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CALMNESS AND EXACT PENALIZATION FOR CONSTRAINED VECTOR SET-VALUED OPTIMIZATION PROBLEMS*

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Abstract: In this paper, we study calmness and exact penalization properties for a class of constrained vector set-valued optimization problems. We establish equivalence relationships between (local) calmness and (local) exact penalization for this class of constrained vector set-valued optimization problems. Some necessary and /or sufficient conditions for (local) calmness are also derived.

Key words: *constrained vector set-valued optimization, calmness, exact penalization*

Mathematics Subject Classification: *90C29, 90C46*

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1 Introduction

Penalty methods have been extensively applied to solve various constrained optimization problems (see, e.g., [2, 6, 8, 9, 18, 22, 25]). The obvious advantage of the validness of exact penalization property lies in the fact that one can solve a constrained optimization problem by solving only one unconstrained optimization problem. Relationship between validness of exact penalization property and calmness for l_1 penalty functions for constrained scalar optimization problems were investigated in, e.g., [6, 23]. Equivalence between them was first established in [3, 4]. This equivalence was further developed for lower order penalty functions in [17, 24] and was then extended to constrained vector optimization problems in [15, 16].

Set-valued optimization has wide applications in vector optimization (see, e.g., [5, 10, 21] and references therein), economics (see, e.g., [1] and references therein) and finance (if the set-valued risk is to be minimized, see, e.g., [12, 13] and references therein). Set-valued optimization has received considerable attention from the optimization community (see, e.g., [7, 11, 20, 26] and references therein). In [14], we obtained equivalence relations between (local) calmness and (local) exact penalization for constrained scalar set-valued optimization problems. The aim of this paper is to introduce and study a more general penalty scheme than those considered in [14, 15] for a class of constrained vector set-valued optimization problems. We derive equivalences between the existence of a (local) exact penalty function and the corresponding (local) calmness for the constrained vector set-valued optimization problems. Some necessary and/or sufficient conditions for the (local) calmness are also given.

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The paper is organized as follows. In Section 2, we present the constrained vector set-valued optimization problem, the associated perturbed problem, the penalty problem, some definitions and notations which will be used in the subsequent sections. In Section 3, we establish equivalence between local calmness and local exact penalization. In Section 4, equivalences between several types of calmness and exact penalization will be derived. In Section 5, we give some necessary and/or sufficient conditions for various notions of (local) calmness.

2 Problems, Definitions and Notations

In this section, we present the constrained vector set-valued optimization problem, its associated perturbed problem, the penalty problem, related definitions and notations which will be used in the subsequent sections.

Let X, Y and Z be normed spaces, $X_1 \subset X$ be a nonempty and closed set, $C \subset Y$ be a nontrivial, closed and convex cone with nonempty interior $intC$. Y is ordered by C, i.e., for any $y_1, y_2 \in Y$, $y_1 \leq_C y_2$ if and only if $y_2 - y_1 \in C$. Fix an $e \in intC$. Let $D \subset Z$ be a closed and convex cone. Let $F : X_1 \to 2^Y \backslash \{\emptyset\}$ be a set-valued map and $G : X_1 \to 2^Z \backslash \{\emptyset\}$ be a nonempty-compact-valued map.

Consider the following constrained vector set-valued optimization problem:

(CVSO) $\quad C - \min \quad F(x) \qquad \text{s.t.} \quad x \in X_1, \quad G(x) \cap -D \neq \emptyset.$

Denote by

$$X_0 := \{x \in X_1 : G(x) \cap -D \neq \emptyset\}$$

the feasible set of (CVSO).

Throughout the paper, we always assume that $X_0 \neq \emptyset$.

Definition 2.1. (i) $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$ is said to be a local (weakly) efficient solution pair of (CVSO) iff there exists a neighbourhood U of $\bar{x}$ such that

$$[F(x) - \{\bar{y}\}] \cap -C\backslash\{0\} = \emptyset, \forall x \in X_0 \cap U \ (resp.\, [F(x) - \{\bar{y}\}] \cap -intC = \emptyset, \forall x \in X_0 \cap U);$$

(ii) $\bar{x} \in X_0$ is said to be a local (weakly) efficient solution of (CVSO) iff there exists $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a local (weakly) efficient solution pair of (CVSO);

(iii) $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$ is said to be a (weakly) efficient solution pair of (CVSO) iff

$$[F(x) - \{\bar{y}\}] \cap -C\backslash\{0\} = \emptyset, \forall x \in X_0 \ (resp.\, [F(x) - \{\bar{y}\}] \cap -intC = \emptyset, \forall x \in X_0).$$

In this case, $\bar{y}$ is called a (weakly) efficient point of (CVSO);

(iv) $\bar{x} \in X_0$ is said to be a (weakly) efficient solution of (CVSO) iff there exists $\bar{y} \in F(\bar{x})$ such that $(\bar{x}, \bar{y})$ is a (weakly) efficient solution pair of (CVSO).

It is obvious that a (weakly) efficient solution (pair) is a local (weakly) efficient solution (pair) and an efficient solution (pair) (point) is a weakly efficient solution (pair) (point).

In this paper, two imaginary points $-\infty, +\infty$ will be used. By $+\infty$, we mean that $y \leq_C +\infty$ for any $y \in Y$ while $-\infty$ means that $-\infty \leq_C y$ for any point $y \in Y$.

Let $Y_1 \subset Y$. $\bar{y} \in Y$ is said to be an infimum point of Y_1 iff

(I) for $\bar{y} \in Y$, if (i) $y - \bar{y} \notin -C\backslash\{0\}, \forall y \in Y_1$ and (ii) there exists a sequence $\{y_n\} \subset Y_1$ such that $y_n \to \bar{y}$;

(II) for $\bar{y} = -\infty$, if there exist sequences $\{y_n\} \subset Y_1$ and $\{t_n\} \subset R^1_+$ with $t_n \to +\infty$ such that $y_n \leq_C -t_n e$;

(III) for $\bar{y} = +\infty$, if $Y_1 = \emptyset$.

Denote by $infY_1$ the set of all infimum points of Y_1. Let V_1 denote the set of all infimum points of (CVSO), i.e., $V_1 = infF(X_0)$. Denote by V_2 and V_3 the sets of efficient points, weakly efficient points of (CSVO), respectively. Clearly, $V_2 \subset V_3$ and $V_2 \subset V_1$.

