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AN SQP AUGMENTED LAGRANGIAN METHOD FOR TWO CLASSES OF NONLINEAR SEMIDEFINITE PROGRAMMING PROBLEMS ARISING IN DISCRETE-TIME FEEDBACK CONTROL

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Abstract: In this article, two nonlinear semi-definite programming problems resulting from the linear-quadratic control problem for discrete-time systems are considered. A sequential quadratic programming augmented Lagrangian method is introduced for solving the two problems. Some properties that the two problems share are discussed. The method is tested on several test problems that demonstrate the applicability of the considered approach for solving this problem class.

Key words: *discrete-time output feedback control, nonlinear semi-definite programming, sequential quadratic programming, trust region methods*

Mathematics Subject Classification: *90C22, 90C30, 93D15, 49N10, 49N35, 65K05*

1 Introduction

In this article, we consider nonlinear semi-definite programming (NSDP) problems of the following form:

$$\min \quad J(X) \qquad \text{s. t.} \quad h(X) = 0,\ g(X) \succ 0, \tag{1.1}$$

where $J : \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times r} \to \mathbb{R}$, $h : \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times r} \to \mathbb{R}^{n\times n}$, $g : \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times r} \to \mathbb{R}^{n\times n}$ are sufficiently smooth matrix functions. This problem is assumed to be nonlinear and generally non-convex.

For more than a decade NSDP has attracted many authors in the optimization community. An interior-point method for non-convex semi-definite programs was introduced by Jarre [5]. Leibfritz and Mostafa [10] proposed an interior-point trust region method for solving special class of NSDP problems resulting from the continuous-time output feedback design problem. Kočvara et al. [7] considered an augmented Lagrangian method for solving similar problem. Sun et al. [24] investigated the rate of convergence of the augmented Lagrangian approach for solving nonlinear semi-definite program. Correa et al. [1] proposed sequential semi-definite programming method for another class of NSDP. Yamashita and Yabe [27] studied local and superlinear convergence of a primal-dual interior point method for NSDP. Freund et al. [2] proposed sequential semi-definite programming approach for solving nonlinear program with nonlinear semi-definite constraints. Xiao et al. [26] considered an inverse problem raised from the semidefinite quadratic programming, where an augmented Lagrangian method was studied for solving that problem. Li and Zhang [11]

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presented a nonlinear Lagrangian method generated by a Löwner operator associated with Log-Sigmoid function for nonconvex semidefinite programming. Mostafa [20] introduced an SQP method globalized using line search for solving certain NSDP problem originating from discrete-time output feedback control systems.

Optimal control problems or design optimization problems are characterized from general nonlinear optimization problems by the fact that the unknown variable is partitioned into two components, one represents the state variable and the other represents the control or the design variable. In this work we consider two NSDPs which are derived from the static output feedback design problem for discrete-time systems. The two NSDP problems are related to each other and have several properties in common. We attempt to highlight the similarities and differences between the two problems. Moreover, we compare the performance of the proposed SQP augmented Lagrangian method globalized by using trust region for the two problems numerically. In order to see the significance of the proposed SQP augmented Lagrangian approach we compare this method numerically with Newton's method globalized by line search [15] applied on particular formulation of the design problem.

This article is organized as follows. In the next section the output feedback design problem is stated and its relationship with the considered NSDP problems are discussed. In Section 3 first and second-order derivatives of the problems functions are obtained and the optimality conditions are discussed. The augmented Lagrangian SQP method globalized by using trust region is introduced in Section 4. Section 5 is devoted to test the performance of the augmented Lagrangian SQP trust method on the considered NSDP problems numerically through several test problems from the literature. Finally we end by a conclusion.

Notations: Throughout the paper $\|\cdot\|$ denotes the Frobenius norm defined as $\|G\| = \sqrt{\langle G, G\rangle}$, where $\langle\cdot,\cdot\rangle$ is the inner product defined by $\langle G_1, G_2\rangle = \mathrm{Tr}(G_1^T G_2)$ for $G_1, G_2 \in \mathbb{R}^{m\times n}$ and $\mathrm{Tr}\,(\cdot)$ is the trace operator. For matrix $G \in \mathbb{R}^{n\times n}$ the notations $G \succ 0$ and $G \succeq 0$ denote that G is strictly positive definite and positive semi-definite, respectively. For function $f(X)$ the notations $f_X(X)\Delta X$ and $f_{XX}(X)(\Delta X, \Delta X)$ denote the first and second order directional derivatives of f at X in the direction of ΔX.

2 Output Feedback Design and NSDP Formulation

An important class of nonlinear semi-definite programming problems can be derived from the static output feedback (SOF) design problem for continuous or discrete-time systems. The SOF design problem for discrete-time systems is stated as follows; for the SOF problem we refer to the two surveys [15,25] and later references among them [12–14,17–19,23]. Consider the linear time-invariant control system with the following state-space realization:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned} \tag{2.1}$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ and $y_k \in \mathbb{R}^{n_y}$ denote the state, the control input and the measured output vectors, respectively. Furthermore, A, B, C are given constant matrices of appropriate dimensions.

The following assumption is often imposed on the initial state vector to remove the dependency of the problem on x_0.

Assumption 2.1. Assume that x_0 is a random variable uniformly distributed on the unit sphere with $E[x_0] = 0$, where $E[\cdot]$ is the expected value.

The following SOF control law is often used to close the system:

$$u_k = Fy_k, \tag{2.2}$$

where $F \in \mathbb{R}^{n_u \times n_y}$ denotes the unknown SOF gain matrix that has to be determined by a suitable numerical procedure. By substituting the control law (2.2) into the control system (2.1) the closed loop counterpart has the following form:

$$x_k = A(F)x_{k-1} = A(F)^k\, x_0, \tag{2.3}$$

where $A(F) := A + BFC$ is the augmented closed-loop matrix.

For an objective function let us consider the following quadratic cost function, which is to be minimized subject to the system (2.1):

$$J(F) = E\Big[\sum_{k=0}^{\infty}\big(x_k^T Q x_k + u_k^T R u_k\big)\Big], \tag{2.4}$$

where $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ are given constant weight matrices.

Assumption 2.2. We assume that $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$ are given constant matrices; $Q \in \mathbb{R}^{n_x \times n_x}$, $R \in \mathbb{R}^{n_u \times n_u}$ and $V \in \mathbb{R}^{n_x \times n_x}$ are given symmetric and positive definite matrices.

The above optimal control problem can be restated as an optimization problem. It is well-known approach in systems and control literature to formulate the above optimal control problem as a matrix optimization problem; see e.g. [15]. We give the derivation for completeness. By substituting x_k of (2.3) into the control law $u_k = Fy_k = FCx_k$ and then substituting both of x_k and u_k into the objective function (2.4) we obtain

$$J(F) = E[x_0^T K(F) x_0] = \operatorname{Tr}\,(K(F)E[x_0 x_0^T]) = \operatorname{Tr}\,(K(F)V),$$

where

$$K(F) = \sum_{k=0}^{\infty}(A(F)^T)^k\, Q(F)\, A(F)^k$$

is the exact solution of the following discrete Lyapunov equation

$$K(F) = A(F)^T K(F) A(F) + Q(F), \tag{2.5}$$

where $Q(F) := Q + C^T F^T R F C$. Moreover, $V = E[x_0 x_0^T] \in \mathbb{R}^{n_x \times n_x}$ is the covariance matrix which is assumed to be positive definite; see [15]. In order to have unique solution for (2.5) one must choose F from the following set of stabilizing output feedback gains:

$$\mathcal{D}_s = \{F \in \mathbb{R}^{n_u \times n_y} : \rho(A(F)) < 1\}, \tag{2.6}$$

where $\rho(\cdot)$ is the spectral radius.

Consequently, the static output feedback problem can be written as follows:

$$\text{(P1)} \qquad \min_{F \in \mathcal{D}_s} \quad \operatorname{Tr}(K(F)V),$$

where F is considered as an independent variable and $K(F)$ solves the discrete Lyapunov equation (2.5).

An equivalent formulation of P1 is the following optimization problem; see below:

$$\text{(P2)} \qquad \min_{F \in \mathcal{D}_s} \quad \operatorname{Tr}(L(F)Q(F)),$$

where $L(F)$ solves the discrete Lyapunov equation:

$$L(F) = A(F)L(F)A(F)^T + V. \tag{2.7}$$

The two problems P1 and P2 have been studied by many authors; see e.g. the above mentioned citations. The objective functions of the problems P1 and P2 are equal as shown in the following lemma.

Lemma 2.3. *Let $F \in \mathcal{D}_s$ and let $K(F)$ and $L(F)$ be solutions of the discrete Lyapunov equations* (2.5) *and* (2.7), *respectively. Then*

$$\mathrm{Tr}(L(F)Q(F)) = \mathrm{Tr}(K(F)V).$$

Proof. Let us rewrite (2.5) and (2.7) as:

$$Q(F) = K(F) - A(F)^T K(F)A(F)$$
$$V = L(F) - A(F)L(F)A(F)^T,$$

where $K(F)$ and $L(F)$ solutions of the discrete Lyapunov equations (2.5) and (2.7) are symmetric. Pre-multiplying the first equation by $L(F)$ and the second one by $K(F)$ then applying the trace operator give:

$$\begin{aligned}\mathrm{Tr}(L(F)Q(F)) &= \mathrm{Tr}(L(F)K(F) - L(F)A(F)^T K(F)A(F)) \\ &= \mathrm{Tr}(L(F)K(F)) - \mathrm{Tr}(L(F)A(F)^T K(F)A(F)) \\ &= \mathrm{Tr}(K(F)L(F)) - \mathrm{Tr}(K(F)A(F)L(F)A(F)^T) \\ &= \mathrm{Tr}(K(F)V),\end{aligned}$$

where the property $\mathrm{Tr}(M_1M_2) = \mathrm{Tr}(M_2M_1)$ is used for any two matrices M_1, M_2 of appropriate dimensions. □

Note that the objective functions of P1 and P2 are shown to be equal under the assumption that $K(F)$ and $L(F)$ are solutions of discrete Lyapunov equations.

