E-02 |
| EG2 | 10000 | 1979/3584/5207/2.95E+01 | 567/1172/1672/9.42E+00 | 141/353/261/1.89E+00 |
| DIXMAANA | 1000 | 9/19/10/1.56E-02 | 8/17/9/1.56E-02 | 9/19/10/0.00E+00 |
| DIXMAANA | 10000 | 9/19/10/9.38E-02 | 7/15/8/7.81E-02 | 8/17/9/9.38E-02 |
| DIXMAANB | 1000 | 21/55/35/3.12E-02 | 20/54/34/3.12E-02 | 22/58/38/1.56E-02 |
| DIXMAANB | 10000 | 22/57/38/2.97E-01 | 22/58/38/2.97E-01 | 22/59/39/2.97E-01 |
| DIXMAANC | 1000 | 30/81/54/4.69E-02 | 26/70/45/3.12E-02 | 26/70/46/3.12E-02 |
| DIXMAANC | 10000 | 31/83/55/4.06E-01 | 27/72/48/3.59E-01 | 24/66/43/3.28E-01 |
| DIXMAANE | 1000 | 176/339/191/1.88E-01 | 162/319/169/1.72E-01 | 171/328/187/1.72E-01 |
| DIXMAANE | 10000 | 465/917/486/4.69E+00 | 451/902/458/4.56E+00 | 468/925/494/4.80E+00 |
| PPQ | 1000 | 159/319/160/2.58E+00 | 160/321/161/2.61E+00 | 159/319/160/2.59E+00 |
| PPQ | 10000 | 28/57/33/5.21E+01 | 29/59/34/5.37E+01 | 29/59/34/5.37E+01 |
| BT | 1000 | 47/96/49/1.56E-02 | 47/95/48/1.56E-02 | 40/81/41/1.56E-02 |
| BT | 10000 | 37/75/38/1.72E-01 | 53/110/59/2.66E-01 | 41/83/42/2.03E-01 |
| APQ | 1000 | 189/379/190/6.25E-02 | 189/379/190/6.25E-02 | 189/379/190/6.25E-02 |
| APQ | 10000 | 600/1201/601/1.95E+00 | 600/1201/601/2.05E+00 | 600/1201/601/2.05E+00 |
| TPQ | 1000 | 177/355/178/7.81E-02 | 177/355/178/9.38E-02 | 177/355/178/7.81E-02 |
| TPQ | 10000 | 562/1125/563/2.48E+00 | 562/1125/563/2.56E+00 | 562/1125/563/2.56E+00 |
| EDENSCH | 1000 | 29/49/42/1.56E-02 | 27/46/39/1.56E-02 | 27/46/38/1.56E-02 |
| EDENSCH | 10000 | 27/47/42/1.41E-01 | 26/43/38/1.41E-01 | 26/44/40/1.41E-01 |
| VARDIM | 1000 | 36/74/38/1.56E-02 | 36/73/38/1.56E-02 | 38/78/41/1.56E-02 |
| VARDIM | 10000 | 46/93/47/1.72E-01 | 46/93/47/1.88E-01 | 46/93/47/1.88E-01 |
| S1 | 1000 | 2000/4001/2002/7.19E-01 | 2000/4001/2002/7.50E-01 | 1999/3999/2001/7.50E-01 |
| LIARWHD | 1000 | 20/42/27/0.00E+00 | 24/52/37/1.56E-02 | 24/53/36/0.00E+00 |
| LIARWHD | 10000 | 26/62/43/1.25E-01 | 25/56/36/1.09E-01 | 25/55/36/1.25E-01 |
| DIAGONAL6 | 1000 | 5/11/6/0.00E+00 | 5/11/6/0.00E+00 | 5/11/6/0.00E+00 |
| DIAGONAL6 | 10000 | 5/11/6/4.69E-02 | 5/11/6/4.69E-02 | 5/11/6/6.25E-02 |
| DIXON3DQ | 1000 | 1989/3979/1991/7.03E-01 | 1993/3987/1995/7.19E-01 | 1993/3987/1995/7.03E-01 |
| DIXMAANF | 1000 | 227/484/295/2.50E-01 | 338/680/401/3.59E-01 | 206/436/260/2.34E-01 |
| DIXMAANF | 10000 | 697/1402/903/7.69E+00 | 1384/2912/1614/1.51E+01 | 555/1165/702/6.23E+00 |
| DIXMAANG | 1000 | 234/478/270/2.66E-01 | 275/567/304/3.28E-01 | 227/464/255/2.50E-01 |

