Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 9, No. 3, pp. 413–425, July 2013

ESSENTIAL SOLUTIONS OF PARAMETRIC VECTOR OPTIMIZATION PROBLEMS

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Abstract: The concepts of essential solutions and essential solution sets for parametric vector optimization problems are introduced, and the relations among the concepts of essential solutions, essential solution sets and lower semicontinuity of solution mappings are discussed. The characterization of essential solutions is presented, and some sufficient conditions for closedness of solution mappings are obtained. Finally, some corollaries of the main results are given as applications for some special optimization models.

Key words: *essential solution, essential solution set, vector optimization, stability, parametric optimization*

Mathematics Subject Classification: *90C31, 90C29*

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1 Introduction

For parametric variational problems, a solution is said to be essential if it is stable with respect to the parameters, that is, there exists a sequence of parametric solutions converging to the essential solution.

In 1950, Fort [8] first introduced the notion of essential fixed points. A fixed point x of a mapping f is called to be essential if each mapping sufficiently near f has a fixed point arbitrarily near x. The method of essential fixed points has been widely used in various fields recently [11, 17, 18, 20, 23-26]. For example, it has been first applied in vector optimization [25] in 1992, and in scalar optimization [17] and equilibrium problems [26, 27], simultaneously, in 1999. The authors of [10, 12, 14, 15, 18, 23, 24, 27] also considered essential solution sets instead of essential solutions and searched for some essential components of the solution sets. For example, Yang and Yu [24] showed that every multi-objective generalized game satisfying some continuity and convexity conditions has at least one essential component of its weak Pareto-Nash equilibrium point sets. In addition, Reference [16] is a valuable book on stability of optimization problems.

The stability of optimization problems can also be transformed to the continuity [5-7, 9, 21, 22] and variational properties [1-4] of the feasible mappings and the solution mappings. The authors of [5-7, 9, 21, 22] established necessary and sufficient conditions for lower semicontinuity and/or upper semicontinuity of the feasible mappings and the solution mappings of vector optimization problems.

In this paper, we introduce the definitions of essential solutions and essential solution sets of parametric vector optimization problems. Moreover, we show that the existence of

ISSN 1348-9151

essential solutions follows from the lower semicontinuity of solution mappings, and implies the existence of essential solution set in Section 2. In Section 3, the characterization of essential solutions is presented. Furthermore, we obtain the sufficient conditions for lower semicontinuity and closedness of solution mappings. The model considered in this paper contains some optimization problems as special cases. In section 4, we give some corollaries of our main results as applications for some special cases.

2 Preliminaries

Let (X, d_X), (Λ, d_Λ) be metric spaces and $(Y, \|\cdot\|)$, $(Z, \|\cdot\|)$ be Banach spaces. For $\varepsilon > 0$, $B_X(x, \varepsilon)$ and $B_Y(y, \varepsilon)$ denote the ε-neighborhood of $x \in X$ and $y \in Y$, respectively, i.e.,

$$B_X(x, \varepsilon) = \{x' \in X : d_X(x, x') < \varepsilon\}$$

and

$$B_Y(y, \varepsilon) = \{y' \in Y : \|y - y'\| < \varepsilon\}.$$

Given a closed convex pointed cone $C \subset Z$ with a nonempty interior, the partial order $\preceq_C$ ($\prec_C$) in Z is defined as $z \preceq_C z'$ ($z \prec_C z'$) if and only if $z' - z \in C$ ($z' - z \in \text{int}C$, respectively) for $z, z' \in Z$. Let h be Hausdorff metric on the collection Φ of all nonempty closed subsets of Y, which is defined as

$$h(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}, \quad \forall A, B \in \Phi,$$

where $d(a, B) = \inf_{b \in B} \|a - b\|$ and $d(A, b) = \inf_{a \in A} \|a - b\|$. It follows from [13, Theorem 4.3.8] that (Φ, h) is a complete metric space. For $\varepsilon > 0$, $B_h(A, \varepsilon)$ denotes the ε-neighborhood of $A \in \Phi$, i.e.,

$$B_h(A, \varepsilon) = \{B \in \Phi : h(A, B) < \varepsilon\}.$$

We consider the following parametric vector optimization problems:

$$(VP) \quad \min_{\preceq_C} f(x, \lambda)$$

$$s.t. \quad g(x) \in K,$$

where $f : X \times \Lambda \to Z$ and $g : X \to Y$ are continuous mappings and $K \in \Phi$. We discuss the stability of solutions of (VP), and regard λ and K as the parameters to be perturbed in the present paper. Denote the product parameter space by $\Lambda \times \Phi$ with the metric defined as

$$\rho((\lambda, A), (\lambda', A')) = \max\{d_\Lambda(\lambda, \lambda'), h(A, A')\}, \quad \forall (\lambda, A),\ (\lambda', A') \in \Lambda \times \Phi.$$

For $\varepsilon > 0$, $B_\rho((\lambda, A), \varepsilon)$ denotes the ε-neighborhood of $(\lambda, A) \in \Lambda \times \Phi$, i.e.,

$$B_\rho((\lambda, A), \varepsilon) = \{(\lambda', A') \in \Lambda \times \Phi : \rho((\lambda', A'), (\lambda, A)) < \varepsilon\}.$$

Denote the parametric vector optimization problem (VP) by (λ, K). Let

$$F(K) = \{x \in X : g(x) \in K\}$$

be the feasible set of (λ, K).

A point $\bar{x} \in F(K)$ is called a solution of (λ, K) if there is no $x \in F(K)$ such that $f(x, \lambda) \preceq_C f(\bar{x}, \lambda)$ and $f(x, \lambda) \neq f(\bar{x}, \lambda)$. A point $\bar{x} \in F(K)$ is called a weak solution of

(λ, K) if there is no $x \in F(K)$ such that $f(x, \lambda) \prec_C f(\bar{x}, \lambda)$. Denote the solution set by $S(\lambda, K)$ and the weak solution set by $S^w(\lambda, K)$ for $(\lambda, K) \in \Lambda \times \Phi$.

This paper deals with the stability of (VP) on the domain of S, defined as

$$\text{dom}S = \{(\lambda, K) \in \Lambda \times \Phi : S(\lambda, K) \neq \emptyset\}.$$

Associated with the parameter space $\Lambda \times \Phi$, we consider the set-valued mappings $F : \Phi \to 2^X$ and $S : \Lambda \times \Phi \to 2^X$, where 2^X denotes the collection of all nonempty subsets of X. The stability of parametric vector optimization problems (λ, K) can be transformed to the continuity and variational properties of the feasible mapping F and the solution mapping S.

A mapping $g : X \to Y$ is said to be open whenever the image of any open set in X is an open set in Y.

Let $T : X \to 2^Y$ be a set-valued mapping. We now recall some well-known definitions.

(i) T is upper semicontinuous (usc for brevity) at $x \in X$ if for every open set $V \subset Y$ containing $T(x)$, there exists an open neighborhood U of x such that $T(x) \subset V$ for all $x \in U$.

(ii) T is said to be lower semicontinuous (lsc for brevity) at $x \in X$ if for any open set $V \subset Y$ satisfying $V \cap T(x) \neq \emptyset$, there exists an open neighborhood U of x such that $V \cap T(x) \neq \emptyset$ for all $x \in U$. It is well known that T is lsc at $x \in X$ if and only if for any sequence $x_n \in X$ converging to x and any $y \in T(x)$ there exist $y_n \in T(x_n)$ for all n such that $y_n \to y$.

(iii) T is continuous at $x \in X$ if it is both upper and lower semicontinuous at x.

(iv) T is closed at $x \in X$ if for all sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ satisfying $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, and $y_n \in T(x_n)$, one has $y \in T(x)$.

We also say that T is usc (lsc, closed) on X if T is usc (lsc, closed, respectively) at each $x \in X$.