The following theorem provides an equivalence between the asymptotic stability of the control system (2.1) and coupled conditions on positive definiteness; see e.g. [4, Theorem 4.1.3] for the proof.

Theorem 2.4 ([4, Theorem 4.1.3])**.** *Let $A \in \mathbb{R}^{n_x\times n_x}, B \in \mathbb{R}^{n_x\times n_u}, C \in \mathbb{R}^{n_y\times n_x}$ be given matrices. Then an $F \in \mathcal{D}_s$ exists if and only if there exist $P \in \mathbb{R}^{n_x\times n_x}$ and $F \in \mathbb{R}^{n_u\times n_y}$ such that $(P, F) \in \mathcal{F}_s$, where*

$$\mathcal{F}_s = \{(P, F) : P \succ 0,\ P - A(F)^T PA(F) \succ 0\}. \tag{2.8}$$

According to Theorem 2.4 and by considering K and F for the problem P1, respectively, L and F for the problem P2 to be independent variables then we have the following couple of NSDP problems:

$$(\text{P3}) : \begin{cases} \min\limits_{K,F} & J(K) = \mathrm{Tr}(KV) \\ \text{s. t.} & h(K, F) = 0,\ K \succ 0,\ g(K, F) \succ 0. \end{cases} \tag{2.9}$$

and

$$(\text{P4}): \begin{cases} \min\limits_{L,F} & J(L,F) = \text{Tr}(LQ(F)) \\ \text{s. t.} & \widetilde{h}(L,F) = 0,\ L \succ 0,\ \widetilde{g}(L,F) \succ 0. \end{cases} \tag{2.10}$$

where $h, \widetilde{h}, g, \widetilde{g}\ :\ \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_u \times n_y} \to \mathbb{R}^{n_x \times n_x}$ are nonlinear matrix functions defined by:

$$h(K,F) = A(F)^T K A(F) - K + Q(F) \tag{2.11}$$

$$g(K,F) = K - A(F)^T K A(F) \tag{2.12}$$

$$\widetilde{h}(L,F) = A(F) L A(F)^T - L + V \tag{2.13}$$

$$\widetilde{g}(L,F) = L - A(F) L A(F)^T, \tag{2.14}$$

where $A(F)$ and $Q(F)$ are as defined above.

Obviously the NSDPs P3 and P4 are generalizations of the problems P1 and P2, respectively. On the other hand, the equivalence between the stability condition (2.6) and the positive definite constraints in (2.8) provides a convenient way to fulfill such constraints in the numerical algorithm. For example, we might follow one of the following two alternatives to handle the positive definite constraints. First, we can equivalently solve P3 or P4 as an equality constrained problem in an interior-point framework as considered in [10]; e.g. the NSDP P4 can be reformulated as:

$$\min_{L,F} \quad \phi^\mu(L,F) = J(L,F) - \mu\big[\log(\det(L)) + \log(\det(\widetilde{g}(L,F)))\big] \quad \text{s.t.} \quad \widetilde{h}(L,F) = 0, \tag{2.15}$$

where $\mu > 0$ is given barrier parameter. The second alternative is to replace the positive definite constraints (2.8) by the stability condition (2.6) and fulfill such condition within the numerical algorithm explicitly; see e.g. [19].

3 Derivatives and Optimality Conditions

In order to observe the relationship between both NSDP problems P3 and P4 more we will compare the resulting systems of the Karush–Kuhn–Tucker (KKT) necessary optimality conditions for both problems. First, let us define the Lagrangian functions associated with the two problems:

$$\ell(K,F,U,Z_1,Z_2) = \text{Tr}(KV) + \text{Tr}(U^T h(K,F)) + \text{Tr}(Z_1^T K) + \text{Tr}(Z_2^T g(K,F)), \tag{3.1}$$

$$\begin{aligned} \widetilde{\ell}(L,F,\widetilde{U},\widetilde{Z}_1,\widetilde{Z}_2) &= \text{Tr}(L(Q + C^T F^T R F C)) + \text{Tr}(\widetilde{U}^T \widetilde{h}(L,F)) \\ &\quad + \text{Tr}(\widetilde{Z}_1^T L) + \text{Tr}(\widetilde{Z}_2^T \widetilde{g}(L,F)). \end{aligned} \tag{3.2}$$

where $U, Z_1, Z_2, \widetilde{U}, \widetilde{Z}_1, \widetilde{Z}_2 \in \mathbb{R}^{n_x \times n_x}$ are the Lagrange multipliers matrices associated with the constraints of the two problems.

Derivatives of ℓ and h are obtained in the next lemma. Similarly one can obtain the derivatives of $\widetilde{\ell}$ and $\widetilde{h}$.

Lemma 3.1. *Consider the NSDP problem P3. The objective and the constraint functions are twice continuously differentiable. Moreover, the first-order directional derivatives of the*

Lagrangian function (3.1) at (K, F, U, Z_1, Z_2) *applied in the directions* $\Delta K \in \mathbb{R}^{n_x \times n_x}$ *and* $\Delta F \in \mathbb{R}^{n_u \times n_y}$ *are given by:*

$$\ell_K(\cdot)\Delta K = \mathrm{Tr}\big(\Delta K(V + Z_1 + Z_2 - U + A(F)(U - Z_2)A(F)^T)\big)$$

$$\ell_F(\cdot)\Delta F = 2\,\mathrm{Tr}\big((B^T KA(F)(U - Z_2) + RFCU)C^T \Delta F^T\big)$$

and the second-order directional derivatives of ℓ *are given by:*

$$\begin{aligned}
\ell_{KK}(\cdot)(\Delta K, \Delta K) &= 0 \\
\ell_{KF}(\cdot)(\Delta K, \Delta F) &= 2\,\mathrm{Tr}(B^T \Delta KA(F)(U - Z_2)C^T \Delta F^T) \\
&= \ell_{FK}(\cdot)(\Delta F, \Delta K) \\
\ell_{FF}(\cdot)(\Delta F, \Delta F) &= 2\,\mathrm{Tr}((B^T KB\Delta FC(U - Z_2) + R\Delta FCU)C^T \Delta F^T).
\end{aligned}$$

Moreover, the directional derivatives of h *are:*

$$\begin{aligned}
h_K(\cdot)\Delta K &= A(F)^T \Delta KA(F) - \Delta K \\
h_F(\cdot)\Delta F &= C^T \Delta F^T (B^T KA(F) + RFC) + (B^T KA(F) + RFC)^T \Delta FC. \qquad (3.3)
\end{aligned}$$

Proof. By using directional derivatives and differentiating ℓ and h with respect to their arguments the above derivatives are obtained. □

The next theorem states the KKT optimality conditions of the problem P3.

Theorem 3.2 (See e.g. [2, Sec. 5])**.** *Suppose that* $(\bar{K}, \bar{F})$ *is a local minimizer for the NSDP problem* (2.9) *and let* (2.9) *be regular at* $(\bar{K}, \bar{F})$. *Then there exist Lagrange multipliers* $\bar{U}, \bar{Z}_1, \bar{Z}_2 \in \mathbb{R}^{n_x \times n_x}$ *such that at* $(\bar{K}, \bar{F}, \bar{U}, \bar{Z}_1, \bar{Z}_2)$ *the following conditions are satisfied:*

$$A(\bar{F})(\bar{U} - \bar{Z}_2)A(\bar{F})^T - \bar{U} + \bar{Z}_2 + \bar{Z}_1 + V = 0 \qquad (3.4)$$

$$2\,(B^T \bar{K} A(\bar{F})(\bar{U} - \bar{Z}_2) + R\bar{F}C\bar{U})C^T = 0 \qquad (3.5)$$

$$A(\bar{F})^T \bar{K} A(\bar{F}) - \bar{K} + Q(\bar{F}) = 0 \qquad (3.6)$$

$$\mathrm{Tr}(\bar{K}^T \bar{Z}_1) = 0,\ \mathrm{Tr}(g(\bar{K}, \bar{F})^T \bar{Z}_2) = 0 \qquad (3.7)$$

$$\bar{K} \succ 0,\ g(\bar{K}, \bar{F}) \succ 0,\ \bar{Z}_1 \succeq 0, \bar{Z}_2 \succeq 0. \qquad (3.8)$$

Proof. From Lemma 3.1 and by applying the KKT conditions on the derivatives ℓ_K, ℓ_F and ℓ_U then (3.4)–(3.6) are, respectively, obtained. Moreover, (3.7) represents the complementarity condition and (3.8) represents feasibility with respect to the semi-definite constraints. □

Similarly for the problem P4 we have the following result.