Let $z \in Z$. Consider the following perturbed problem of (CVSO):

$(CVSO_z)$ $\qquad$ inf $\;F(x)$ $\qquad$ s.t. $\;x \in X_1, \quad G(x) \cap (-D + \{z\}) \neq \emptyset.$

Denote by X_z and $V_i(z), i = 1, 2, 3$ the feasible set and sets of infimum points, efficient points and weakly efficient points of $(CVSO_z)$, respectively.

Clearly, $V_i(0) = V_i, i = 1, 2, 3.$

Let $\sigma : R^1_+ \to R^1_+ \cup \{+\infty\}$ be a proper, nondecreasing, upper semicontinuous function satisfying $argmin(\sigma) = \{0\}$.

Definition 2.2. Let $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$ be a local (weakly) efficient solution pair of (CVSO).

(i) (CVSO) is said to be locally σ-calm at $(\bar{x}, \bar{y})$ iff there exists $M > 0$ such that for any sequences $\{z_n\} \subset Z\backslash\{0\}$ with $z_n \to 0$, $\{x_n\} \subset X$ with each $x_n \in X_{z_n}$ and $x_n \to \bar{x}$, it holds that

$$\left[\frac{F(x_n) - \{\bar{y}\}}{\sigma(\|z_n\|)} + \{Me\}\right] \cap -intC = \emptyset;$$

(ii) (CVSO) is said to be locally σ-calm at $\bar{x}$ iff (CVSO) is locally σ-calm at each of its local (weakly) efficient solution pair $(\bar{x}, \bar{y})$;

(iii) (CVSO) is said to be uniformly locally σ-calm at $\bar{x}$ iff there exists $M > 0$ such that for any local (weakly) efficient solution pair $(\bar{x}, \bar{y})$, any sequences $\{z_n\} \subset Z\backslash\{0\}$ with $z_n \to 0$, $\{x_n\} \subset X$ with each $x_n \in X_{z_n}$ and $x_n \to \bar{x}$, it holds that

$$\left[\frac{F(x_n) - \{\bar{y}\}}{\sigma(\|z_n\|)} + \{Me\}\right] \cap -intC = \emptyset.$$

Remark 2.3. (i) Definition 2.2 does not depend on the choice of $e \in intC$.

(ii) When $Y = R^1$, $C = R^1_+$, $\sigma(t) = t^\alpha, t \geq 0$, where $\alpha > 0$, Definition 2.2 reduces to the local calmness of order α defined in [14].

(iii) If F and G are single-valued and $\sigma(t) = t^\alpha, t \geq 0$, where $\alpha > 0$, Definition 2.2 reduces to the local calmness of order α defined in [15].

(iv) It is easily seen that uniform local σ-calmness at $\bar{x}$ implies local σ-calmness at $\bar{x}$, which in turn implies local σ-calmness at $\bar{x}$ at each (weakly) efficient solution pair $(\bar{x}, \bar{y})$.

Definition 2.4. Let $i \in \{1, 2, 3\}$. (CVSO) is said to be type i uniformly σ-calm iff there exist $M > 0$ and a neighbourhood W of $0 \in Z$ such that

$$\left[\frac{V_i(z) - V_i}{\sigma(\|z\|)} + \{Me\}\right] \cap -intC = \emptyset, \forall z \in W\backslash\{0\}.$$

Remark 2.5. (i) The notion of uniform local σ-calmness of (CVSO) has nothing to do with the choice of $e \in intC$.

(ii) When $Y = R^1$, $C = R^1_+$, $\sigma(t) = t^\alpha, t \geq 0$, where $\alpha > 0$, Definition 2.2 reduces to the calmness of order α defined in [14].

(iii) If F and G are single-valued and $\sigma(t) = t^\alpha, t \geq 0$, where $\alpha > 0$, Definition 2.2 reduces to the calmness of order α defined in [15].

(iv) Type 1 uniform σ-calmness implies type 2 uniform σ-calmness and Type 3 uniform σ-calmness implies type 2 uniform σ-calmness.

Definition 2.6. Let $i \in \{i = 1, 2, 3\}, \bar{y} \in V_i$. (CVSO) is said to be type i σ-calm at $\bar{y}$ iff there exist $M > 0$ and a neighbourhood W of $0 \in Z$ such that

$$\left[\frac{V_i(z) - \{\bar{y}\}}{\sigma(\|z\|)} + \{Me\}\right] \cap -intC = \emptyset, \forall z \in W \backslash \{0\}.$$

If (CVSO) is type i σ-calm at each $\bar{y} \in V_i$, we say that it is type i σ-calm.

It is obvious that type i uniform σ-calmness implies type i-calmness, which in turn implies type i σ-calmness at each $\bar{y} \in V_i$. Moreover, similar remarks can be made on Definition 2.6 as those on Definition 2.4.

At the end of this section, we introduce the penalty problem:

$$(\sigma - PP_r) \qquad \inf_{x \in X_1} F(x) + \{r\sigma(d(G(x), -D))e\},$$

where $r > 0$ is the penalty parameter and

$$d(G(x), -D) = inf\{\|z + d\| : z \in G(x), d \in D\}.$$

Denote by $V_i^\sigma(r), i = 1, 2, 3$ the sets of infimum points, efficient points and weakly efficient points of $(\sigma - PP_r)$, respectively.

$\boxed{3}$ Equivalence between Local Calmness and Local Exact Penalization

In this section, we derive equivalence relations between local calmness and local exact penalization.

The following function $\xi : Y \to R^1$ will be frequently used in the sequel:

$$\xi(y) = min\{t : te - y \in C\}, \forall y \in Y.$$

It is known from [5] that ξ is a strictly monotone (i.e., $\xi(y_1) \leq \xi(y_2), \forall y_1, y_2 \in Y$ with $y_2 - y_1 \in C$ and $\xi(y_1) < \xi(y_2)$, if $y_2 - y_1 \in intC$), positively homogeneous, subadditive, convex and continuous function. Moreover, $\xi(y) \geq 0$ iff $y \notin -intC$ and $\xi(y + te) = \xi(y) + t, \forall y \in Y$ and $t \in R^1$.