Definition 2.1. A point $\bar{x} \in S(\lambda, K)$ is said to be an essential solution of (λ, K) if for any open neighborhood U of $\bar{x}$ in X, there exists an open neighborhood V of (λ, K) in $\Lambda \times \Phi$ such that

$$U \cap S(\lambda', K') \neq \emptyset, \quad \forall (\lambda', K') \in V.$$

Definition 2.2. A nonempty closed subset $E(\lambda, K)$ of $S(\lambda, K)$ is said to be an essential solution set of (λ, K) if for any open set $U \supset E(\lambda, K)$, there exists an open neighborhood V of (λ, K) in $\Lambda \times \Phi$ such that

$$U \cap S(\lambda', K') \neq \emptyset, \quad \forall (\lambda', K') \in V.$$

Remark 2.3. 1) Note that any essential solution of (λ, K) can be arbitrarily approximated by some solution of (λ', K') when (λ', K') is sufficiently close to (λ, K).

2) The notion of an essential solution set is a natural generalization of the concept of an essential solution. A given $\bar{x} \in X$ is an essential solution of (λ, K) if and only if the singleton $\{\bar{x}\}$ is an essential solution set of (λ, K).

3) The solution mapping S is lsc at (λ, K) if and only if every solution $x \in S(\lambda, K)$ is essential.

4) The lower semicontinuity of S implies the existence of essential solutions. Further, the existence of essential solutions implies the existence of essential solution sets.

The following example shows that the existence of essential solution set cannot ensure the existence of essential solution.

Example 2.4. Let $\Lambda = Y = Z = R^2$,

$$X = K = \{(x_1, x_2) \in R^2 : x_2 = 1 - x_1, 0 \le x_1 \le 1\},$$

$$g(x) = x, \quad \forall x \in X,$$

$$f(x, \lambda) = x, \quad \forall (x, \lambda) \in X \times \Lambda,$$

and

$$C = R^2_+ = \{(x_1, x_2) \in R^2 : 0 \le x_1, 0 \le x_2\},$$

where R^2 denote the real plane with Euclidean norm. Then we have

$$S(\lambda, K) = F(K) = X.$$

It is obvious that $S(\lambda, K)$ is an essential solution set.

Now we show that every solution $\widetilde{x} \in S(\lambda, K)$ is not a essential solution. It follows from $S(\lambda, K) = X$ and the definition of X that there exists $\bar{x} \in S(\lambda, K)$ such that $\|\widetilde{x} - \bar{x}\| > \frac{1}{4}$. Let

$$K_n = \{(x_1, x_2) \in R^2 : x_2 - \bar{x}_2 = (\frac{1}{n} - 1)(x_1 - \bar{x}_1),\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$$

for each positive integer n. It is obvious that

$$(\lambda, K_n) \to (\lambda, K).$$

However, we have

$$F(K_n) = S(\lambda, K_n) = \{\bar{x}\}.$$

Hence $\widetilde{x}$ is not an essential solution.

The next example shows that the existence of an essential solution does not guarantee the lower semicontinuity of the solution mapping S.

Example 2.5. Let $X = \Lambda = C = R_+ := [0, +\infty)$, $Y = Z = R$,

$$f(x, \lambda) = \lambda x, \quad \forall (x, \lambda) \in R_+ \times R_+,$$

$$g(x) = x, \quad \forall x \in R_+,$$

$$K_0 = R_+,$$

and

$$\lambda_0 = 0.$$

Then we have $F(K_0) = R_+$ and $S(\lambda_0, K_0) = R_+$.

Let $\lambda_n = \frac{1}{n}$ and $K_n = K_0 = R_+$ for each positive integer n. Then we have $(\lambda_n, K_n) \to (\lambda_0, K_0)$, $F(K_n) = K_n = R_+$ and $S(\lambda_n, K_n) = \{0\}$. It implies that S is not lsc at (λ_0, K_0).

We claim that $0 \in S(\lambda_0, K_0)$ is an essential solution of (λ_0, K_0). In fact, if 0 is not an essential solution, there exist a positive real number $\epsilon > 0$ and a sequence (λ_k, K_k) converging to (λ_0, K_0) such that

$$S(\lambda_k, K_k) \cap B_X(0, \epsilon) = \emptyset, \tag{2.1}$$

where $\lambda_k \in R_+$, $K_k \subset R_+$ for all positive integer k. Note that $F(K_k) = K_k$. Let $a_k = \min K_k$. Then we have $a_k \in S(\lambda_k, K_k)$. It follows from (2.1) that $a_k \ge \epsilon$ for each positive integer k, in contradiction with $K_k \to K_0$.