Lemma 3.3. *Let* $(\bar{L}, \bar{F})$ *be a local minimizer for the NSDP problem P4 and let P4 be regular at* $(\bar{L}, \bar{F})$. *Then there exist Lagrange multipliers* $\widetilde{U}, \widetilde{Z}_1, \widetilde{Z}_2 \in \mathbb{R}^{n_x \times n_x}$ *such that at*

$(\bar{L}, \bar{F}, \widetilde{U}, \widetilde{Z}_1, \widetilde{Z}_2)$ *it holds that:*

$$A(\bar{F})^T(\widetilde{U} - \widetilde{Z}_2)A(\bar{F}) - \widetilde{U} + \widetilde{Z}_2 + \widetilde{Z}_1 + Q(\bar{F}) = 0 \tag{3.9}$$

$$2\,(B^T(\widetilde{U} - \widetilde{Z}_2)A(\bar{F}) + R\bar{F}C)\bar{L}C^T = 0 \tag{3.10}$$

$$A(\bar{F})\bar{L}A(\bar{F})^T - \bar{L} + V = 0 \tag{3.11}$$

$$\mathrm{Tr}(\bar{L}^T\widetilde{Z}_1) = 0,\ \ \mathrm{Tr}(\widetilde{g}(\bar{L}, \bar{F})^T\widetilde{Z}_2) = 0 \tag{3.12}$$

$$\bar{L} \succ 0,\ \ \widetilde{g}(\bar{L}, \bar{F}) \succ 0,\ \ \widetilde{Z}_1 \succeq 0,\ \ \widetilde{Z}_2 \succeq 0. \tag{3.13}$$

Observe that, if we replace the positive definite constraints by the stability condition $F \in \mathcal{D}_s$ and assume that $\bar{F} \in \mathcal{D}_s$ then the KKT system reduces to the first three equations (3.4)–(3.6) for the problem P3 and (3.9)–(3.11) for the problem P4. Moreover, the two systems coincide and in this case $\bar{L}$ plays the role of the dual variable in the first system while $\bar{K}$ is the dual variable for the second system.

4 Augmented Lagrangian SQP Trust Region Method

We consider an SQP augmented Lagrangian method globalized by trust region for solving P3 and P4. This method is close to the method ALSQP introduced in [16] for solving different NSDP originating from the $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem for continuous-time systems. In order to simplify the presentation let us define the Lagrangian function associated with the equality constraint of the problem (1.1) by:

$$\ell(X, U) = J(X) + Tr(U^T h(X)), \tag{4.1}$$

where $U \in \mathbb{R}^{n\times n}$ is the associated Lagrange multiplier. Moreover, the augmented Lagrangian function is defined by:

$$\mathcal{L}^\sigma(X, U) = \ell(X, U) + \frac{\sigma}{2}\|h(X)\|^2, \tag{4.2}$$

where $\sigma \geq 0$ is the penalty parameter.

First and second-order directional derivatives of $\mathcal{L}^\sigma$ can be obtained from the derivatives of ℓ using the fact that:

$$\mathcal{L}^\sigma_X(X, U)\Delta X = \ell_X(X, U + \sigma h(X))\Delta X$$

and

$$\mathcal{L}^\sigma_{XX}(X, U)(\Delta X, \Delta X) = J_{XX}(X)(\Delta X, \Delta X) + \mathrm{Tr}((U + \sigma h)^T h_{XX}(\Delta X, \Delta X)) + \sigma\mathrm{Tr}(h_X \Delta X^T h_X \Delta X).$$

The SQP augmented Lagrangian trust region method solves successively quadratic program of the following form:

$$\min_{\Delta X} \quad q_k(\Delta X) \quad \text{s.t.} \quad h_X(X_k)\Delta X + h(X_k) = 0, \quad \|\Delta X\| \leq \delta_k, \quad X_k + \Delta X \in \mathcal{F}_s, \tag{4.3}$$

where $\delta_k > 0$ is the trust region radius at iteration k, $X_k \in \mathbb{R}^{n_x\times n_x} \times \mathbb{R}^{n_u\times n_y}$ is given and $q_k(\Delta X)$ is the quadratic model of the augmented Lagrangian function (4.2) which takes the following form:

$$q_k(\Delta X) = \mathcal{L}^\sigma(X_k, U_k) + \mathcal{L}^\sigma{}_X(X_k, U_k)\Delta X + \frac{1}{2}\mathcal{L}^\sigma{}_{XX}(X_k, U_k)(\Delta X, \Delta X). \tag{4.4}$$

In particular, by setting $X = (K, F)$ the constrained trust region problem corresponding to P3 takes the following form:

$$\min_{(\Delta K,\Delta F)} q_k(\Delta K, \Delta F) \quad \text{s.t.} \quad h_K(K_k, F_k)\Delta K + h_F(K_k, F_k)\Delta F + h(K_k, F_k) = 0,$$
$$\|(\Delta K, \Delta F)\| \le \delta_k,\ (K_k + \Delta K, F_k + \Delta F) \in \mathcal{F}_s, \tag{4.5}$$

where $K_k \in \mathbb{R}^{n_x\times n_x}$ and $F_k \in \mathbb{R}^{n_u\times n_y}$ are given and $\delta_k > 0$ is the trust region radius at iteration k.

Similarly by taking $X = (L, F)$ the constrained trust region problem corresponding to the NSDP problem P4 will be:

$$\min_{(\Delta L,\Delta F)} \widetilde{q}_k(\Delta L, \Delta F) \quad \text{s.t.} \quad \widetilde{h}_L(L_k, F_k)\Delta L + \widetilde{h}_F(L_k, F_k)\Delta F + \widetilde{h}(L_k, F_k) = 0,$$
$$\|(\Delta L, \Delta F)\| \le \delta_k,\ (L_k + \Delta L, F_k + \Delta F) \in \mathcal{F}_s, \tag{4.6}$$

where $L_k \in \mathbb{R}^{n_x\times n_x}$ and $F_k \in \mathbb{R}^{n_u\times n_y}$ are given.

Since any of the above trust region problems may not have a solution when the current iterate is infeasible due to inconsistent constraints; see e.g. [21], we may avoid this situation by using the tangent space strategy with step decomposition. The trust region method presented in [19] follows this strategy. We apply the algorithm given there for solving the two trust region problems (4.5) and (4.6), but with replacing the regular Lagrangian function by the augmented Lagrangian.

Throughout this section the following assumptions hold:

Assumption 4.1. We assume that:

(i) the functions J, h, $\widetilde{h}$, g, and $\widetilde{g}$ are twice continuously differentiable.

(ii) there exist strict feasible point $X_0 = (K_0, F_0)$ for P3 or $X_0 = (L_0, F_0)$ for P4.

(iii) the mappings h_K and $\widetilde{h}_L$ are invertible for given $X = (K, F)$ and $X = (L, F)$, respectively.

Based on the above assumption where h_K^{-1} and $\widetilde{h}_L^{-1}$ are assumed to exist, we have the following result.

Lemma 4.2. *Let $(X, U) \in \mathbb{R}^{n_x\times n_x} \times \mathbb{R}^{n_u\times n_y} \times \mathbb{R}^{n_x\times n_x}$ and $\Delta X \in \mathbb{R}^{n_x\times n_x} \times \mathbb{R}^{n_u\times n_y}$ be given. The range space of the operators T_1 and T_2, defined below, coincide with the null spaces of the Jacobian matrices ∇h^T and $\nabla\widetilde{h}^T$:*

$$T_1(K, F) = (-h_K^{-1}(K, F)h_F(K, F), \mathcal{I}),\ T_2(L, F) = (-\widetilde{h}_L^{-1}(L, F)\widetilde{h}_F(L, F), \mathcal{I}), \tag{4.7}$$

where $\mathcal{I}$ is the identity operator.

Proof. The operators $\nabla_K h(K, F)$ and $\nabla_L \widetilde{h}(L, F)$ are linear as well as bijective. Then both are invertible and consequently the linearized equality constraints of the QP subproblems (4.5) and (4.6) imply that:

$$\Delta K = -\nabla_K h^{-1}(K, F)\nabla_F h(K, F)\Delta F - \nabla_K h^{-1}(K, F)h(K, F)$$

$$\Delta L = -\nabla_L \widetilde{h}^{-1}(L, F)\nabla_F \widetilde{h}(L, F)\Delta F - \nabla_L \widetilde{h}^{-1}(L, F)\widetilde{h}(L, F).$$

This leads to the following decomposition of the steps:

$$\Delta X = (\Delta K, \Delta F) = T_1(K,F)\Delta F + N_1(K,F), \tag{4.8}$$

$$\Delta X = (\Delta L, \Delta F) = T_2(L,F)\Delta F + N_2(L,F), \tag{4.9}$$

where

$$N_1(K,F) = (-h_K^{-1}(K,F)h(K,F), 0),\ N_2(L,F) = (-\widetilde{h}_L^{-1}(L,F)\widetilde{h}(L,F), 0), \tag{4.10}$$

where 0 is the zero matrix.