Theorem 3.1. *Let $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$ be a local (weakly) efficient solution pair of (CVSO). Assume that there exist $r_0 > 0$, $m_0 \in R^1$ and a neighbourhood U_0 of $\bar{x}$ such that*

$$[F(x) + r_0\{\sigma(d(G(x), -D))e - m_0 e\}] \cap -intC = \emptyset, \forall x \in X_1 \cap U_0. \tag{3.1}$$

Then the following two statements are equivalent.

(i) *(CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$;*

(ii) *there exists $\bar{r} > r_0$ such that $(\bar{x}, \bar{y})$ is a local (weakly) efficient solution pair to $(\sigma - PP_r)$ whenever $r \geq \bar{r}$.*

Proof. We prove only the case of local weakly efficient solution pair since the local efficient solution case can be analogously proved.

$(i) \Rightarrow (ii)$ Suppose to the contrary that (ii) does not hold. Then, there exist $0 < r_n \uparrow +\infty$, $\{x_n\} \subset X_1$ with $x_n \to \bar{x}$ and $y_n \in F(x_n)$ such that

$$y_n - \bar{y} + r_n\sigma(d(G(x_n), -D))e \in -intC. \tag{3.2}$$

From this, it is easily seen that $x_n \notin X_0$ when n is sufficiently large. Otherwise, $(\bar{x}, \bar{y})$ cannot be a local weakly efficient solution pair of (CVSO). Consequently, $d(G(x_n), -D) \neq 0$ when n is sufficiently large. From (3.1), we have

$$\xi(y_n) + r_0\sigma(d(G(x_n), -D)) \geq m_0$$

when n is sufficiently large. This together with (3.2) implies

$$\sigma(d(G(x_n), -D)) < \frac{\xi(\bar{y}) - m_0}{r_n - r_0} \to 0 \tag{3.3}$$

as $n \to +\infty$. We assert that

$$0 < s_n = d(G(x_n), -D) \to 0 \tag{3.4}$$

as $n \to +\infty$. Indeed, suppose to the contrary that there exist a subsequence $\{n_k\}$ of $\{n\}$ and some $\delta > 0$ such that $s_{n_k} \geq \delta$. Then, $\sigma(s_{n_k}) \geq \sigma(\delta) > 0$ by the nondecreasingness of σ, contradicting (3.4). It follows that for any $1 > \epsilon > 0$, there exist $z_n \in G(x_n)$ and $d_n \in D$ such that $\|z_n + d_n\| \leq (1+\epsilon)s_n \to 0$ as $n \to +\infty$. Let $z'_n = z_n + d_n$. Then

$$\|z'_n\| \leq (1+\epsilon)s_n \to 0. \tag{3.5}$$

Clearly, $z_n \in G(x_n) \cap (z'_n - D)$, namely, $x_n \in X_{z'_n}$. It follows from (3.2) that

$$\frac{y_n - \bar{y}}{2\sigma(d(G(x_n), -D))} + \frac{r_n}{2}e \in -intC$$

when n is sufficiently large. This together with (3.4), (3.5), the nondecreasingness of σ and the upper semicontinuity of σ at $s_n > 0$ yields

$$\frac{y_n - \bar{y}}{\sigma(\|z'_n\|)} + \frac{r_n}{2}e \leq_C \frac{y_n - \bar{y}}{\sigma((1+\epsilon)s_n)} + \frac{r_n}{2}e \leq_C \frac{y_n - \bar{y}}{2\sigma(d(G(x_n), -D))} + \frac{r_n}{2}e \in -intC$$

when $\epsilon > 0$ is so small that $\sigma((1+\epsilon)s_n) \leq 2\sigma(s_n)$, contradicting the fact that (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$.

$(ii) \Rightarrow (i)$ Suppose to the contrary that (i) does not hold. Then, there exist $0 < M_n \to +\infty$, $\{z_n\} \subset Z\backslash\{0\}$ with $z_n \to 0$, $\{x_n\} \subset X$ with $x_n \in X_{z_n}$, $x_n \to \bar{x}$, and $y_n \in F(x_n)$ such that

$$\frac{y_n - \bar{y}}{\sigma(\|z_n\|)} + M_n e \in -intC.$$

Thus,

$$y_n + M_n\sigma(\|z_n\|)e \leq_C \bar{y}. \tag{3.6}$$

By $x_n \in X_{z_n}$, we have $G(x_n) \cap (z_n - D) \neq \emptyset$. That is, $z_n \in G(x_n) + D$. Thus, $d(G(x_n), -D) \leq \|z_n\|$. It follows that

$$\sigma(d(G(x_n), -D)) \leq \sigma(\|z_n\|).$$

This combined with (3.6) yields

$$y_n + M_n\sigma((d(G(x_n), -D))e \leq_C \bar{y}. \tag{3.7}$$

Clearly, $d(G(x_n), -D)) \neq 0$ when n is suffieciently large. Otherwise, $(\bar{x}, \bar{y})$ cannot be a local weakly efficient solution pair of (CVSO). It follows from (3.7) that

$$y_n + (M_n - 1)\sigma(d(G(x_n), -D))e - \bar{y} \in -intC$$

when n is sufficiently large. Therefore, $(\bar{x}, \bar{y})$ is a local weakly efficient solution pair to $(\sigma - PP_r)$, contradicting (ii). The proof is complete. □

The following example verifies the results of Theorem 3.1.

Example 3.2. Let $X = X_1 = Z = R^1$, $D = R^1_+$, $Y = R^2$ and $C = R^2_+$. Let $F(x) = \{(-s, s) \in R^2 : s \leq x\}, \forall x \in R^1$, $G(x) = [x^3, 0], x \leq 0$; $G(x) = [x^3, 2x^3], x > 0$. Suppose that $\sigma(t) = t^{1/3}, t \geq 0$. It is routine to check that $(\bar{x}, \bar{y}) = (0, (0, 0))$ is a locally efficient solution pair to (CSVO), condition (3.1) holds and (i) and (ii) of Theorems 3.1 hold.

The next corollary follows immediately from Theorem 3.1.