Example 2.6. 1) When X is a compact metric space, $Z = R^m$ and $C = R^m_+$, for the unconstrained vector optimization problems, Xiang and Zhou [23] showed that each set of all solutions corresponding to the same optimal values is essential under functional perturbations (parametric perturbations in the present paper, correspondingly). They also gave characterizations of an essential solution, an essential solution set and an essential component, respectively.

2) For the weak solution of general vector optimization problems, Peng [18] analyzed the relationship between the existence of essential weak solutions and the stability of the weak solutions set. He also showed that the set of all stable problems (every weak solution is essential) is an dense G_δ subset of the set of all problems with the given topology. However, he assumed that the decision space X is a compact metric space.

3 Main Results

First, we discuss the continuity of the feasible mapping F.

Lemma 3.1. *If the mapping $g : X \to Y$ is open, then the feasible mapping F is lsc on Φ.*

Proof. For any $K \in \Phi$, suppose that W is an open set in X such that $W \cap F(K) \neq \emptyset$. It follows from the definition of $F(K)$ that there is $x \in W$ such that $g(x) \in K$, i.e., $g(W) \cap K \neq \emptyset$. Then we can take $y_0 \in g(W) \cap K$. Noting that $g(W)$ is an open set in Y since g is an open mapping, there exists a real number $\epsilon > 0$ such that $B_Y(y_0, \epsilon) \subset g(W)$.

For every $K' \in B_h(K, \epsilon) \subset \Phi$, we claim that $B_Y(y_0, \epsilon) \cap K' \neq \emptyset$. In fact, if there exists $K' \in B_h(K, \epsilon)$ such that $B_Y(y_0, \epsilon) \cap K' = \emptyset$, we have $d(y_0, K') \geq \epsilon$. Then, from $y_0 \in K$, we can deduce that $h(K, K') \geq \epsilon$, which contradict $K' \in B_h(K, \epsilon)$.

Therefore, $B_Y(y_0, \epsilon) \subset g(W)$ and $B_Y(y_0, \epsilon) \cap K' \neq \emptyset$ imply $g(W) \cap K' \neq \emptyset$, i.e., $W \cap F(K') \neq \emptyset$. This means that F is lsc at K. The proof is complete. □

The following example shows that the assumption that g is an open mapping is essential in Lemma 3.1.

Example 3.2. Let $X = Y = Z = R^2$,

$$K = R^2_+,$$

$$g(x) = \begin{cases} (1,1), & x \in R^2_+ \\ (0,0), & x \notin R^2_+ \end{cases},$$

and

$$K_n = \{(x_1, x_2) \in R^2 : \frac{1}{n} \leq x_1,\ \frac{1}{n} \leq x_2\}$$

for each positive integer n. Note that g is not an open mapping. It is easy to check that

$$K_n \to K,$$

$$F(K) = R^2,$$

and

$$F(K_n) = R^2_+.$$

It implies that F is not lsc at K.

Lemma 3.3. *If the mapping $g : X \to Y$ is continuous, then the feasible mapping F is closed on Φ.*

Proof. For any $K \in \Phi$, let $\{K_n\} \subset \Phi$ and $\{x_n\} \subset X$ be sequences with $x_n \in F(K_n)$ such that $K_n \to K$ and $x_n \to \bar{x}$ as $n \to \infty$. It is sufficient to show that $\bar{x} \in F(K)$.

If $\bar{x} \notin F(K)$, i.e., $g(\bar{x}) \notin K$, there exists $\epsilon > 0$ such that $B_Y(g(\bar{x}), \epsilon) \cap K = \emptyset$ (since K is closed). It follows from the continuity of g that there exists $\eta > 0$ such that

$$g(x) \in B_Y(g(\bar{x}), \frac{\epsilon}{2}), \quad \forall x \in B_X(\bar{x}, \eta).$$

Noting that $x_n \to \bar{x}$ and $K_n \to K$, we have $x_n \in B_X(\bar{x}, \eta)$ and $K_n \in B_h(K, \frac{\epsilon}{2})$ for sufficiently large n. It follows that $g(x_n) \in B_Y(g(\bar{x}), \frac{\epsilon}{2})$, which implies that $g(x_n) \notin K_n$. This contradicts $x_n \in F(K_n)$. The proof is complete. □

For $(\lambda, K) \in \Lambda \times \Phi$, we say that (λ, K) satisfies the *local inf-compactness condition* at $\bar{x} \in X$ if there exists a positive real number α such that for all $(\lambda', K') \in B_\rho((\lambda, K), \alpha)$ and each $x \in B_X(\bar{x}, \alpha)$, the sets

$$L(x, \lambda', K') := \{z \in F(K') : f(z, \lambda') \preceq_C f(x, \lambda')\}$$

are contained in a compact subset D of X.