The null space of the Jacobian ∇h^T is given by:

$$\begin{aligned} &\left\{(\Delta K, \Delta F) \in \mathbb{R}^{n_x\times n_x + n_u\times n_y} : h_K(K,F)\Delta K + h_F(K,F)\Delta F = 0\right\} \\ &= \left\{T_1(K,F)\Delta F,\ \Delta F \in \mathbb{R}^{n_u\times n_y}\ \right\} = \mathcal{R}(T_1(K,F)), \end{aligned} \tag{4.11}$$

where $\mathcal{R}(T_1)$ is the range space of T_1 as defined in (4.7); Similarly the null space of $\nabla\widetilde{h}^T$ coincides with the range space of the operator T_2. □

From the step decomposition given by Lemma 4.2 the linearized equality constraints of the QP problems (4.5) and (4.6) decompose into four linear matrix equations as given in the following lemma.

Lemma 4.3. *Let $X = (K,F) \in \mathcal{F}_s$ for P3 and $X = (L,F) \in \mathcal{F}_s$ for P4 be given and let h_K and $\widetilde{h}_L$ be invertible. Then the linearized equality constraints of the QP problems* (4.5) *and* (4.6) *decompose into the following linear matrix equations:*

$$h_K(K,F)\Delta K^n + h(K,F) = 0 \tag{4.12}$$

$$h_K(K,F)\Delta K^t + h_F(K,F)\Delta F = 0 \tag{4.13}$$

$$\widetilde{h}_L(L,F)\Delta L^n + \widetilde{h}(L,F) = 0 \tag{4.14}$$

$$\widetilde{h}_L(L,F)\Delta L^t + \widetilde{h}_F(L,F)\Delta F = 0. \tag{4.15}$$

Proof. From the step decomposition (4.8)–(4.9) the linearized equality constraints in (4.5) and (4.6) can be, respectively, rewritten as:

$$h_K(K,F)\Delta K^t + h_F(K,F)\Delta F + h_K(K,F)\Delta K^n + h(K,F) = 0 \tag{4.16}$$

$$\widetilde{h}_L(L,F)\Delta L^t + \widetilde{h}_F(L,F)\Delta F + \widetilde{h}_L(L,F)\Delta L^n + \widetilde{h}(L,F) = 0. \tag{4.17}$$

Since $T_1(K,F)\Delta F$ and $T_2(L,F)\Delta F$ are the null spaces of the Jacobian matrices ∇h^T and $\nabla\widetilde{h}^T$, respectively, then according to (4.11) we have

$$h_K(K,F)\Delta K^t + h_F(K,F)\Delta F = 0, \qquad \forall \quad \Delta F$$

$$\widetilde{h}_L(L,F)\Delta L^t + \widetilde{h}_F(L,F)\Delta F = 0 \qquad \forall \quad \Delta F,$$

that correspond to (4.13) and (4.15), respectively. Thus, (4.16) and (4.17) reduce to (4.12) and (4.14), respectively. □

By substituting the derivatives of Lemma 3.1 in (4.12)–(4.15) these equations take the form of discrete Lyapunov equations.

In order to solve the trust region problems (4.5) and (4.6) we use the the reduced Hessian approach with step decomposition; see e.g. [21]. In that approach the trust region problem decomposes into two unconstrained trust region subproblems. The first subproblem is of the following form:

$$\min_{\|\Delta K^n\|\leq\theta\delta_k} \frac{1}{2}\|h_K(K_k,F_k)\Delta K^n + h(K_k,F_k)\|^2, \tag{4.18}$$

$$\min_{\|\Delta L^n\|\leq\theta\delta_k} \frac{1}{2}\|\widetilde{h}_L(L_k,F_k)\Delta L^n + \widetilde{h}(L_k,F_k)\|^2, \tag{4.19}$$

where $\theta \in (0,1)$ is a parameter. An efficient way for solving these two subproblems is to solve the linear matrix equations (4.12) and (4.14) and then scale the obtained solutions to satisfy the trust region constraint. The resulting solutions ΔK^n and ΔL^n of the two subproblems are called the *normal* steps for the two subproblems, respectively.

The tangential step components $\Delta K^t(\Delta F) = T_1(\cdot)\Delta F$ and $\Delta L^t(\Delta F) = T_2(\cdot)\Delta F$ depend on the step component ΔF. Consequently, if ΔF is known in advance then ΔK^t and ΔL^t can be obtained by solving (4.13) and (4.15), respectively. The tangential trust region subproblems are, respectively:

$$\min_{\Delta F} \quad \phi_k^1(\Delta F) \ \text{ s.t. } \ \|\Delta F\| \leq \delta_k, \ \ (K_k + \Delta K^n + T_1(\cdot)\Delta F, F_k + \Delta F) \in \mathcal{F}_s, \tag{4.20}$$

$$\min_{\Delta F} \quad \phi_k^2(\Delta F) \ \text{ s.t. } \ \|\Delta F\| \leq \delta_k, \ \ (L_k + \Delta L^n + T_2(\cdot)\Delta F, F_k + \Delta F) \in \mathcal{F}_s, \tag{4.21}$$

where $\phi_k^1(\Delta F)$ and $\phi_k^1(\Delta F)$ are the quadratic models $q_k(\Delta K, \Delta F)$ and $\widetilde{q}_k(\Delta L, \Delta F)$ projected into the null spaces of the Jacobian of the corresponding linearized equality constraints, which can be rewritten in following compact forms:

$$\phi_k^1(\Delta F) = \langle \Delta F, T_1^T\nabla J + T_1^T\nabla^2_{XX}\mathcal{L}^\sigma N_1\rangle + \frac{1}{2}\langle \Delta F, T_1^T\nabla^2_{XX}\mathcal{L}^\sigma T_1\Delta F\rangle \tag{4.22}$$

$$\phi_k^2(\Delta F) = \langle \Delta F, T_2^T\nabla J + T_2^T\nabla^2_{XX}\mathcal{L}^\sigma N_2\rangle + \frac{1}{2}\langle \Delta F, T_2^T\nabla^2_{XX}\mathcal{L}^\sigma T_2\Delta F\rangle \tag{4.23}$$

where $T_1^T\nabla^2_{XX}\mathcal{L}^\sigma T_1$ and $T_2^T\nabla^2_{XX}\mathcal{L}^\sigma T_2$ are the reduced Hessian of the augmented Lagrangian for P3 and P4, respectively.

By applying the optimality conditions on the trust region subproblem (4.20) we obtain the following coupled linear system of equations.

Lemma 4.4. *Let* $(K, F, U) \in \mathcal{F}_s \times \mathbb{R}^{n_x\times n_x}$ *be given. The step* $\Delta F \in \mathbb{R}^{n_u\times n_y}$ *solution of the trust subproblem* (4.20) *solves the following linear matrix equation:*

$$\begin{aligned} &(B^TKB + R)\Delta FC(U + \sigma h(.))C^T + B^T\Delta K^t(\Delta F)A(F)(U + \sigma h(.))C^T \\ &+(B^TKA(F) + RFC)\Delta W(\Delta F)C^T + \lambda\Delta F = -B^T\Delta K^nA(F)(U + \sigma h(.))C^T \\ &-(B^TKA(F) + RFC)\big(Y + (U + \sigma h(.))\big)C^T, \end{aligned} \tag{4.24}$$

where $\lambda > 0$ *is the Lagrange multiplier associated with the trust region constraint and* ΔK^n, $\Delta K^t(\Delta F)$, U, Y *and* $\Delta W(\Delta F)$ *solve, respectively, the discrete Lyapunov equations* (4.12), (4.13) *and*

$$U = A(F)UA(F)^T + V \tag{4.25}$$

$$Y = A(F)YA(F)^T + M_1 + M_1^T \tag{4.26}$$

$$\Delta W = A(F)\Delta WA(F)^T + B\Delta FC(U + \sigma h(.))A(F)^T + A(F)(U + \sigma h(.))(B\Delta FC)^T, \tag{4.27}$$

where

$$M_1 = A(F)(U + \sigma h)A(F)^T - (U + \sigma h) + V.$$

Proof. First, we form the Lagrangian function associated with the trust region constraint of the problem (4.20):

$$\begin{aligned}
\ell^1(\Delta F, \lambda) &= \phi_k^1(\Delta F) + \lambda(\|\Delta F\|^2 - \delta_k^2) \\
&= \mathrm{Tr}(\Delta K^t(\Delta F)M_1) + 2\mathrm{Tr}((B^T KA(F) + RFC)(U + \sigma h)C^T \Delta F^T) \\
&+ 2\mathrm{Tr}(B^T \Delta K^n A(F)(U + \sigma h)C^T \Delta F^T) + \mathrm{Tr}((B^T KB + R)\Delta FC(U + \sigma h)C^T \Delta F^T) \\
&+ 2\mathrm{Tr}(B^T \Delta K^t(\Delta F)A(F)(U + \sigma h)C^T \Delta F^T) + \lambda(\mathrm{Tr}(\Delta F^T \Delta F) - \delta_k^2).
\end{aligned}$$