Corollary 3.3. *Let $\bar{x} \in X_0$ be a local (weakly) efficient solution pair of (CVSO). Assume that for each local (weakly) efficient solution pair $(\bar{x}, \bar{y})$, there exist $r_0(\bar{y}) > 0$, $m_0(\bar{y}) \in R^1$ and a neighbourhood $U_0(\bar{y})$ of $\bar{x}$ such that (3.1) holds (with r_0, m_0 and U_0 replaced by $r_0(\bar{y}), m_0(\bar{y})$ and $U_0(\bar{y})$, respectively).*

Then the following two statements are equivalent.

(i) *(CVSO) is locally σ-calm at $\bar{x}$;*

(ii) *for each local (weakly) efficient solution pair $(\bar{x}, \bar{y})$, there exists $\bar{r}(\bar{y}) > r_0(\bar{y})$ such that $(\bar{x}, \bar{y})$ is a local (weakly) efficient solution pair to $(\sigma - PP_r)$ whenever $r \geq \bar{r}(\bar{y})$.*

Similar to the proof of Theorem 3.1, we can prove the next theorem (see also the proof of Theorem 4.2 in the next section).

Theorem 3.4. *Let $\bar{x} \in X_0$ be a local (weakly) efficient solution of (CVSO). Assume that there exist $r_0 > 0$, $m_0 \in R^1$ and a neighbourhood U_0 of $\bar{x}$ such that (3.1) holds. Further assume that there exists $M_0 > 0$ such that*

$$\bar{y} \leq_C M_0 e \tag{3.8}$$

for any $\bar{y} \in Y$ with $(\bar{x}, \bar{y})$ being a local (weakly) efficient solution pair of (CVSO).

Then the following two statements are equivalent.

(i) *(CVSO) is uniformly local σ-calm at $\bar{x}$;*

(ii) *there exist $\bar{r} > r_0$ and a neighbourhood U_1 of $\bar{x}$ such that, for any $\bar{y}$ with $(\bar{x}, \bar{y})$ being a local (weakly) efficient solution pair, it holds that*

$$[F(x) + r\sigma(d(G(x), -D))e - \bar{y}] \cap -C \backslash \{0\} = \emptyset, \forall x \in U_1 \cap X_1$$

(resp. $[F(x) + r\sigma(d(G(x), -D))e - \bar{y}] \cap -intC = \emptyset, \forall x \in U_1 \cap X_1$) whenever $r \geq \bar{r}$.

$\boxed{4}$ Equivalence between Calmness and Exact Penalization

In this section, we establish equivalences between (uniform) type i $(i = 1, 2, 3)$ calmness of (CVSO) and exact penalization of $(\sigma - PP_r)$.

Definition 4.1. Let $z \in Z$. $(CVSO_z)$ is said to be inf-externally stable (resp. externally stable, weakly externally stable) iff for any $x \in X_z$ and any $y \in F(x)$, there exists $\bar{y} \in V_1$(resp. V_2, V_3) such that $\bar{y} \leq_C y$.

Theorem 4.2. *Assume that $(CVSO_z)$ is inf-externally stable when $\|z\|$ is sufficiently small. Suppose that there exist $r_0 > 0$, $m_0 \in R^1$ such that*

$$[F(x) + \{r_0\sigma(d(G(x), -D))e - m_0 e\}] \cap -intC = \emptyset, \forall x \in X_1 \tag{4.1}$$

holds. Further assume that there exists $M_0 > 0$ such that (3.8) holds for any $\bar{y} \in V_1$. Then, the following two statements are equivalent:

(i) *(CVSO) is type 1 uniformly σ-calm;*

(ii) *there exists $\bar{r} > r_0$ such that $V_1 \subset V_1^\sigma(r)$ whenever $r \geq \bar{r}$.*

Proof. $(i) \Rightarrow (ii)$ Suppose to the contrary that (ii) does not hold. Then, there exist $0 < r_n \uparrow +\infty$ and $\bar{y}_n \in V_1$ such that $\bar{y}_n \notin V_1^\sigma(r_n)$. Since $\bar{y}_n \in V_1$, there exist $\{x_{n,k}\} \subset X_0$ and $y_{n,k} \in F(x_{n,k})$ such that $y_{n,k} \to \bar{y}_n$ as $k \to +\infty$. Note that $y_{n,k} \in F(x_{n,k}) + \{r_n\sigma(d(G(x_{n,k}), -D))e\} = F(x_{n,k})$. This combined with $\bar{y}_n \notin V_1^\sigma(r_n)$ yields that there exist $x_n \in X_1$ and $y_n \in F(x_n)$ such that

$$y_n + r_n\sigma(d(G(x_n), -D))e - \bar{y}_n \in -C\backslash\{0\}.$$

This together with (3.8) yields,

$$\xi(y_n) - M_0 + r_n\sigma(d(G(x_n), -D)) \le \xi(y_n - \bar{y}_n) + r_n\sigma(d(G(x_n), -D)) \le 0. \tag{4.2}$$

On the other hand, from (4.1), we deduce that

$$\xi(y_n) + r_0\sigma(d(G(x_n), -D)) \ge m_0. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$m_0 - M_0 + (r_n - r_0)\sigma(d(G(x_n), -D)) \le 0.$$

That is,

$$\sigma(d(G(x_n), -D)) < \frac{M_0 - m_0}{r_n - r_0}$$

when n is sufficiently large. Consequently, $\lim_{n\to+\infty} \sigma(d(G(x_n), -D)) = 0$. Hence, $\lim_{n\to+\infty} d(G(x_n), -D) = 0$. Arguing as in the proof of Theorem 3.1 (i)$\Rightarrow$(ii), we obtain a sequence $\{z'_n\} \subset Z\backslash\{0\}$ with $x_n \in X_{z'_n}$ and $z'_n \to 0$ such that

$$\frac{y_n - \bar{y}'_n}{\sigma(\|z'_n\|)} + r_n e \le_C \frac{y_n - \bar{y}'_n}{\sigma(d(G(x_n), -D))} + r_n e \in -C\backslash\{0\}. \tag{4.4}$$

By the inf-external stability of $(CVSO_{z'_n})$ (when n is sufficiently large) and $y_n \in F(x_n)$, $x_n \in X_{z'_n}$, we obtain $y'_n \in V_1(z'_n)$ such that $y'_n \le_C y_n$. This together with (4.4) yields

$$\frac{y'_n - \bar{y}'_n}{\sigma(\|z'_n\|)} + (r_n - 1)e \in -intC$$

when n is sufficiently large, contradicting (i).