Continued on next page

**Table 1 – continued from previous page**

| Problem | N | CG-DESCENT<br>iter/fn/gn/time | MPRP<br>iter/fn/gn/time | HZPR<br>iter/fn/gn/time |
|---|---|---|---|---|
| DIXMAANG | 10000 | 568/1129/631/6.17E+00 | 1186/2396/1247/1.28E+01 | 691/1371/772/7.61E+00 |
| DIXMAANH | 1000 | 259/562/343/3.12E-01 | 35/145/124/7.81E-02 | 240/535/332/2.97E-01 |
| DIXMAANH | 10000 | 61/206/184/1.22E+00 | 61/195/174/1.16E+00 | 758/1576/957/8.47E+00 |
| DIXMAANI | 1000 | 157/313/160/1.56E-01 | 160/317/165/1.56E-01 | 157/314/159/1.72E-01 |
| DIXMAANI | 10000 | 469/919/497/4.78E+00 | 444/891/448/4.53E+00 | 458/912/474/4.70E+00 |
| DIXMAANJ | 1000 | 215/454/277/2.50E-01 | 259/522/307/2.81E-01 | 245/509/304/2.66E-01 |
| DIXMAANJ | 10000 | 699/1468/946/7.97E+00 | 827/1699/934/8.81E+00 | 650/1354/842/7.34E+00 |
| DIXMAANK | 1000 | 231/486/284/2.81E-01 | 212/456/264/2.66E-01 | 233/497/298/2.81E-01 |
| DIXMAANK | 10000 | 653/1329/770/7.59E+00 | 1593/3180/1735/1.80E+01 | 612/1258/746/7.31E+00 |
| DIXMAANL | 1000 | 5903/11878/7483/8.34E+00 | 14912/31009/16324/5.14E+01 | 5002/10585/5723/6.97E+00 |
| ENGVAL1 | 1000 | 29/52/39/1.56E-02 | 30/53/41/1.56E-02 | 27/48/36/1.56E-02 |
| ENGVAL1 | 10000 | 26/44/38/1.25E-01 | 28/50/40/1.41E-01 | 25/45/36/1.25E-01 |
| FLETCHCR | 1000 | 2946/6017/3073/1.33E+00 | 2926/6022/3100/1.36E+00 | 2930/6037/3108/1.38E+00 |
| COSINE | 1000 | 13/29/26/3.12E-02 | 11/26/23/1.56E-02 | 11/27/22/1.56E-02 |
| COSINE | 10000 | 13/33/30/1.88E-01 | 12/28/27/1.72E-01 | 12/28/26/1.56E-01 |
| DENSCHNB | 1000 | 8/17/9/0.00E+00 | 6/13/7/0.00E+00 | 8/17/9/1.56E-02 |
| DENSCHNB | 10000 | 8/17/9/3.12E-02 | 6/13/7/1.56E-02 | 8/17/9/3.12E-02 |
| DENSCHNF | 1000 | 30/71/56/3.12E-02 | 21/50/40/1.56E-02 | 22/53/43/1.56E-02 |
| DENSCHNF | 10000 | 26/63/52/1.56E-01 | 21/50/40/1.25E-01 | 22/53/43/1.41E-01 |
| SINQUAD | 1000 | 551/1223/761/7.97E-01 | 1483/3175/1880 /2.05E+00 | 290/772/570/5.47E-01 |
| SINQUAD | 10000 | 2203/4763/2770/3.03E+01 | 3982/8502/4903/5.41E+01 | 3745/9359/6538/6.44E+01 |
| BIGGSB1 | 1000 | 500/1001/501/1.72E-01 | 500/1001/501/1.72E-01 | 500/1001/501/1.88E-01 |