By differentiating ℓ^1 with respect to ΔF and applying the optimality conditions we obtain (4.24) coupled with the discrete Lyapunov equations (4.12), (4.13), (4.25), (4.26), and (4.27). In order to evaluate the derivative ℓ^1 with respect to ΔF the following properties of the trace operator are used:

$$\frac{\partial}{\partial \Delta F} \mathrm{Tr}\,(M_1 \Delta F M_2) = M_1^T M_2^T, \quad \frac{\partial}{\partial \Delta F} \mathrm{Tr}\,(M_1 \Delta F^T M_2) = M_2 M_1,$$

for any given matrices M_1 and M_2 of appropriate dimensions. Note that for the two terms that contain $\Delta K^t(\Delta F)$ we replace $\Delta K^t(\Delta F)$ solution of (4.13) by its exact value:

$$\Delta K^t(\Delta F) = \sum_{k=0}^{\infty} (A(F)^T)^k \Big(C^T \Delta F^T (B^T KA(F)+RFC)+(B^T KA(F)+RFC)^T \Delta FC\Big) A(F)^k, \tag{4.28}$$

and then perform the differentiation, e.g. the derivative of the first term in ℓ^1 is carried out as follows:

$$\begin{aligned}
&\frac{\partial}{\partial \Delta F} \mathrm{Tr}(M_1 \Delta K^t(\Delta F)) \\
&\qquad = \frac{\partial}{\partial \Delta F} \mathrm{Tr}\Big[M_1 \sum_{k=0}^{\infty} (A(F)^T)^k \Big(C^T \Delta F^T (B^T KA(F) + RFC) \\
&\qquad + (B^T KA(F) + RFC)^T \Delta FC\Big) A(F)^k\Big] \\
&\qquad = (B^T KA(F) + RFC) \sum_{k=0}^{\infty} A(F)^k \big(M_1 + M_1^T\big)(A(F)^T)^k\, C^T \\
&\qquad = (B^T KA(F) + RFC)YC^T,
\end{aligned}$$

where

$$Y = \sum_{k=0}^{\infty} A(F)^k \big(M_1 + M_1^T\big)(A(F)^T)^k$$

solves the discrete Lyapunov equation (4.26). □

Similarly by applying the first-order optimality conditions on the subproblem (4.21) the following result is obtained.

Lemma 4.5. *Let $(L, F, \widetilde{U}) \in \mathcal{F}_s \times \mathbb{R}^{n_x \times n_x}$ be given. The step $\Delta F \in \mathbb{R}^{n_u \times n_y}$ solution of the trust subproblem* (4.21) *solves the following linear matrix equation:*

$$\begin{aligned}(B^T(\widetilde{U} + \sigma\widetilde{h}(.))B + R)\Delta F C L C^T + (B^T(\widetilde{U} + \sigma\widetilde{h}(.))A(F) + RFC)\Delta L^t(\Delta F)C^T \\ + B^T \Delta\widetilde{W}(\Delta F)A(F)LC^T + \lambda\Delta F = -(B^T(\widetilde{U} + \sigma\widetilde{h}(.))A(F) + RFC)\big(L + \Delta L^n + \widetilde{Y}\big)C^T),\end{aligned} \tag{4.29}$$

where ΔL^n, $\Delta L^t(\Delta F)$, $\widetilde{U}$, $\widetilde{Y}$ and $\Delta\widetilde{W}(\Delta F)$ solve, respectively, the discrete Lyapunov equations (4.14), (4.15) *and*

$$\widetilde{U} = A(F)^T \widetilde{U} A(F) + \widetilde{h}(L, F) \tag{4.30}$$

$$\widetilde{Y} = A(F)^T \widetilde{Y} A(F) + M_2 + M_2^T \tag{4.31}$$

$$\begin{aligned}\Delta\widetilde{W} &= A(F)^T \Delta\widetilde{W} A(F) + C^T \Delta F^T \big(B^T(\widetilde{U} + \sigma\widetilde{h}(.))A(F) + RFC\big) \\ &\quad + \big(B^T(\widetilde{U} + \sigma\widetilde{h}(.))A(F) + RFC\big)^T \Delta F C,\end{aligned} \tag{4.32}$$

where

$$M_2 = A(F)^T(\widetilde{U} + \sigma\widetilde{h})A(F) - (\widetilde{U} + \sigma\widetilde{h}) + Q(F).$$

We apply for solving the linear matrix equation (4.24) coupled with the discrete Lyapunov equations (4.12), (4.13), (4.25), (4.26) and (4.27) a modified Steihaug conjugate gradient (CG) trust region algorithm (see e.g. [21, Algorithm 4.3]). At every iteration of the CG algorithm and for given trust-region radius δ_k a maximal parameter $\tau > 0$ is calculated that make sure that the positive definite constraint:

$$(K_k + \Delta K_k^n + T_1(K_k, F_k)\Delta F, F_k + \Delta F) \in \mathcal{F}_s$$

is satisfied for the subproblem (4.20) and

$$(L_k + \Delta L_k^n + T_2(L_k, F_k)\Delta F, F_k + \Delta F) \in \mathcal{F}_s$$

is satisfied for the subproblem (4.21). The Matlab function cholinc of the incomplete Cholesky factorization is utilized to achieve that goal. The CG trust region algorithm is stated below in which G denotes the approximate solution of (4.24) coupled with (4.12), (4.13), (4.25), (4.26) and (4.27) while E and H denote the residual and the direction required by the CG method, respectively.

Algorithm 4.6 (CG trust region algorithm for calculating $(\Delta K^t, \Delta F)$ solution of (4.20))**.** Let $(K_k, F_k) \in \mathcal{F}_s$, U_k, Y_k, $\delta_k > 0$ and $\sigma > 0$ be given. Let $\Delta K_k^n \in \mathbb{R}^{n_x \times n_x}$ be a solution of (4.18). Set $G := 0_{n_u \times n_y}$,

$$E := -B^T \Delta K^n A(F)(U_k + \sigma h_k(.))C^T - (B_k^T K A(F_k) + RF_kC)\big(Y_k + (U_k + \sigma h_k(.))\big)C^T,$$

$H := E$, $\epsilon_{\rm cg} = 0.01\,\|E\|$.
Repeat at most $n_u \times n_y$.

1. Solve (4.13) and (4.27) for $\Delta K^t(H)$ and $\Delta W(H)$, respectively.
2. Calculate the curvature $\mathcal{K} = \mathrm{Tr}(H^T\mathcal{U}(H))$, where $\mathcal{U}(\cdot)$ is the left hand side of (4.24). Then set $\xi = \|E\|^2/\mathcal{K}$.
3. Calculate the following parameter:

$$\tilde{\tau} = \max\Big\{\tau > 0\ :\ \|G+\tau H\| \le \delta_k, K_k + \Delta K_k^n + \Delta K^t(\tau)\succ 0,\ \Delta S(\tau) \succ 0\Big\},$$

such that

$$\Delta S(\tau) = \Delta K^t(\Delta F) - A(F_k + G + \tau H)^T \Delta K^t(\Delta F) A(F_k + G + \tau H). \tag{4.33}$$

4. If $\xi > \tilde{\tau}$ or $\mathcal{K} \le 0$, then set $\Delta F = G + \tilde{\tau}H$, and exit; otherwise, set $G^+ = G + \xi H$.
5. Update the residual: $E^+ = E - \xi\,\mathcal{U}(H)$, and set $\epsilon_{cg} = \min\{\epsilon, \|E^0\|\}$.
6. If $\frac{\|E^+\|}{\|E^0\|} \le \epsilon_{cg}$, set $\Delta F = G^+$ and exit; otherwise go to the next step.
7. Compute $\zeta = \frac{\|E^+\|^2}{\|E\|^2}$, set $H^+ = E^+ + \zeta H$, and go to step 1.

End(repeat)

In step 3 of the above algorithm the maximal $\bar{\tau}$ is calculated as follows. For given δ_k, G and H we solve for $\tau > 0$ the scalar quadratic equation

$$\|G + \tau\, H\|^2 = \delta_k^2.$$

If the computed $\tau > 0$ is such that

$$K_k + \Delta K_k^n + \Delta K^t(\tau)\succ 0 \quad \text{and} \quad \Delta S(\tau) \succ 0,$$

where $\Delta S(\tau)$ is given by (4.33), then we set $\bar{\tau} := \tau$, update the step and exit the CG method. Otherwise, we decrease τ in a backtracking loop until we reach $\bar{\tau} > 0$ satisfying these positive definite constraints. The Matlab function cholinc of the incomplete Cholesky factorization can be used to check those positive definite constraints. According to Theorem 2.4 one can equivalently compute $\bar{\tau} > 0$ such that $F_k + (G + \bar{\tau}H) \in \mathcal{D}_s$, namely that fulfills $\rho(A + B(F_k + (G + \bar{\tau}H))C) < 1$.

The same algorithm is applied to solve the linear matrix equation (4.29) coupled with the discrete Lyapunov equations (4.14), (4.15), (4.30), (4.31) and (4.32).

The update of the computed trial step by the SQP augmented Lagrangian method depends on the value of the ratio $r_k = ared_k/pred_k$, where $ared_k$ is the actual reduction that occurs in the merit function (chosen in our case as the augmented Lagrangian function (4.2)) and $pred_k$ is the predicted reduction in the quadratic model. The two quantities for P3 are defined by:

$$ared_k = \mathcal{L}^\sigma(K_k, F_k, U_k) - \mathcal{L}^\sigma(K_k + \Delta K, F_k + \Delta F, U_{k+1}) \tag{4.34}$$

$$pred_k = \phi_k^1(0) - \phi_k^1(\Delta F) + \sigma\big(\|h^k\|^2 - \|h^k + h_K^k(\cdot)\Delta K + h_F^k(\cdot)\Delta F\|^2\big), \tag{4.35}$$

where $\phi^1(\Delta F)$ is given by (4.22). Similarly for P4 the reductions $ared_k$ and $pred_k$ are:

$$ared_k = \mathcal{L}^\sigma(L_k, F_k, \widetilde{U}_k) - \mathcal{L}^\sigma(L_k + \Delta L, F_k + \Delta F, \widetilde{U}_{k+1}) \tag{4.36}$$

$$pred_k = \phi_k^2(0) - \phi_k^2(\Delta F) + \sigma\big(\|\widetilde{h}^k\|^2 - \|\widetilde{h}^k + \widetilde{h}_L^k(\cdot)\Delta L + \widetilde{h}_F^k(\cdot)\Delta F\|^2\big). \tag{4.37}$$

The framework of the SQP augmented Lagrangian trust region method for solving P3 is written in the following lines. The method stops if a KKT point is reached. Therefore it is convenient to use the following stopping criterion:

$$\|\nabla_K \ell^k\| + \|\nabla_F \ell^k\| + \|h^k\| \le \epsilon, \tag{4.38}$$

where $\epsilon > 0$ is the tolerance. The algorithm can be applied for solving P4 analogously.