(ii)$\Rightarrow$(i) Suppose to the contrary that (i) does not hold. Then, there exist $\{z_n\} \subset Z\backslash\{0\}$ with $z_n \to 0$, $0 < M_n \to +\infty$, $\bar{y}_n \in V_1$ and $\bar{y}'_n \in V_1(z_n)$ such that

$$\frac{\bar{y}'_n - \bar{y}_n}{\sigma(\|z_n\|)} + M_n e \in -intC.$$

It follows that there exist $x_n \in X_{z_n}$ and $y_n \in F(x_n)$ such that

$$\frac{y_n - \bar{y}_n}{\sigma(\|z_n\|)} + M_n e \in -intC. \tag{4.5}$$

On the other hand, by (ii), there exists $\bar{r} > r_0$ such that $\bar{y}_n \in V_1^\sigma(\bar{r})$. Thus,

$$y_n + \bar{r}\sigma(d(G(x_n), -D))e - \bar{y}_n \notin -C\backslash\{0\}. \tag{4.6}$$

Note that $G(x_n) \cap (z_n - D) \ne \emptyset$. Consequently, $d(G(x_n), -D) \le \|z_n\|$. It follows that $\sigma(d(G(x_n), -D)) \le \sigma(\|z_n\|)$. This combined with (4.6) yields

$$y_n + \bar{r}\sigma(\|z_n\|)e - \bar{y}_n \notin -C\backslash\{0\},$$

contradicting (4.5) when n is sufficiently large. The proof is complete. □

The following example verifies Theorem 4.2.

Example 4.3. Let $X = Z = R^1$, $D = R^1_+$, $X_1 = [-1, 1]$, $Y = R^2$ and $C = R^2_+$. Let $F(x) = \{(-s, s) \in R^2 : s \le x\}, \forall x \in R^1$, $G(x) = [x^3, 0], x \le 0$; $G(x) = [x^3, 2x^3], x > 0$. Suppose that $\sigma(t) = t^{1/3}, t \ge 0$. It can be routinely checked that all the conditions and results of Theorem 4.2 hold.

Analogous to the proof of Theorem 4.2, we can prove the next two theorems.

Theorem 4.4. *Let $i \in \{2, 3\}$. Assume that $(CVSO_z)$ is externally stable for $i = 2$ and weakly externally stable for $i = 3$, when $\|z\|$ is sufficiently small. Suppose that there exist $r_0 > 0$, $m_0 \in R^1$ such that (4.1) holds. Further assume that there exists $M_0 > 0$ such that (3.8) holds for any $\bar{y} \in V_i$. Then, the following two statements are equivalent:*
(i) *(CVSO) is type i uniformly σ-calm;*
(ii) *there exists $\bar{r} > r_0$ such that $V_i \subset V_i^\sigma(r)$ whenever $r \ge \bar{r}$.*

Theorem 4.5. *Let $i \in \{1, 2, 3\}$. Let $\bar{y} \in V_i$. Assume that $(CVSO_z)$ is inf-externally stable for $i = 1$, externally stable for $i = 2$ and weakly externally stable for $i = 3$, when $\|z\|$ is sufficiently small. Suppose that there exist $r_0 > 0$, $m_0 \in R^1$ such that (4.1) holds. Then, the following two statements are equivalent:*
(i) *(CVSO) is type i σ-calm at $\bar{y}$;*
(ii) *there exists $\bar{r} > r_0$ such that $\bar{y} \in V_i^\sigma(r)$ whenever $r \ge \bar{r}$.*

The following corollary follows immediately from Theorem 4.5.

Corollary 4.6. *Let $i \in \{1, 2, 3\}$. Assume that $(CVSO_z)$ is inf-externally stable for $i = 1$, externally stable for $i = 2$ and weakly externally stable for $i = 3$, when $\|z\|$ is sufficiently small. Suppose that there exist $r_0 > 0$, $m_0 \in R^1$ such that (4.1) holds. Then, the following two statements are equivalent:*
(i) *(CVSO) is type i σ-calm;*
(ii) *for each $\bar{y} \in V_i$, there exists $\bar{r}(\bar{y}) > r_0$ such that $\bar{y} \in V_i^\sigma(r)$ whenever $r \ge \bar{r}(\bar{y})$.*

5 Necessary and/or Sufficient Conditions for (Local) Calmness

In this section, we derive necessary and/or sufficient conditions for (local) σ-calmness of (CVSO).

Let C^* be the positive polar cone of C defined by

$$C^* = \{\lambda \in Y^* : \lambda(c) \ge 0, \forall c \in C\},$$

where Y^* is the dual space of Y. Let

$$C^{*0} = \{\lambda \in C^* : \lambda(e) = 1\}.$$

It is known from [15] that C^{*0} is weak*−compact. Moreover, from [5], we have that

$$\xi(y) = max_{\lambda \in C^{*0}} \lambda(y), \forall y \in Y.$$

Let $\bar{x} \in X_0$ and $\bar{y} \in Y$. Consider the following scalar set-valued optimization problem

$$(CSSO(\bar{x}, \bar{y})) \qquad \min \quad F_1(x) \qquad \text{s.t.} \quad x \in X_1, \quad G(x) \cap -D \neq \emptyset,$$

where

$$F_1(x) = \{\xi(y - \bar{y}) : y \in F(x)\} = \{max_{\lambda \in C^{*0}} \lambda(y - \bar{y}) : y \in F(x)\}, \forall x \in X_1. \tag{5.1}$$

Recall from [14] that $\bar{x} \in X_0$ is called a (local) solution of $(CSSO(\bar{x},\bar{y}))$ with (local) optimal value $\bar{t}$ if $\bar{t} \in F_1(\bar{x})$ (resp. there exists a neighbourhood U of $\bar{x}$) such that $\bar{t} \leq t, \forall y \in F_1(x), \forall x \in X_0$ (resp. $\bar{t} \leq t, \forall y \in F_1(x), \forall x \in X_0 \cap U$).

It is easily checked that $(\bar{x},\bar{y})$ is a (local) weakly efficient solution of (CVSO) if and only if $\bar{x}$ is a (local) solution to $(CSSO(\bar{x},\bar{y}))$ with (local) optimal value $\bar{t} = 0$.