| BIGGSB1 | 10000 | 5000/10001/5001/1.71E+01 | 5001/10003/5003/1.77E+01 | 5000/10001/5001/1.78E+01 |
| PPQ2 | 1000 | 0/1/1/1.56E-02 | 0/1/1/0.00E+00 | 0/1/1/1.56E-02 |
| PPQ2 | 10000 | 0/1/1/1.17E+00 | 0/1/1/1.16E+00 | 0/1/1/1.16E+00 |
| SQ2 | 1000 | 53/107/54/1.56E-02 | 53/107/54/3.12E-02 | 53/107/54/1.56E-02 |
| SQ2 | 10000 | 177/355/178/5.78E-01 | 177/355/178/6.09E-01 | 177/355/178/5.94E-01 |
| GENROSE | 1000 | 6457/13308/7001/2.80E+00 | 4238/8498/4296/1.84E+00 | 6554/13334/6906/2.89E+00 |
| NONDIA | 1000 | 3336/7054/4709/1.42E+00 | 3041/6327/4693/1.36E+00 | 109/232/143/4.69E-02 |
| PENALTY1 | 1000 | 14/29/15/0.00E+00 | 14/29/15/0.00E+00 | 14/29/15/0.00E+00 |
| PENALTY1 | 10000 | 16/33/17/7.81E-02 | 16/33/17/6.25E-02 | 16/33/17/6.25E-02 |
| POWER | 1000 | 10456/20913/10457/3.62E+00 | 10454/20909/10455/3.77E+00 | 10457/20915/10458/3.75E+00 |
| FREUOTH | 1000 | 120/233/157/7.81E-02 | 69/130/105/4.69E-02 | 48/100/76/3.12E-02 |
| FREUOTH | 10000 | 114/207/185/8.28E-01 | 52/104/89/3.91E-01 | 53/101/94/4.22E-01 |
| SROSENBR | 1000 | 32/96/72/1.56E-02 | 36/88/62/1.56E-02 | 42/129/100/1.56E-02 |
| SROSENBR | 10000 | 36/101/77/1.72E-01 | 38/92/65/1.72E-01 | 43/138/111/2.34E-01 |
| WOODS | 1000 | 369/784/428/1.56E-01 | 332/711/394/1.41E-01 | 309/659/364/1.41E-01 |
| WOODS | 10000 | 211/466/272/9.06E-01 | 263/564/311/1.14E+00 | 138/321/200/6.41E-01 |
| DQRTIC | 1000 | 31/63/32/1.56E-02 | 31/63/32/1.56E-02 | 31/63/32/1.56E-02 |
| DQRTIC | 10000 | 37/75/38/1.41E-01 | 36/73/37/1.41E-01 | 36/73/37/1.41E-01 |
| NONCVXU2 | 1000 | 2529/4386/3203/4.14E+00 | 2449/4237/3114/4.05E+00 | 2661/4408/3579/4.52E+00 |
| BROYDN7D | 1000 | 1007/2304/1333/8.75E-01 | 985/2321/ 1366/8.91E-01 | 544/1290/768/5.00E-01 |
| BROWNAL | 1000 | 6/19/19/0.00E+00 | 6/17/17/0.00E+00 | 6/17/17/0.00E+00 |
| BROWNAL | 10000 | 7/22/22/9.38E-02 | 7/19/19/9.38E-02 | 7/19/19/9.38E-02 |
| GENHUMPS | 1000 | 778/1658/892/2.11E+00 | 654/1407/764/ 1.94E+00 | 655/1438/805/1.89E+00 |
| BDEXP | 1000 | 14/29/15/4.69E-02 | 14/29/15/4.69E-02 | 14/29/15/4.69E-02 |
| BDEXP | 10000 | 14/29/15/3.59E-01 | 14/29/15/3.91E-01 | 14/29/15/3.75E-01 |