Algorithm 4.7 (SQPaL: The SQP augmented Lagrangian trust region method)**.** Initialization: Let $(K_0, F_0, U_0) \in \mathcal{F}_s \times \mathbb{R}^{n_x \times n_x}$ be given starting point. Let $\epsilon \in (0,1)$ be the tolerance and $0 < c_1 < c_2 < 1$ be given parameters. Set $k := 0$.
Until the convergence condition (4.38) is satisfied, do

1. Compute the normal step ΔK^n solution of the subproblem (4.18).
2. Given ΔK^n, compute the tangential step $T_1(\cdot)\Delta F = (\Delta K^t(\Delta F), \Delta F)$ solution of the subproblem (4.20) such that $(K_k + \Delta K^n + \Delta K^t(\Delta F), F_k + \Delta F) \in \mathcal{F}_s$.
3. Set $(\Delta K, \Delta F) = (\Delta K^n, 0) + (\Delta K^t(\Delta F), \Delta F)$, and then compute a new multiplier U_{k+1} solution of the linear matrix equation (4.25).
4. Compute the quantities $ared_k$ and $pred_k$ by using (4.34) and (4.35), respectively. Then update the penalty parameter σ by the scheme given in [16, Algorithm 3.2].
5. Update the trust region radius and the computed trial step: Compute the ratio r_k.

   If $r_k < c_1$,
   set $\delta_{k+1} := \delta_k/2$ and $(K_{k+1}, F_{k+1}) := (K_k, F_k)$;
   Else if $c_1 \le r_k < c_2$,
   set $\delta_{k+1} := \delta_k$ and $(K_{k+1}, F_{k+1}) = (K_k, F_k) + (\Delta K, \Delta F)$;
   Else if $r_k \ge c_2$,
   set $\delta_{k+1} := 2\,\delta_k$ and $(K_{k+1}, F_{k+1}) = (K_k, F_k) + (\Delta K, \Delta F)$.
   End (If)

6. Set $k := k + 1$ and go to Step 1.

End (do)

Starting point In order to find feasible starting point with respect to the positive definite constraints, i.e. $(K_0, F_0) \in \mathcal{F}_s$ and $(L_0, F_0) \in \mathcal{F}_s$ we use Theorem 2.4 for that purpose. For P3, e.g., the feasible point $(K_0, F_0) \in \mathcal{F}_s$ is obtained as follows. If $\rho(A) < 1$, then F_0 might be the zero matrix and K_0 is obtained as the positive definite solution of the discrete Lyapunov equation (3.6). In case $\rho(A) \ge 1$ then $F_0 \in \mathcal{D}_s$ is required. The matrix $F_0 \in \mathcal{D}_s$ can be obtained e.g. by using the method described in [17, Fstab]. Moreover, we can also apply the parametrization approach described in [19] so that $F_0 = 0$ can be chosen to start the iteration sequence. The same strategy can be used to find $(L_0, F_0) \in \mathcal{F}_s$ as we solve P4.

5 Numerical Results

This section is devoted to present the implementation of the method SQPaL for solving the NSDP problems P3 and P4. It is particularly important to analyze the performance of the method SQPaL on the two problems and to compare the achieved stationary point of this method when applied on either of the two formulations. Moreover, the method SQPaL applied on P3 is compared numerically with Newton's method with line search [15] for solving P1; we denote it for short by NLS.

The considered test problems are from the benchmark collection COMPlib [9]. All these test problems are for continuous-time models where we use the Matlab function `c2d` to convert the continuous-time system into the discrete-time counterpart, where the sampling time period was taken $\Delta T = 0.1$. For all test problems the constant weight matrices Q and R and the covariance matrix V are chosen as $Q = V = I_{n_x}$ and $R = 1.5 I_{n_u}$. The parameters of the trust region method have been assigned the values $c_1 = 1 \times 10^{-4}$ and $c_2 = 0.1$. The tolerance is chosen as $\epsilon = 1.0 \times 10^{-5}$.

In the following we consider five test problems in details. For each test problem we list the constant data matrices A, B, and C of the discrete-time model. For each test problem we compare the method SQPaL on the two formulations P3 and P4 with respect to number of iterations. Moreover, the method SQPaL applied on P3 is compared vs. the method NLS applied on the corresponding problem P1. For each test problem we follow the procedure given at the end of the last section to compute feasible starting point with respect to the positive definite constraints; i.e. $(K_0, F_0) \in \mathcal{F}_s$ and $(L_0, F_0) \in \mathcal{F}_s$ for P3 and P4, respectively.

Example 5.1. The first example is the terrain following model [9, TF1]. The data matrices for the discrete-time model are as follows:

$$A = \begin{bmatrix} 0.9048 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0952 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0.0048 & 0.1000 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0952 & 0.9048 & 0 & 0 \\ -0.0082 & 0.0034 & 0 & 0.0048 & 0.0951 & 0.9997 & 0 \\ 0.0000 & 0.0002 & 0.0050 & 0 & 0 & 0 & 1.0000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0952 & 0 \\ 0.0048 & 0 \\ 0.0002 & 0 \\ 0 & 0.0090 \\ 0 & 0.0004 \\ -0.0004 & 0.0000 \\ 0.0000 & 0 \end{bmatrix}, C^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The uncontrolled system is discrete-time Schur unstable, where $\rho(A) = 1$. Therefore, $F_0 \in \mathcal{D}_s$ is required to find $(K_0, F_0) \in \mathcal{F}_s$ and $(L_0, F_0) \in \mathcal{F}_s$ as explained above. By using

$$F_0 = \begin{bmatrix} -0.9 & -2.0 & -0.6 & -0.6 \\ -1.9 & 0.0 & -1.1 & -1.9 \end{bmatrix}$$

the method SQPaL achieved the stationary points $(K_*, F_*) \in \mathcal{F}_s$ and $(L_*, F_*) \in \mathcal{F}_s$ after 10 iterations for the two problems P3 and P4, where the same F_* and the same J_* are obtained.

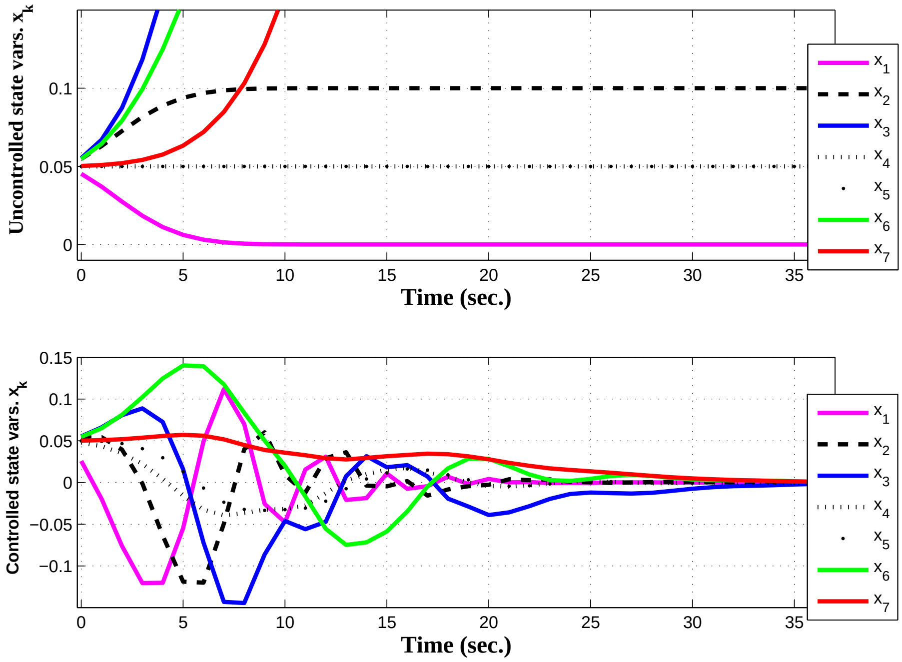

Figure 1: Open and closed–loop systems for the terrain following model of Example 5.1.

The final SOF gain is:

$$F_* = \begin{bmatrix} -0.8803 & -1.9931 & -0.5817 & -0.6291 \\ -1.7402 & -0.0205 & -1.0863 & -1.7129 \end{bmatrix}.$$

On the other hand, the method NLS could not achieve the prescribed accuracy and stops at the eleventh iteration with $||\nabla J(F)|| = 4.3292\times10^{-5}$.

