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# SEMISMOOTH NEWTON METHOD FOR AFFINE DIFFERENTIAL VARIATIONAL INEQUALITIES*

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**Abstract:** In this paper we study a semismooth Newton method for affine linear differential variational inequalities (DVIs). These problems comprise the solution of two problems: an ordinary differential equation (ODE) and an infinite dimensional variational inequality (VI). Such problems have various applications in engineering as well as in economic sciences and were recently introduced in a paper by Pang and Stewart [23]. The method we propose is based on a Runge-Kutta time discretization scheme of the underlying ODE and a suitable reformulation of the resulting finite dimensional problem as a system of nonlinear, nonsmooth equations. We discuss the existence of solutions to the infinite and the finite problem and analyze the convergence properties of the resulting numerical scheme. Moreover, we provide some numerical results.

**Key words:** *variational inequality, optimal control problems, Runge-Kutta methods, semismooth Newton, differential*

**Mathematics Subject Classification:** *90C33, 49M15, 49J15,65K15*

## 1 Introduction

The general form of the DVI as introduced in the paper by Pang and Stewart [23] is of the form

$$\begin{aligned} \dot{x}(t) &= \Phi(t, x(t), u(t)), \qquad t \in [0, T] \\ u(t) &\in \mathcal{S}(K, F(t, x(t), \cdot)), \\ 0 &= \Psi(x(0), x(T)) \end{aligned} \tag{1.1}$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are supposed to be continuous nonlinear functions, $\mathcal{S}(K, F(t, x(t), \cdot)$ denotes the solution set of the VI: find a continuous function $\bar{u}(t) : [0, T] \to K$ such that

$$\int_0^T F(t, x(t), \bar{u}(t))^T (u(t) - \bar{u}(t))\, dt \geq 0 \tag{1.2}$$

holds for any continuous function $u(t) : [0, T] \to K$. In the following, we will assume that $K \subseteq \mathbb{R}^m$ is a convex set. Moreover, in our theoretical analysis, we assume that the ODE depends only on some given initial data $x^0 \in \mathbb{R}^n$, i.e. $\Psi(x(0), x(T)) = x(0) - x^0$. Finally,

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we will assume that the differential variational inequality is affine linear, where the precise form of the considered DVI will be given in the next section.

Given in the form (1.1), the solution of a DVI combines two problems to be solved in parallel: the solution of an ODE and the solution of an infinite dimensional VI. In that sense, the analysis of these problems comprise the theory and numerical solution of ODEs, but also the theory and numerical solution of VIs.

Despite their many applications in various fields (see below), DVIs have not been analytically discussed and numerically analyzed in detail in the literature so far. Pang and Stewart give an extensive introduction into the field in a recent paper [23]. They discuss several variants and reformulations of the general DVI and present relations to neighbouring fields as well as existence results and convergence properties of some Euler type time-stepping schemes for a variety of different DVI classes. Other papers consider e.g. the existence and uniqueness results of some special class of DVIs, namely differential complementarity problems (DCP)[28] or solution sensitivities of initial conditions for DVIs [24]. All mentioned papers do not aim to discuss numerical methods (in detail).

In contrast to these papers, we focus on the numerical solution of DVIs. As general Runge-Kutta methods [12] are more common to solve different types of ODEs, we therefore first extend the convergence theory given in [23] only for simple Euler type time-stepping schemes to general s-stage Runge-Kutta methods for the affine-linear case in Section 2. In Section 3 we then apply a common nonsmooth reformulation of the resulting finite dimensional VI using the so-called *natural map* [8]. The numerical method we present as next is based on the application of a semismooth Newton method as introduced in [26] to the nonlinear, nonsmooth system of equations. The method is proved to be reasonable and convergent under suitable assumptions. Finally, in Section 4 we present some numerical results for two examples taken from the literature which represent models of a two-player differential Nash game.

## 1.1 Applications

Applications of DVIs arise in engineering sciences [1, 13, 22, 27] as well as in economic sciences. In our presentation here we will focus on differential Nash games that appear e.g. in management sciences [2, 6, 9, 17] and give rise to DVIs by their associated set of necesserary conditions.

The concept of Nash games and Nash equilibria were introduced by J.F. Nash [20, 21] in the 1950's and form a powerful tool to model strategic behaviour in various situations. They are nowadays widely-used in particular in the economic sciences. A noncooperative Nash game is given by a set of $N$ players, where each player aims to solve his own optimization problem. However, the players cannot solve their optimization problems independently from each other as the problems are coupled through the associated objective functions and/or the feasible sets that depend on the other players' strategy choices. A Nash equilibrium is a situation (or to be more precise a vector of the players strategy choices), where each player does not feel the incentive to change his chosen strategy seperately, if all other players stick to their strategy.