Definition 5.1 ([14])**.** Let $\bar{x} \in X_0$ and $\bar{y} \in Y$. Consider $(CSSO(\bar{x},\bar{y}))$. Suppose that $\bar{x} \in X_0$ is a local solution to $(CSSO(\bar{x},\bar{y}))$ with local optimal value $\bar{t}$. $(CSSO(\bar{x},\bar{y}))$ is said to be locally σ-calm at $\bar{x}$ iff there exists $M > 0$ such that for any sequences $z_n \in Z\backslash\{0\}$ with $z_n \to 0$, $x_n \in X$ with $x_n \in X_{z_n}$ and $x_n \to \bar{x}$, and $t_n \in F_1(x_n)$, there holds

$$\frac{t_n - \bar{t}}{\sigma(\|z_n\|)} + M \geq 0. \tag{5.2}$$

Definition 5.2 ([14])**.** Let $\bar{x} \in X_0$ and $\bar{y} \in Y$. Consider $(CSSO(\bar{x},\bar{y}))$. Suppose that $\bar{x} \in X_0$ is a solution to $(CSSO(\bar{x},\bar{y}))$ with optimal value $\bar{t}$. $(CSSO(\bar{x},\bar{y}))$ is said to be σ-calm iff there exist $M > 0$ and a neighbourhood W of $0 \in Z$ such that

$$\frac{\bar{v}(z) - \bar{t}}{\sigma(\|z\|)} + M \geq 0, \forall z \in W\backslash\{0\}, \tag{5.3}$$

where $\bar{v}(z)$ is the optimal value of the following perturbed problem of $(CSSO(\bar{x},\bar{y}))$:

$$(CVSO_z(\bar{x},\bar{y})) \qquad \min \quad F_1(x) \qquad \text{s.t.} \quad x \in X_1, \quad G(x) \cap (z - D) \neq \emptyset$$

and F_1 is defined by (5.1).

Theorem 5.3. *Let $(\bar{x},\bar{y}) \in X_0 \times F(\bar{x})$ be a local weakly efficient solution pair to (CVSO). Then, the following two statements are equivalent.*

(i) *(CVSO) is locally σ-calm at $(\bar{x},\bar{y})$;*

(ii) *$(CSSO(\bar{x},\bar{y}))$ is locally σ-calm at $\bar{x}$ with $\bar{t} = 0$.*

Proof. (i)⇒(ii) Suppose to the contrary that there exist $0 < M_n \to +\infty$, $z_n \in Z\backslash\{0\}$ with $z_n \to 0$, $x_n \in X_{z_n}$ with $x_n \to \bar{x}$ and $t_n \in F_1(x_n)$ such that

$$\frac{t_n}{\sigma(\|z_n\|)} + M_n < 0. \tag{5.4}$$

Then, from $t_n \in F_1(x_n)$ and (5.4), there exists $y_n \in F(x_n)$ such that

$$\frac{\xi(y_n - \bar{y})}{\sigma(\|z_n\|)} + M_n < 0. \tag{5.5}$$

Namely,

$$\xi(\frac{y_n - \bar{y}}{\sigma(\|z_n\|)}) + M_n < 0.$$

It follows that

$$\frac{y_n - \bar{y}}{\sigma(\|z_n\|)} + M_n e \in -intC, \tag{5.6}$$

contradicting the fact that (CVSO) is locally σ-calm at $(\bar{x},\bar{y})$.

(ii)⇒(i) Suppose to the contrary that there exist $0 < M_n \to +\infty$, $z_n \in Z\backslash\{0\}$ with $z_n \to 0$, $x_n \in X_{z_n}$ with $x_n \to \bar{x}$ and $y_n \in F(x_n)$ such that (5.6) holds. Then, (5.5) holds. Let $t_n = \xi(y_n - \bar{y})$. Then, (5.4) holds, contradicting the fact that $(CSSO(\bar{x},\bar{y}))$ is locally σ-calm at $\bar{x}$ with $\bar{t} = 0$. □

The next corollary follows immediately from Theorem 5.3.

Corollary 5.4. *Let $\bar{x} \in X_0$ be a local weakly efficient solution to (CVSO). Then, (CVSO) is locally σ-calm at $\bar{x}$ if and only if for any local weakly efficient solution pair $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$, $(CSSO(\bar{x}, \bar{y}))$ is locally σ-calm at $\bar{x}$ with $\bar{t} = 0$.*

The following theorems can proved analogously to Theorem 5.3.

Theorem 5.5. *Let $\bar{x} \in X_0$ be a local weakly efficient solution to (CVSO). Then, (CVSO) is uniformly local σ-calm at $\bar{x}$ if and only if there exists $M > 0$ such that for any local weakly efficient solution pair $(\bar{x}, \bar{y})$ of (CVSO), and any sequences $z_n \in Z \backslash \{0\}$ with $z_n \to 0$, $x_n \in X$ with $x_n \in X_{z_n}$ and $x_n \to \bar{x}$, and $t_n \in F_1(x_n)$, (5.2) holds with $\bar{t} = 0$.*

Theorem 5.6. *Let $i \in \{2, 3\}$ and $\bar{y} \in V_i$ with $\bar{y} \in F(\bar{x})$ for some $\bar{x} \in X_0$. Then,*

(i) *if $(CSSO(\bar{x}, \bar{y}))$ is σ-calm with $\bar{t} = 0$, then (CVSO) is σ-calm at $\bar{y}$;*

(ii) *assume that $(CVSO_z)$ is externally stable for $i = 2$ and $(CVSO_z)$ is weakly externally stable for $i = 3$ when $\|z\|$ is sufficiently small, then the converse of (i) is also true.*

The next corollary follows directly from Theorem 5.6.

Corollary 5.7. *Let $i \in \{2, 3\}$.*

(i) *If for each $\bar{y} \in V_i$ with $\bar{y} \in F(\bar{x})$ and $\bar{x} \in X_0$, $(CSSO(\bar{x}, \bar{y}))$ is σ-calm with $\bar{t} = 0$, then (CVSO) is type i σ-calm;*

(ii) *assume that $(CVSO_z)$ is externally stable for $i = 2$ and $(CVSO_z)$ is weakly externally stable for $i = 3$ when $\|z\|$ is sufficiently small, then the converse of (i) is also true.*

Theorem 5.8. *Let $i \in \{2, 3\}$.*

(i) *If there exist $M > 0$ and a neighbourhood W of $0 \in Z$ such that for any $\bar{y} \in V_i$ with $\bar{y} \in F(\bar{x})$ and $\bar{x} \in X_0$, (5.3) holds with $\bar{t} = 0$;*

(ii) *assume that $(CVSO_z)$ is externally stable for $i = 2$ and $(CVSO_z)$ is weakly externally stable for $i = 3$ when $\|z\|$ is sufficiently small, then the converse of (i) is also true.*

The next proposition presents a sufficient condition for the (local) σ-calmness of (CVSO) at its (local) weakly efficient solution pair, which is obtainable by linear scalarization.