Table 1: Example 5.1: Performance of the method SQPaL for solving P3.

k	J_k	$\|\nabla_X \ell_k\| + \|h_k\|$	$\|h_k\|$	$\rho(A(F_k))$	δ_k
0	3.8803e+003	7.9575e+003	2.6354e−012	9.9221e−001	7.9575e+003
1	3.4902e+003	1.2265e+003	3.9286e−001	9.9342e−001	1.5915e+004
2	3.5961e+003	6.1220e+002	6.7903e−001	9.9274e−001	1.5915e+004
3	3.6366e+003	2.0956e+002	1.9427e−002	9.9272e−001	3.1830e+004
⋮	⋮	⋮	⋮	⋮	⋮
8	3.6213e+003	4.5600e−002	6.8503e−008	9.9250e−001	1.0186e+006
9	3.6213e+003	7.2942e−005	2.8078e−008	9.9250e−001	2.0371e+006
10	3.6213e+003	2.2967e−009	4.5357e−013	9.9250e−001	4.0742e+006

Table 1 shows the convergence behavior of the method SQPaL for solving P3. The columns from the left to the right are the iteration counter k, the objective function J_k, the convergence criterion $\|\nabla_X \ell_k\| + \|h_k\|$, the norm of the equality constraint $\|h_k\|$, the spectral radius of the closed-loop system matrix $\rho(A(F_k))$, and the trust region radius δ_k.

In order to show the benefit of computing the optimal output feedback controller $F_* \in \mathcal{D}_s$ we have plotted the state variables of the discrete-time system for the open and closed-loop systems. Note that this model is an open-loop unstable system. Figure 1 shows the state variables for the open and closed-loop systems. The effect of the achieved controller can be observed where all state variables in the second figure converge to the zero state.

Example 5.2. The second test problem represents the lateral axis dynamic for the L–1011 aircraft model [9, AC17]. The data matrices of the discrete-time model are the following:

$$A = \begin{bmatrix} 0.7385 & 0.0802 & 0.0001 & -0.0027 \\ -0.0845 & 0.9751 & 0.0035 & 0.0002 \\ 0.0025 & -0.0258 & 1.0000 & 0.0911 \\ 0.0538 & -0.4984 & -0.0009 & 0.8277 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0025 \\ 0.0001 \\ -0.0075 \\ -0.1459 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The uncontrolled system is discrete-time Schur stable, where $\rho(A) = 0.9723 < 1$. Starting from $F_0 = 0$ the method SQPaL requires 10 and 27 iterations to reach the stationary point for P3 and P4, respectively. Moreover, the method NLS requires 12 iterations to converge to the stationary point. The achieved discrete-time output feedback gain by both methods is:

$$F_* = \begin{bmatrix} 1.1736 & 1.7594 \end{bmatrix}.$$

Table 2: Example 5.2: Performance of the method SQPaL for solving P3 (right) vs. the method NLS for solving P1 (left).

k	$J(F_k)$	$\|\nabla J(F_k)\|$	k	$J(K_k)$	$\|\nabla_X \ell_k\| + \|h_k\|$
0	1.0558e+003	5.5993e+003	0	1.0558e+003	5.5993e+003
1	6.8745e+002	2.4118e+003	1	7.7677e+002	3.3367e+003
2	4.6698e+002	1.0248e+003	2	4.1066e+002	1.1128e+003
3	3.3987e+002	4.3055e+002	3	3.1023e+002	4.2220e+002
⋮	⋮	⋮	⋮	⋮	⋮
10	1.9781e+002	9.7820e-002	8	1.9770e+002	1.6972e-001
11	1.9781e+002	3.8009e-004	9	1.9781e+002	6.3819e-004
12	1.9781e+002	5.7479e-009	10	1.9781e+002	6.5305e-009

Table 2 compares the method SQPaL for solving P3 vs. the method NLS for solving P1, where fast local rate of convergence is observed.

Example 5.3. This test problem represents a transport aircraft application [9, AC9]. The data matrices for the discrete-time model are as follows:

$$A = \begin{bmatrix} 0.9986 & 0.0171 & -0.0019 & -0.0561 & -0.0015 & -0.0003 & 0.0013 & -0.0012 & -0.0000 & -0.0999 \\ -0.0015 & 0.9279 & 0.0923 & 0.0002 & -0.0117 & -0.0019 & 0.0014 & 0.0052 & 0.0000 & 0.0001 \\ 0.0001 & 0.0073 & 0.9168 & -0.0000 & -0.1388 & -0.0271 & -0.0001 & -0.0005 & -0.0000 & -0.0000 \\ 0.0000 & 0.0004 & 0.0958 & 1.0000 & -0.0093 & -0.0015 & -0.0000 & -0.0000 & -0.0000 & -0.0000 \\ 0 & 0 & 0 & 0 & 0.1353 & 0.0460 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0067 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.9565 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9565 & 0.0004 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0004 & 0.9565 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0008 & 0.0001 & -0.0001 & -0.0050 \\ -0.0037 & 0.0001 & 0.0003 & 0.0000 \\ -0.0730 & -0.0000 & -0.0000 & -0.0000 \\ -0.0024 & -0.0000 & -0.0000 & -0.0000 \\ 0.4175 & 0 & 0 & 0 \\ 0.9933 & 0 & 0 & 0 \\ 0 & 0.0922 & 0 & 0 \\ 0 & 0 & 0.1119 & 0 \\ 0 & 0 & -4.7751 & 0 \\ 0 & 0 & 0 & 0.1000 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 0.0065 & 1.0000 & -0.0137 & 0 & 0 \\ 0.3203 & 0 & 0.1780 & -13.5800 & 0 \\ -0.0336 & 0 & 0.0002 & 0 & 1.0000 \\ 0 & 0 & -0.5610 & 13.5800 & 0 \\ -0.1032 & 0 & -0.0373 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.0065 & -1.0000 & 0.0137 & 0 & 0 \\ -0.0236 & 0 & -0.0131 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The method SQPaL requires an initial $F_0 \in \mathcal{D}_s$ to obtain feasible point with respect to the positive definite constraints, since $\rho(A) = 1.0012 > 1$. Having a starting point as explained above the method SQPaL achieves the stationary point after 73 and 130 iterations for P3 and P4, respectively, while the method NLS fails to converge. The initial and optimal output feedback controllers are, respectively:

$$F_0 = \begin{bmatrix} 0.2800 & -0.3100 & -0.3800 & 0.0157 & 0.5400 \\ 0.0600 & 0.6300 & -0.0700 & -0.0113 & 0.1900 \\ 0.0000 & 0.0000 & 0.0000 & 0.0001 & -0.0000 \\ -0.2100 & 0.9300 & 0.2800 & -0.0085 & -0.3700 \end{bmatrix},$$

$$F_* = \begin{bmatrix} 1.5751 & -0.5930 & -1.1304 & 0.0155 & 0.8881 \\ -0.7038 & 0.7094 & 0.5765 & -0.0048 & -0.2839 \\ 0.0063 & -0.0002 & 0.0043 & 0.0002 & 0.0005 \\ -2.2541 & 1.2479 & 0.8306 & -0.0234 & -0.7498 \end{bmatrix}.$$

Table 3 shows the convergence behavior of the method SQPaL for the two problems P3 and P4.

Table 3: Example 5.3: Performance of the method SQPaL on the problems P3 and P4, respectively.

k	J_k	$\|\nabla_X \ell_k\| + \|h_k\|$	k	J_k	$\|\nabla_X \widetilde{\ell}_k\| + \|\widetilde{h}_k\|$
0	2.8065e+002	1.4437e+002	0	2.8065e+002	1.4437e+002
1	2.8054e+002	1.2017e+002	1	2.8054e+002	1.2117e+002
2	2.6552e+002	9.4544e+001	2	2.7232e+002	3.0755e+002
3	2.6655e+002	6.1488e+001	3	2.7435e+002	3.8596e+001
⋮	⋮	⋮	⋮	⋮	⋮
71	2.5455e+002	1.4834e−001	128	2.5455e+002	2.8663e−004
72	2.5455e+002	4.5868e−003	129	2.5455e+002	2.1954e−004
73	2.5455e+002	7.2132e−006	130	2.5455e+002	1.0139e−008

Example 5.4. This example is the decentralized interconnected system [9, DIS1]. The data matrices for the discrete-time model are as follows:

$$A = \begin{bmatrix} 1.0130 & -0.0044 & 0.0057 & 0.0043 & 0.0110 & 0.1989 & 0.0020 & 0.0076 \\ -0.0514 & 0.9778 & -0.0022 & -0.0014 & -0.0058 & -0.0908 & 0.0977 & -0.0040 \\ 0 & 0 & 0.9726 & 0.0289 & 0 & 0 & 0 & 0.0519 \\ 0 & 0 & 0 & 0.9675 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0275 & 0.0031 & 0.8462 & 0 & 0 & 0.0084 \\ -0.0150 & 0.0124 & -0.0006 & -0.0004 & -0.0013 & 0.8420 & 0.0204 & 0.0010 \\ -0.0321 & 0.0272 & -0.0018 & -0.0014 & -0.0029 & -0.0558 & 0.8341 & -0.0023 \\ 0 & 0 & 0 & 0.0220 & 0 & 0 & 0 & 0.8205 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.0078 & 0.0024 & 0.0017 & 0.0001 \\ 0.0583 & -0.0008 & 0.0014 & -0.0001 \\ 0 & 0.0171 & 0 & 0.0007 \\ 0 & 0.1426 & 0 & 0 \\ 0 & 0.0004 & 0 & 0.0012 \\ 0.0004 & -0.0001 & 0.0153 & -0.0002 \\ 0.0010 & -0.0002 & 0.0374 & -0.0008 \\ 0 & 0.0017 & 0 & 0.0225 \end{bmatrix}, C^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The uncontrolled system is discrete-time Schur stable, where $\rho(A) = 0.9912 < 1$. The method SQPaL achieves the prescribed accuracy in 10 and 12 iterations for P3 and P4, respectively. Moreover, the method NLS achieves the same SOF gain matrix F_* after 12 iterations:

$$F_* = \begin{bmatrix} 0.4592 & -0.5589 & 0.0619 & 0.0782 \\ -0.2417 & -0.0200 & -0.3875 & -0.7075 \\ -0.5865 & -0.2234 & -0.0984 & -0.1185 \\ -0.0406 & 0.0019 & -0.0663 & -0.0271 \end{bmatrix}.$$

Table 4 compares the performance of the method SQPaL for solving the problem P3 vs. the method NLS for solving the problem P1.