A special class of Nash games are the so-called differential Nash games, where each player $\nu \in \{1, .., N\}$ aims to solve an infinite dimensional optimal control problem of the form [6]

$$\begin{array}{rll} \max_{u_\nu, x} & J_\nu(x, u_\nu, u_{-\nu}) & \\ \text{subject to} & \dot{x}(t) = g(x, u_\nu, u_{-\nu}) & t \in [0, T], \\ & x(0) = x^0 & \\ & u_\nu(t) \in U_\nu \subseteq \mathbb{R}^{m_\nu} & \end{array} \tag{1.3}$$

where $u_\nu$ denotes the control of player $\nu$, $u_{-\nu}$ contains the control vectors of the remaining players $(1, 2, .., \nu-1, \nu+1, .., N)$, $x$ is the state (being the same for all players),

$$J_\nu(x, u_\nu, u_{-\nu}) = \int_0^T \theta_\nu(x, u_\nu, u_{-\nu})(t)\, dt + \sigma_\nu(x(T))$$

the objective functional of player $\nu$ and $U_\nu$ the feasible set of player $\nu$. Under suitable assumptions, by Pontryagin's maximum principle [18], using the Hamiltonian functions

$$H_\nu(x, p_\nu, u_\nu, u_{-\nu}) = p_\nu g(x, u_\nu, u_{-\nu}) + \theta_\nu(x, u_\nu, u_{-\nu}) \tag{1.4}$$

for each player $\nu = 1, .., N$ we obtain a set of necessary conditions:

$$\begin{array}{rcll} \dot{x}(t) & = & g(x, u_\nu, u_{-\nu}), & x(0) = x^0, \\ \dot{p}_\nu(t) & = & -D_x H_\nu(x, p_\nu, u_\nu, u_{-\nu}), & p_\nu(T) = \nabla \sigma_\nu(x(T)), \\ u_\nu(t) & \in & \mathcal{S}(U_\nu, D_{u_\nu} H_\nu(x, p_\nu, \cdot, u_{-\nu})), & \end{array} \tag{1.5}$$

If $U_\nu \neq \mathbb{R}^{p_\nu}$, then (1.5) correspond to a DVI of the form (1.1). Hence, one way to solve the differential Nash game is to solve the associated DVI that corresponds to the concatenated system of necessary conditions for player $\nu = 1, .., N$ (though the state equation appears only once), which again yields a DVI of the form (1.1).

We refer the interested reader to the two monographs [2, 6]. Moreover, an overview of several applications of differential Nash games in management sciences can be found in the survey [9].

## 2 Runge-Kutta Time-Stepping Scheme for Affine DVIs

In this section, we will first describe and analyse a Runge-Kutta based time-stepping scheme for affine differential variational inequalities (ADVI), where we suppose that the VI depends on a closed convex set $K$ (in particular a polyhedral set) and $F(t, x(t), u(t))$ is an affine linear function both in $x$ and $u$, i.e. we analyse

$$\begin{array}{rcll} \dot{x}(t) & = & f(t, x(t)) + B(t, x(t))\, u(t), & (t, x) \in \Omega \\ u(t) & \in & \mathcal{S}(K, G), & \\ x(0) & = & x^0 & \end{array} \tag{2.1}$$

where $B(t, x(t)) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ and $G(x(t), u(t)) := d + Dx(t) + Eu(t)$, with $d \in \mathbb{R}^m$, $D \in \mathbb{R}^{m \times n}$ and $E \in \mathbb{R}^{m \times m}$. Moreover, $\mathcal{S}(K, G)$ denotes the solution set of the AVI: find a continuous function $\bar{u}(t) : [0, T] \to K$ such that

$$\int_0^T G(x(t), \bar{u}(t))^T (u(t) - \bar{u}(t))\, dt \geq 0 \tag{2.2}$$

holds for any continuous function $u(t) : [0, T] \to K$. Such problems arise e.g. as initial-value problem involving an ODE with discontinuous right-hand side, linear complementarity systems, simulations of dynamics involving frictional contact problems [1, 22, 27] or boundary-value problems of the type (1.1) that can be transformed into an initial-value problem.

The existence of solutions to (2.1) can be obtained by application of Theorem 6.1 in [23].

**Proposition 2.1** ((Pang,Stewart)[23])**.** *Let $K$ be a polyhedron with $0 \in K$, $E$ be positive semidefinite and suppose that either*

1. *$(K, E)$ is an $R_0$-pair or*
2. *$(d + Dx) \subseteq int\mathcal{K}(K, E)^*$ for all $(t, x) \in \Omega$.*

*holds. Then (2.1) has a weak solution.*

**Remark 2.2.** $(K, E)$ is an $R_0$-pair e.g. if $K$ is a convex cone and $E$ is symmetric and strictly copositive on $K$, i.e. $y^T E y > 0$ holds for all $y \in K\backslash\{0\}$. Moreover, if $(K, E)$ is an $R_0$-pair, the solution set $\mathcal{S}(K, r + E)$ is bounded for any $r = d + Dx, r \in \mathbb{R}^m$. For more information on the $R_0$-property and the conditions of Lemma 2.1 refer e.g. [8].