Now, we establish sufficient conditions for a solution to be essential.

Theorem 3.4. *For $(\lambda, K) \in \Lambda \times \Phi$ and $\bar{x} \in S(\lambda, K)$, suppose that the following conditions hold.*

(i) $g : X \to Y$ *is a continuous open mapping,*

(ii) $f : X \times \Lambda \to Z$ *is a continuous mapping,*

(iii) (λ, K) *satisfies the local inf-compactness condition at $\bar{x}$ and*

$$\{x \in F(K) : f(x, \lambda) = f(\bar{x}, \lambda)\} = \{\bar{x}\}.$$

Then $\bar{x}$ is an essential solution of (λ, K).

Proof. Suppose that $\bar{x}$ is not an essential solution. Then there exist $\epsilon_0 > 0$ and a sequence (λ_n, K_n) converging to (λ, K) such that

$$S(\lambda_n, K_n) \cap B_X(\bar{x}, \epsilon_0) = \emptyset. \tag{3.1}$$

Noting that $\bar{x} \in F(K)$ and F is lsc at K by Lemma 3.1, there exist $x_n \in F(K_n)$ for all n such that $x_n \to \bar{x}$. It follows from (3.1) that $x_n \notin S(\lambda_n, K_n)$ for sufficiently large n. By the local inf-compactness condition, the continuity of f and the closedness of F, it is easy to check that

$$L(x_n, \lambda_n, K_n) = \{z \in F(K_n) : f(z, \lambda_n) \preceq_C f(x_n, \lambda_n)\}$$

are closed subsets of the compact set D for sufficiently large n. Consider the following vector optimization problem

$$\begin{aligned} &\min_{\preceq_C} f(x, \lambda_n) \\ s.t. \quad & x \in L(x_n, \lambda_n, K_n). \end{aligned} \tag{3.2}$$

From the compactness of $L(x_n, \lambda_n, K_n)$, the continuity of f, and [19, Theorem 3.2.7], it follows that the solution set S_n of problem (3.2) is not empty.

We claim that $S_n \subset S(\lambda_n, K_n)$. In fact, if there exists $\hat{x} \in S_n$ with $\hat{x} \notin S(\lambda_n, K_n)$, there exist $x \in F(K_n)$ such that

$$f(\hat{x}, \lambda_n) - f(x, \lambda_n) \in C \setminus \{0\}. \tag{3.3}$$

Noting that $x \in L(x_n, \lambda_n, K_n)$ since $f(x, \lambda_n) \preceq_C f(\hat{x}, \lambda_n) \preceq_C f(x_n, \lambda_n)$, the inclusion relation (3.3) contradicts $\hat{x} \in S_n$.

Since $S_n \neq \emptyset$, we can take $x'_n \in S_n \subset L(x_n, \lambda_n, K_n) \subset F(K_n)$ for each n. Then we have

$$f(x_n, \lambda_n) - f(x'_n, \lambda_n) \in C \tag{3.4}$$

by the definition of $L(x_n, \lambda_n, K_n)$. Since the local inf-compactness condition implies that x'_n belong to the compact set D for sufficiently large n, we may assume without loss of generality that there exists $x' \in D$ such that $x'_n \to x'$. Note that the set-valued mapping F is closed by Lemma 3.3. Then we have $x' \in F(K)$.

Now we show that

$$f(\bar{x}, \lambda) - f(x', \lambda) \in C. \tag{3.5}$$

If not, there exists $\epsilon > 0$ such that

$$B_Y(f(\bar{x}, \lambda) - f(x', \lambda), \epsilon) \cap C = \emptyset$$

since C is nonempty and closed. It follows from the continuity of f that

$$f(x_n, \lambda_n) - f(x'_n, \lambda_n) \notin C$$

for all sufficiently large n, in contradiction with (3.4).