Example 5.5. The current application represents a power system model [9, PSM]. The

Table 4: Example 5.4: Performance of the method SQPaL for solving P3 (right) vs. the method NLS for solving P1 (left).

k	J_k	$\|\nabla J(F_k)\|$	k	J_k	$\|\nabla_X \ell_k\| + \|h_k\|$
0	5.2471e+002	2.0740e+003	0	5.2471e+002	2.0740e+003
1	3.8550e+002	1.0186e+003	1	3.1466e+002	8.9195e+002
2	2.8487e+002	4.6819e+002	2	2.2005e+002	3.0413e+002
3	2.3464e+002	2.3604e+002	3	1.8952e+002	9.9353e+001
⋮	⋮	⋮	⋮	⋮	⋮
10	1.8337e+002	3.9305e-002	8	1.8337e+002	8.1195e−003
11	1.8337e+002	2.7226e-003	9	1.8337e+002	2.4018e−005
12	1.8337e+002	6.0144e-006	10	1.8337e+002	7.4535e−011

data matrices for the discrete-time system are as follows:

$$A = \begin{bmatrix} 0.9728 & 0.0497 & 0.4143 & -0.4853 & 0.0133 & 0.0001 & 0.0020 \\ -0.2938 & 0.2793 & -0.0778 & 0.0873 & -0.0017 & -0.0000 & -0.0002 \\ -0.0526 & 0.1556 & 0.7078 & 0.0098 & -0.0001 & -0.0000 & -0.0000 \\ 0.0538 & 0.0010 & 0.0119 & 0.9734 & -0.0538 & -0.0010 & -0.0119 \\ 0.0133 & 0.0001 & 0.0020 & 0.4853 & 0.9728 & 0.0497 & 0.4143 \\ -0.0017 & -0.0000 & -0.0002 & -0.0873 & -0.2938 & 0.2793 & -0.0778 \\ -0.0001 & -0.0000 & -0.0000 & -0.0098 & -0.0526 & 0.1556 & 0.7078 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.4875 & -0.0022 \\ 0.0875 & 0.0002 \\ 0.0098 & 0.0000 \\ -0.0133 & 0.0133 \\ -0.0022 & -0.4875 \\ 0.0002 & 0.0875 \\ 0.0000 & 0.0098 \end{bmatrix}, C^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The uncontrolled system is discrete-time Schur stable, where $\rho(A) = 0.9495 < 1$. The method SQPaL requires 8 and 10 iterations for solving P3 and P4, respectively, while the method NLS needs 11 iterations to achieve the same SOF gain:

$$F_* = \begin{bmatrix} 0.7065 & 0.0278 & 0.0469 \\ 0.0469 & -0.0278 & 0.7065 \end{bmatrix}.$$

Table 5 shows the convergence behavior of the method SQPaL on the two problems P3 and P4. The method performs on P3 better than P4 and it converges to the same stationary point for both problems.

Table 7 compares the performance of the method SQPaL for solving the problem P3 vs. the method NLS for solving the problem P1 on 37 test problems from the benchmark collection [9] with respect to number of iterations. The results show the significance of the considered SQP augmented Lagrangian trust region approach for solving the NSDP problem over Newton's method applied on the particular formulation P1.

Moreover, we ran the method SQPaL on the two formulations P3 and P4 and compared the performance in that case over 50 test problems from the benchmark [9]. Table 6 shows the overall performance. Note that, if the method reaches the KKT point in less number of

Table 5: Example 5.5: Performance of the method SQPaL for solving P3 and P4.

k	J_k	$\|\nabla_X \ell_k\| + \|h_k\|$	k	J_k	$\|\nabla_X \widetilde{\ell}_k\| + \|\widetilde{h}_k\|$
0	9.5905e+001	3.9351e+002	0	9.5905e+001	3.9351e+002
1	5.9856e+001	1.3536e+002	1	7.8027e+001	2.3830e+002
2	4.8746e+001	5.4628e+001	2	6.7374e+001	1.5050e+002
3	4.1310e+001	1.7390e+001	3	5.8523e+001	9.4358e+001
⋮	⋮	⋮	⋮	⋮	⋮
6	4.1298e+001	2.3175e−001	8	4.1265e+001	2.8351e−001
7	4.1380e+001	6.6619e−003	9	4.1380e+001	5.3183e−003
8	4.1382e+001	2.2771e−006	10	4.1382e+001	1.3448e−006

Table 6: Comparison between the performance of the method SQPaL on the NSDP formulations P3 and P4 over 50 test problems from the benchmark collection [9].

	Problem P3	Problem P4
Number of wins	42	33
Total number of iterations	643	1044
Iterations average	12.86	20.88

iterations on a test problem it records win. We also report the total number of iterations and the iteration average. We observe that the method SQPaL has better performance when applied to solve P3 than P4. In particular, the stationary point is often reached with less number of iterations.

6 Conclusion

In this article, an SQP augmented Lagrangian trust region method is proposed for solving two nonlinear semidefinite programming problems originating from the static output feedback design problem for discrete-time systems. Both problems share similar properties in common but differ in their structures. The proposed method SQPaL is compared vs. Newton's method with line search applied on a particular formulation of the design problem. The performance of the method SQPaL is compared numerically on the considered couple of nonlinear semidefinite programming problems as well. Based on the considered set of test problems the main conclusion that one can draw is that the SQP augmented Lagrangian trust region method has better performance than Newton's method. Moreover the method SQPaL performs better as it solves the NSDP problem P3 than the NSDP problem P4. Finally, the two problems P3 and P4 share similar characteristics but it is recommended to consider the formulation P3 than P4.

Table 7: Performance of the method SQPaL on the problem P3 vs. the method NLS on the corresponding problem P1.

Problem	Problem dimension			Stability indicator		# iterations	
	n_x	n_u	n_y	$\rho(A)$	$\rho(A(F_*))$	NLS	SQPaL
AC1	5	3	3	1.0000	9.72e-001	12	8
AC4	4	1	2	1.2942	9.95e-001	11	10
AC8	9	1	5	1.0012	9.61e-001	12	11
AC9	10	4	5	1.0012	9.58e-001	*	73
AC15	4	2	3	0.9990	9.68e-001	24	21
AC17	4	1	2	0.9723	9.47e-001	12	10
HE1	4	2	1	1.0280	9.91e-001	5	3
HE2	4	2	2	0.9971	9.70e-001	16	16
HE3	8	4	6	1.0088	9.61e-001	14	11
REA1	4	2	3	1.2203	9.07e-001	8	4
REA2	4	2	2	1.2227	9.18e-001	6	4
DIS1	8	4	4	0.9912	9.59e-001	12	10
DIS2	3	2	2	1.1824	7.83e-001	7	5
DIS3	6	4	4	0.9620	9.05e-001	16	16
DIS4	6	4	6	1.1551	8.71e-001	9	6
AGS	12	2	2	0.9786	9.79e-001	10	4
HF1	130	1	2	0.9981	9.96e-001	9	6
UWV	8	2	2	0.9989	3.07e-001	22	17
IH	21	11	10	1.0000	9.79e-001	8	6
CSE1	20	2	10	1.0000	9.95e-001	11	7
EB1	10	1	1	0.9990	9.83e-001	9	10
EB2	10	1	1	0.9990	9.83e-001	9	10
EB3	10	1	1	0.9990	9.83e-001	49	36
TF1	7	2	4	1	9.92e-001	*	10
PSM	7	2	3	0.9495	9.06e-001	11	8
NN2	2	2	2	1.0000	9.50e-001	5	4
NN8	3	2	2	0.9971	9.24e-001	14	11
NN16	8	4	4	1.0000	9.81e-001	9	7
HF2D10	5	2	3	1.0133	9.22e-001	5	3
HF2D11	5	2	3	1.0253	8.01e-001	5	8
HF2D12	5	2	4	0.9830	9.45e-001	12	10
HF2D13	5	2	4	0.9756	8.01e-001	14	10
HF2D14	5	2	4	1.0227	9.62e-001	14	11
HF2D15	5	2	4	1.1689	8.73e-001	9	6
HF2D17	5	2	4	1.0557	8.07e-001	9	6
HF2D18	5	2	2	1.0285	9.96e-001	9	7
TMD	6	2	4	1	9.87e-001	8	6

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