Proposition 5.9. *Suppose that there exists $\lambda \in C^{*0}$ such that $\bar{x}$ is a (local) solution to the scalar set-valued optimization problem*

$(CSSO_\lambda)$ $\quad$ *min* $\quad F_\lambda(x)$ $\quad$ *s.t.* $\quad x \in X_1, \quad G(x) \cap -D \neq \emptyset,$

where

$$F_\lambda(x) = \{\lambda(y) : y \in F(x)\}, \forall x \in X_1. \tag{5.7}$$

Then, for any $\bar{y} \in F(\bar{x})$ with $\lambda(\bar{y}) = inf_{t \in F_\lambda(\bar{x})} t$, $(\bar{x}, \bar{y})$ is a (local) weakly efficient solution pair of (CVSO). Moreover, if $(CSSO_\lambda)$ is locally σ-calm at $\bar{x}$ with (local) optimal value $\bar{t} = \lambda(\bar{y})$, then (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$.

Proof. It is easily shown by contradiction that for any $\bar{y} \in F(\bar{x})$ with $\lambda(\bar{y}) = inf_{t \in F(\bar{x})} t$, $(\bar{x}, \bar{y})$ is a (local) weakly efficient solution pair of (CVSO). The rest of the proof is similar to that of Theorem 5.3 (ii)$\Rightarrow$(i). $\square$

Now we consider the case when (CVSO) is convex. More specifically, we make the following assumption.

Assumption A. $X_1 \subset X$ is nonempty, closed and convex, F is nonempty-valued, C-convex on X_1 (i.e., $\forall x_1, x_2 \in X_1$, $\forall \theta \in [0, 1]$, $\theta F(x_1) + (1 - \theta) F(x_2) \subset F(\theta x_1 + (1 - \theta) x_2) + C$), G

is nonempty-compact-valued and D-convex on X_1 (i.e., $\forall x_1, x_2 \in X_1$, $\forall \theta \in [0,1]$, $\theta G(x_1) + (1-\theta)G(x_2) \subset G(\theta x_1 + (1-\theta)x_2) + D$).

It is known from [19] that under Assumption A, X_0 is convex and $d(G(x), -D))$ is convex on X_1.

The next lemma follows immediately from ([26], Theorem 4.1).

Lemma 5.10. *Let Assumption A hold. Then, $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$ is a weakly efficient solution pair to (CVSO) if and only if there exists $\lambda \in C^{*0}$ such that $\bar{x}$ is a solution to $(CSSO_\lambda)$ with optimal value $\lambda(\bar{y})$.*

We need the following assumption.

Assumption B. The function $\sigma : R^1_+ \to R^1_+ \cup \{+\infty\}$ is proper, nondecreasing and convex with argmin$\sigma = \{0\}$.

Note that under Assumptions A and B, the set-valued map

$$F_r(x) := F(x) + r\{\sigma(d(G(x), -D))e\}$$

is C-convex on X_1 for any $r > 0$.

The following theorem gives necessary and sufficient conditions for the local σ-calmness of the constrained convex vector set-valued optimization problem (CVSO) at a weakly efficient solution pair.

Theorem 5.11. *Let Assumptions A and B hold and $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$.*

(i) *If $(\bar{x}, \bar{y})$ is a weakly efficient solution pair and (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$, then there exist $\lambda \in C^{*0}$ and $\bar{r} > 0$ such that $\bar{x}$ is a minimizer of the scalar set-valued function $F_\lambda(x) + \bar{r}\{\sigma(d(G(x), -D)\}$ with optimal value $\lambda(\bar{y})$, where $F_1(x)$ defined by (5.7);*

(ii) *The converse of (i) is also true.*

Proof. (i) By Theorem 3.1, there exists $\bar{r} > 0$ such that $(\bar{x}, \bar{y})$ is a local weakly efficient solution pair of the set-valued map $F_{\bar{r}}$ on X_1. Note that $F_{\bar{r}}$ is C-convex on X_1. Thus, $(\bar{x}, \bar{y})$ is a weakly efficient solution pair of the set-valued map $F_{\bar{r}}$ on X_1. Consequently, there exists $\lambda \in C^{*0}$ such that $\bar{x}$ is a minimizer of the scalar set-valued function $F_\lambda(x) + \bar{r}\{\sigma(d(G(x), -D)\}$ with optimal value $\lambda(\bar{y})$.

(ii) Since $\bar{x} \in X_0$ is a minimizer of the scalar set-valued function $F_\lambda(x)+\bar{r}\{\sigma(d(G(x), -D)\}$ with optimal value $\lambda(\bar{y})$, we have

$$\lambda(\bar{y}) \leq \lambda(y) + \bar{r}\sigma(d(G(x), -D), \forall y \in F(x), \forall x \in X_1.$$

That is,

$$\lambda(y - \bar{y} + (\bar{r}/\lambda(e))\sigma(d(G(x), -D)e) \geq 0.$$

Hence,

$$y - \bar{y} + (\bar{r}/\lambda(e))\sigma(d(G(x), -D)e \notin -intC, , \forall y \in F(x), \forall x \in X_1.$$

Consequently, $(\bar{x}, \bar{y})$ is a weakly efficient solution pair of the set-valued map $F_{\bar{r}/\lambda(e)}$ on X_1. Thus, $(\bar{x}, \bar{y})$ is a local weakly efficient solution pair of the set-valued map $F_{\bar{r}/\lambda(e)}$ on X_1. By Theorem 3.1, (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$. □

Remark 5.12. It is easily seen from the proof of Theorem 5.11 (ii) that we can prove the following conclusion without Assumptions A and B: if there exist $\lambda \in C^{*0}$ and $\bar{r} > 0$ such that $\bar{x}$ is a local minimizer of the scalar set-valued function $F_\lambda(x) + \bar{r}\{\sigma(d(G(x), -D))\}$ with local optimal value $\lambda(\bar{y})$, where $F_\lambda(x)$ is defined by (5.7), then (CVSO) is locally σ-calm at its local weakly efficient solution pair $(\bar{x}, \bar{y})$.