Noting that $\bar{x} \in S(\lambda, K)$ and $x' \in F(K)$, the inclusion relation (3.5) implies $f(\bar{x}, \lambda) = f(x', \lambda)$. Then we have $x' = \bar{x}$ by condition (iii). It follows from $x'_n \to x'$ that $x'_n \in B_X(\bar{x}, \epsilon_0)$ for sufficiently large n. On the other hand, we have $x'_n \in S_n \subset S(\lambda_n, K_n)$. It contradicts with (3.1). The proof is complete. □

By Definition 2.1 and 2.2, every essential solution is an essential solution set itself. The following corollary naturally holds.

Corollary 3.5. *Under the assumptions of Theorem 3.4, every closed subset $E(\lambda, K) \subset S(\lambda, K)$ with $\bar{x} \in E(\lambda, K)$ is an essential solution set.*

Applying Theorem 3.4, we directly obtain the following result by 3) of Remark 2.3.

Corollary 3.6. *Let $(\lambda, K) \in \Lambda \times \Phi$ be such that conditions (i)-(iii) of Theorem 3.4 are satisfied for all $\bar{x} \in S(\lambda, K)$. Then S is lsc at (λ, K).*

Next, we establish sufficient conditions for the closedness of the weak solution mapping S^w and the solution mapping S, respectively.

Theorem 3.7. *Suppose that $f : X \times \Lambda \to Z$ is a continuous mapping and the feasible mapping F is lsc at K. Then the weak solution mapping S^w is closed on $\Lambda \times \Phi$.*

Proof. For any $(\lambda, K) \in \Lambda \times \Phi$, let $(\lambda_n, K_n) \in \Lambda \times \Phi$ and $x_n \in S^w(\lambda_n, K_n)$ be sequences such that $(\lambda_n, K_n) \to (\lambda, K)$ and $x_n \to x$ as $n \to \infty$. We only need to prove $x \in S^w(\lambda, K)$.

We argue by contradiction. If $x \notin S^w(\lambda, K)$, there exists $\tilde{x} \in F(K)$ such that

$$f(x, \lambda) - f(\tilde{x}, \lambda) \in \text{int}C.$$

Therefore, there exists a positive real number ϵ such that

$$B_Y(f(x,\lambda) - f(\tilde{x},\lambda),\epsilon) \subset \text{int}C.$$

Since the feasible mapping F is lsc at K, there is a sequence $\tilde{x}_n \in F(K_n)$ such that $\tilde{x}_n \to \tilde{x}$.

From the continuity of f, it follows that there exists a positive integer N such that

$$f(x_n,\lambda_n) - f(\tilde{x}_n,\lambda_n) \in B_Y(f(x,\lambda) - f(\tilde{x},\lambda),\epsilon) \subset \text{int}C, \quad \forall n > N,$$

which contradicts $x_n \in S^w(\lambda_n, K_n)$. The proof is complete. □

Applying Lemma 3.1, the next corollary follows directly from Theorem 3.7.

Corollary 3.8. *Let $f : X \times \Lambda \to Z$ be a continuous mapping and $g : X \to Y$ be an open mapping. Then the weak solution mapping S^w is closed on $\Lambda \times \Phi$.*

Theorem 3.9. *For $(\lambda, K) \in \Lambda \times \Phi$, suppose that $f : X \times \Lambda \to Z$ is a continuous mapping, $g : X \to Y$ is an open mapping, and $S(\lambda, K) = S^w(\lambda, K)$. Then the solution mapping S is closed at (λ, K).*

Proof. Let $(\lambda_n, K_n) \in \Lambda \times \Phi$ and $x_n \in S(\lambda_n, K_n) \subset S^w(\lambda_n, K_n)$ be sequences such that $(\lambda_n, K_n) \to (\lambda, K)$ and $x_n \to x$ as $n \to \infty$. By Corollary 3.8, S^w is closed at (λ, K). Then $x \in S^w(\lambda, K) = S(\lambda, K)$. Hence S is closed at (λ, K). □

It is obvious that the local inf-compactness condition naturally holds if the space X is compact. The following example shows that the condition (iii) is essential in Theorem 3.4.