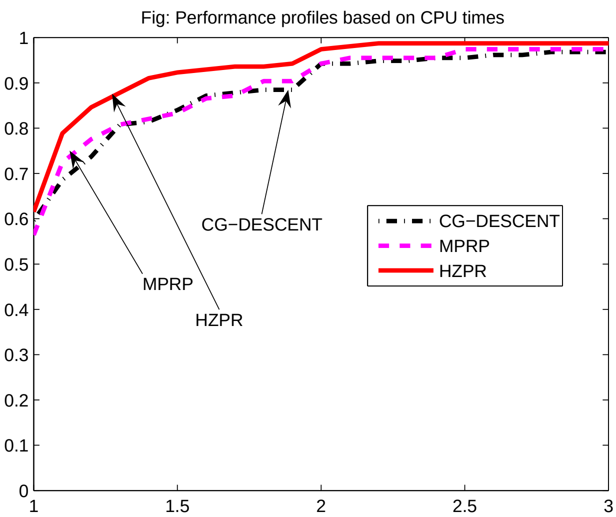


Figure 1: Performance profiles based on CPU time

From the results of our numerical experiments, we can see that the HZPR method performs better than the CG-DESCENT method and the MPRP method, which implies

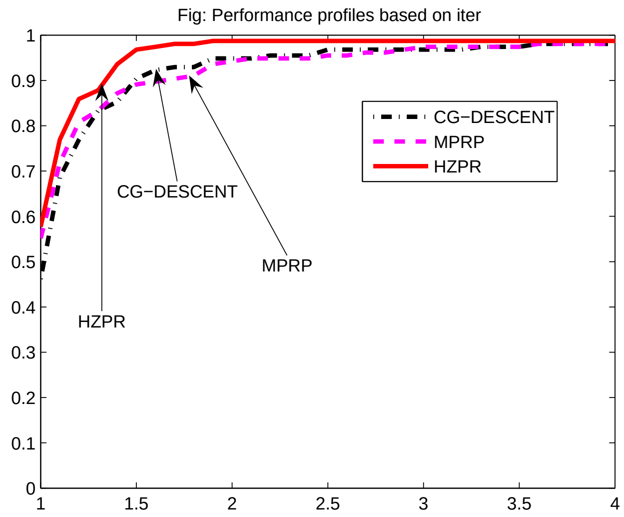


Figure 2: Performance profiles based on iterations

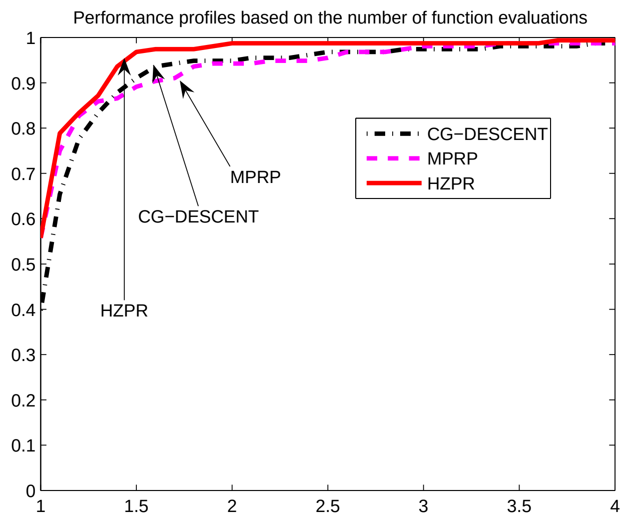


Figure 3: Performance profiles for the number of function evaluations

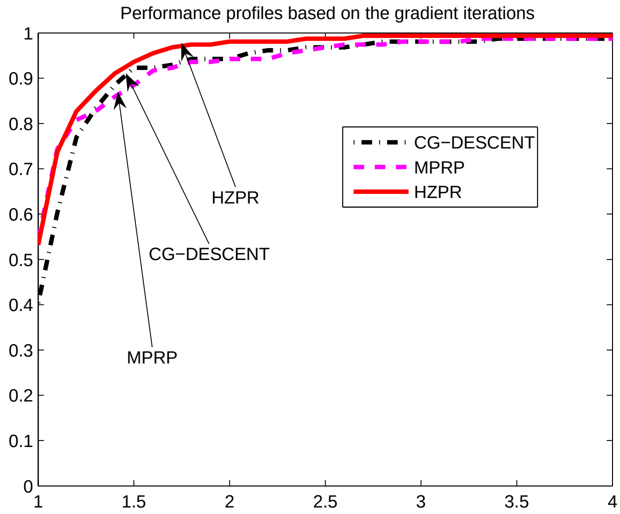


Figure 4: Performance profiles for the number of gradient evaluations

that the HZPR method is efficient in real computation.

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