The following propositions provide sufficient conditions for the local σ-calmness of (CVSO) at its local weakly efficient solution pair.

Proposition 5.13. *Let σ satisfy that $\sigma(t) \geq \beta t, \forall t \geq 0$ for some $\beta > 0$. Let $(\bar{x}, \bar{y}) \in X_0 \times F(\bar{x})$. Suppose that there exist $\lambda \in C^{*0}$ and $\mu \in D^* = \{\mu \in Z^* : \mu(d) \geq 0, \forall d \in D\}$ such that $\bar{x}$ is a local minimizer of the scalar set-valued function $F_\lambda(x) + G_\mu(x)$ on X_1 with local optimal value $\lambda(\bar{y})$, where F_λ is defined by (5.7) and*

$$G_\mu(x) := \{\mu(z) : z \in G(x)\}, \forall x \in X_1. \tag{5.8}$$

Then, $(\bar{x}, \bar{y})$ is a local weakly efficient solution pair of (CVSO) and (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$.

Proof. We first show that $(\bar{x}, \bar{y})$ is a local weakly efficient solution pair of (CVSO). Otherwise, there exist $x_n \in X_0$ and $y_n \in F(x_n)$ such that $x_n \to \bar{x}$ and $y_n - \bar{y} \in -intC$. Therefore,

$$\lambda(y_n) < \lambda(\bar{y}). \tag{5.9}$$

From $x_n \in X_0$, we have $G(x_n) \cap -D \neq \emptyset$. It follows that there exists $z_n \in G(x_n) \cap -D$ such that

$$\mu(z_n) \leq 0. \tag{5.10}$$

From (5.9) and (5.10), we have

$$\lambda(y_n) + \mu(z_n) < \lambda(\bar{y}),$$

contradicting the assumption that $\bar{x}$ is a local minimizer of the scalar set-valued function $F_\lambda(x) + G_\mu(x)$ on X_1 with local optimal value $\lambda(\bar{y})$. Now we prove that $\bar{x}$ is a local minimizer of the scalar set-valued function $F_\lambda(x) + \bar{r}\{\sigma(d(G(x), -D)\}$ with local optimal value $\lambda(\bar{y})$, which further implies that $(\bar{x}, \bar{y})$ is a local weakly efficient solution pair of (CVSO) and (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$ by Remark 5.12. Indeed, since $\bar{x}$ is a local minimizer of the scalar set-valued function $F_\lambda(x) + G_\mu(x)$ on X_1 with local optimal value $\lambda(\bar{y})$, there exists a neighbourhood U of $\bar{x}$ such that

$$\lambda(\bar{y}) \leq \lambda(y) + \mu(z), \forall y \in F(x), z \in G(x), \forall x \in X_1 \cap U. \tag{5.11}$$

By the definition of $d(G(x), -D)$, for any $\epsilon > 0$, there exist $z_\epsilon \in G(x)$ and $d_\epsilon \in D$ such that

$$d(G(x), -D) \geq \|z_\epsilon + d_\epsilon\| - \epsilon.$$

This combined with (5.11) and the fact that $\mu(d_\epsilon) \geq 0$ yields

$$\begin{aligned}
\lambda(\bar{y}) &\leq \lambda(y) + \mu(z_\epsilon) \\
&= \lambda(y) + \mu(z_\epsilon + d_\epsilon) - \mu(d_\epsilon) \\
&\leq \lambda(y) + \|\mu\|\|z_\epsilon + d_\epsilon\| \\
&\leq \lambda(y) + \|\mu\| d(G(x), -D) + \|\mu\|\epsilon \\
&\leq \lambda(y) + \frac{\|\mu\|}{\beta}\sigma(d(G(x), -D)) + \|\mu\|\epsilon, \forall y \in F(x), \forall x \in X_1 \cap U.
\end{aligned}$$

By the arbitrariness of $\epsilon > 0$, we have

$$\lambda(\bar{y}) \leq \lambda(y) + \frac{\|\mu\|}{\beta}\sigma(d(G(x), -D)), \forall y \in F(x), \forall x \in X_1 \cap U.$$

Hence, $\bar{x}$ is a local minimizer of the scalar set-valued function $F_\lambda(x) + \{\frac{\|\mu\|}{\beta}\sigma(d(G(x), -D)\}$ with local optimal value $\lambda(\bar{y})$. The proof is complete. □

The next lemma follows trivially from ([26], Theorem 5.1).

Lemma 5.14. *Let Assumption A hold and $intD \neq \emptyset$. Consider (CVSO). Assume that Slater constraint qualification holds: there exists $x_0 \in X_1$ such that $G(x_0) \cap -intD \neq \emptyset$. Suppose that $(\bar{x}, \bar{y})$ is a weakly efficient solution pair to (CVSO). Then, there exist $\lambda \in C^{*0}$ and $\mu \in D^*$ such that $\bar{x}$ is a minimizer of the scalar set-valued function $F_\lambda(x) + G_\mu(x)$ on X_1 with optimal value $\lambda(\bar{y})$, where F_λ is defined by (5.7) and G_μ is defined by (5.8).*

Proposition 5.15. *Let Assumption A hold and σ satisfy that $\sigma(t) \geq \beta t, \forall t \geq 0$ for some $\beta > 0$. Assume that Slater constraint qualification holds. Then, (CVSO) is locally σ calm at each of its weakly efficient solution pair.*

Proof. Suppose that $(\bar{x}, \bar{y})$ is a weakly efficient solution pair to (CVSO). Then, by Lemma 5.14, there exist $\lambda \in C^{*0}$ and $\mu \in D^*$ such that $\bar{x}$ is a minimizer of the scalar set-valued function $F_\lambda(x) + G_\mu(x)$ on X_1 with optimal value $\lambda(\bar{y})$. By Proposition 5.2, (CVSO) is locally σ-calm at $(\bar{x}, \bar{y})$. □

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