Example 3.10. Let $\Lambda = C = R_+ = [0, +\infty)$, $X = Y = Z = R$,

$$f(x,\lambda) = \begin{cases} \lambda x, & x \in R_+, \lambda \in R_+ \\ 0, & x \notin R_+, \lambda \in R_+ \end{cases},$$

$$g(x) = x, \quad \forall x \in R,$$

$$K_0 = R,$$

and

$$\lambda_0 = 0.$$

Then we have $F(K_0) = R$ and $S(\lambda_0, K_0) = R$.

Let $\lambda_n = \frac{1}{n}$ and $K_n = K_0 = R$ for each positive integer n. Then we have $(\lambda_n, K_n) \to (\lambda_0, K_0)$, $F(K_n) = K_n = R$ and $S(\lambda_n, K_n) = -R_+$. In fact, for the problem (λ_0, K_0) and the solution $1 \in S(\lambda_0, K_0)$, we have

$$L(x, \lambda_0, K_0) = \{z \in F(K_0) : f(z, \lambda_0) \preceq_C f(x, \lambda_0)\} = R,$$

which implies that the problem (λ_0, K_0) satisfies all assumptions of Theorem 3.4 except condition (iii). It is easy to check that the solution $1 \in S(\lambda_0, K_0)$ is not an essential solution of (λ_0, K_0).

4 Applications

Since the vector optimization problem (VP) in Section 2 contains some optimization problems as special cases, we can derive directly some consequences for these special cases from the results of Section 3. In this section, we only discuss two special cases, i.e., scalar optimization problems and cone constrained vector optimization problems.

First, we consider the following scalar optimization problems.

$$(P) \quad \min \varphi(x)$$

$$s.t. \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m,$$

where $\varphi : X \to R$ and $g_i : X \to R$ are continuous real functions.

Let $f : X \times R \to R$ be a continuous function such that

$$f(x, 0) = \varphi(x), \quad \forall x \in X.$$

Definition 4.1. Given a real number $\lambda > 0$, a point $\bar{x} \in X$ is called a (λ, f)-solution of problem (P) if

$$g_i(\bar{x}) \leq \lambda, \quad i = 1, 2, \ldots, m,$$

and

$$f(\bar{x}, \lambda) \leq f(x, \lambda)$$

for all $x \in X$ satisfying $g_i(x) \leq \lambda$.

Let $\Lambda = R$, $C = R_+ := [0, \infty)$ and

$$K_\lambda = \lambda e - R^m_+,$$

where $e = (1, \ldots, 1) \in R^m$.

Consider the following parametric optimization problems

$$(P_\lambda) \quad \min f(x, \lambda)$$

$$s.t. \quad g(x) \in K_\lambda,$$

where $g : X \to R^m$ is defined as $g := \{g_1, \ldots, g_m\}$. Then we can rewrite problem (P) as

$$\min f(x, 0)$$

$$s.t. \quad g(x) \in K_0.$$

Denote the solution set of the parametric optimization problems (P_λ) by $S(\lambda)$, the weak solution set by $S^w(\lambda)$ and the feasible set by $F(\lambda)$. Thus we can write the solution set of (P) as $S(0)$, the weak solution set as $S^w(0)$ and the feasible set as $F(0)$.

Remark 4.2. (i) Obviously, the (λ, f)-solutions of (P) must be solutions of problem (P_λ).

(ii) From Definition 2.1, if a point $\bar{x} \in S(0)$ is an essential solution of (P), there exists a sequence of (λ_n, f)-solutions $\{x_{\lambda_n}\}$ such that $x_{\lambda_n} \to \bar{x}$ as $\lambda_n \to 0$.

Corollary 4.3. *Suppose that $g : X \to R$ is a continuous open function, $f : X \times R \to R$ is a continuous function, (P) satisfies the local inf-compactness condition at $\bar{x} \in S(0)$ and $\bar{x}$ is a unique solution of (P). Then $\bar{x}$ is an essential solution of (P).*

Corollary 4.3 follows directly from Theorem 3.4. It means that any solution $\bar{x}$ satisfying the conditions of Corollary 4.3 is a stable solution of (P).

Note that the equation $S^w(\lambda) = S(\lambda)$ holds trivially for scalar problems. The next result follows immediately from Theorem 3.9.

Corollary 4.4. *Let $f : X \times R \to R$ be a continuous mapping, $g : X \to R$ be an open mapping and $\{x_{\lambda_n}\}$ be a sequence of (λ_n, f)-solutions with $x_{\lambda_n} \to \bar{x}$ as $\lambda_n \to 0$. Then $\bar{x}$ is a solution of problem (P).*

Corollary 4.4 means that we can approximate any solution of (P) by a sequence of (λ, f)-solutions.

Next, we consider the following cone constrained vector optimization problems

$$(CVP) \quad \min_{\preceq_C} \varphi(x)$$

$$s.t. \quad g(x) \preceq_C \theta,$$

where $\varphi : X \to R^m$ and $g : X \to R^m$ are vector functions, $C \subset R^m$ is a closed convex cone with nonempty interior and $\theta = (0, \ldots, 0) \in R^m$.

Let $f : X \times R \to R^m$ be a continuous vector function such that

$$f(x, 0) = \varphi(x), \quad \forall x \in X.$$

Definition 4.5. Given a real number $\lambda > 0$ and $c_0 \in C$, a point $\bar{x} \in X$ is called a (λ, c_0, f)-solution of problem (CVP) if

$$g(\bar{x}) \preceq_C \lambda c_0$$

and there is no $x \in X$ with $g(x) \preceq_C \lambda c_0$ such that

$$f(\bar{x}, \lambda) - f(x, \lambda) \in C \setminus \{0\}.$$

Let $\Lambda = R$ and $K_\lambda = \lambda c_0 - C$. Consider the following parametric cone constrained vector optimization problems

$$(CVP_\lambda) \quad \min_{\preceq_C} f(x, \lambda)$$

$$s.t. \quad g(x) \in K_\lambda.$$

Then we can rewrite problem (CVP) as

$$\min_{\preceq_C} f(x, 0)$$

$$s.t. \quad g(x) \in K_0.$$

Denote the solution set of the parametric optimization problems (CVP_λ) by $S(\lambda, c_0)$, the weak solution set by $S^w(\lambda, c_0)$ and the feasible set by $F(\lambda, c_0)$. Thus we can write the solution set of (CVP) as $S(0, c_0)$, the weak solution set as $S^w(0, c_0)$ and the feasible set as $F(0, c_0)$.

Remark 4.6. (i) It is obvious that the (λ, c_0, f)-solutions of problem (CVP) must be solutions of problem (CVP_λ).

(ii) From Definition 2.1, if a point $\bar{x} \in S(0, c_0)$ is an essential solution of (CVP), there exists a sequence of (λ_n, c_0, f)-solutions $\{x_{\lambda_n}\}$ such that $x_{\lambda_n} \to \bar{x}$ as $\lambda_n \to 0$.

The following results follow immediately from Theorem 3.4 and Theorem 3.9, respectively.

Corollary 4.7. *Suppose that* $g : X \to R$ *is a continuous open function,* $f : X \times R \to R^m$ *is a continuous function,* (CVP) *satisfies the local inf-compactness condition at* $\bar{x} \in S(0, c_0)$ *and* $\{x \in F(0, c_0) : f(x, 0) = f(\bar{x}, 0)\} = \{\bar{x}\}$*. Then* $\bar{x}$ *is an essential solution of* (CVP)*.*

Corollary 4.8. *Let* $f : X \to R$ *be a continuous mapping,* $g : X \to R$ *be a open mapping and* $\{x_{\lambda_n}\}$ *be a sequence of* (λ_n, c_0, f)*-solutions with* $x_{\lambda_n} \to \bar{x}$ *as* $\lambda_n \to 0$*. Suppose that* $S^w(0, c_0) = S(0, c_0)$*. Then* $\bar{x}$ *is a solution of problem* (CVP)*.*

Acknowledgements

The authors are very grateful to the editors and the anonymous referees for their valuable comments and suggestions which improved the original manuscript greatly.

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Manuscript received 30 March 2012
revised 13 September 2012
accepted for publication 19 September

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