The time-stepping scheme for (2.1) that we consider here is based on a general (possibly implicit) s-stage Runge-Kutta method [12] for the initial value problem (IVP) for $x$, i.e.

$$\begin{aligned} X^{(i)}_{n,h} &= x^h_n + h \sum_{j=1}^{s} a_{ij}( f^{(i)}_{n,h} + B^{(i)}_{n,h} u^h_{n+1} ) \\ x^h_{n+1} &= x^h_n + h \sum_{i=1}^{s} \omega_i( f^{(i)}_{n,h} + B^{(i)}_{n,h} u^h_{n+1} ) \end{aligned} \tag{2.3}$$

for all $n = 0, .., N_h - 1$, where $h$ denotes the discretization parameter of an equidistant grid, i.e. $t^h_n = nh$ for $n = 0, .., N_h$ ($N_h$ being the grid size),

$$c_i := \sum_{j=1}^{s} a_{ij}, \qquad f^{(i)}_{n,h} := f(t^h_n + c_i h, X^{(i)}_{n,h}), \qquad \text{and} \qquad B^{(i)}_{n,h} := B(t^h_n + c_i h, X^{(i)}_{n,h})$$

and $x^h_0 = x^0$ for all $h$. Here we used an alternative though equivalent formulation of the standard Runge-Kutta method as it is given e.g. in [12]. The associated Butcher tableau (being the same as for the standard formulation of the method) is given by

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & \omega_1 & \omega_2 & \cdots & \omega_s \end{array} \qquad = \qquad \begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \omega^T \end{array}$$

A suitable discretization of (2.2) is

$$u^h_{n+1} \in \mathcal{S}(K, G^h_{n+1}) \quad \text{with} \quad G^h_{n+1}(u) := d + Dx^h_{n+1} + Eu. \tag{2.4}$$

In the following we will make the general assumptions

**Assumption 2.3.**

1. $f$ and $B$ are smooth Lipschitz continuous functions with constants $L_f$ and $L_B$, respectively.
2. $B(t, x(t))$ is bounded on $\Omega$, i.e. there exists $C_B > 0$, such that $\|B\|_{C(\Omega)} \leq C_B$
3. The Runge-Kutta scheme is at least of first order, i.e $\sum_{i=1}^{s} \omega_i = 1$.

Note that by the Lipschitz-continuity of $f$ and $B$ there exist constants $c_f, c_B > 0$ for all $(t, x) \in \Omega$ such that

$$\|f(t, x)\| \leq c_f(1 + \|x\|) \qquad \text{and} \qquad \|B(t, x)\| \leq c_B(1 + \|x\|).$$

**Proposition 2.4.** *Let $E$ be positive definite and $K$ be closed and convex. Moreover, assume that the previous state $x_n^h$ is given. Then there exists $\bar{h} > 0$ such that for all $h \in (0, \bar{h}]$ it holds:*

1. *If $f(t,x) = Rx$ and $B(t,x) = B$ with $R \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, respectively, then $x_{n+1}^h$ is uniquely defined by $u_{n+1}^h$ and (2.3) and moreover $\mathcal{S}(K, G_{n+1})$ is nonempty and single-valued.*

2. *If $K$ is moreover bounded and (2.3) implicitly defines some continuous function $\Theta^h$, i.e. $x_{n+1}^h = \Theta^h(u_{n+1}^h)$, then $\mathcal{S}(K, G_{n+1}^h)$ is nonempty and compact.*

3. *If $K = \mathbb{R}_+^m$ and (2.3) implicitly defines a continuous function, i.e. $x_{n+1}^h = \Theta^h(u_{n+1}^h)$ and either*

   (a) $u^T D\Theta^h(u) \geq 0$ *for all $u \in K$ or*

   (b) $D\Theta^h(\cdot)$ *satisfies the coercivity condition*

$$\lim_{\substack{u\in K\\ \|u\|\to\infty}} \frac{u^T D\Theta^h(u)}{\|u\|} = \infty$$

   *then $\mathcal{S}(K, G_{n+1}^h)$ is nonempty and bounded.*

*Proof.* The first part follows since under the given conditions, (2.3) implicitly defines an affine linear function $\Theta^h(u_{n+1}^h)$ for sufficiently small $h$. Inserting $\Theta^h(u_{n+1}^h)$ in (2.4) we thus obtain a variational inequality $VI(K, \tilde{F})$ for $u_{n+1}^h$, where $\tilde{F}$ is affine linear and furthermore by the positive definiteness of $E$ strongly monotone. Therefore Theorem 2.3.3 in [8] can be applied, which gives the existence and uniqueness of a solution $u_{n+1}^h$.

Again replacing $x_{n+1}^h$ by $\Theta^h(u_{n+1}^h)$ in (2.4), the second part is a direct consequence of Corollary 2.2.5 in [8].

Finally, the third part follows by replacing $x_{n+1}^h$ by $\Theta^h(u_{n+1}^h)$ in (2.4), which yields the NCP: find $u$ such that

$$0 \leq u \quad \perp \quad d + D\Theta^h(u) + Eu \geq 0,$$

where by assumption $E$ is positive definite. Hence Proposition 2.2.12 in [8] can be applied, by which it follows that $\mathcal{S}(K, G_{n+1}^h)$ is nonempty and bounded . □

Knowing that under suitable conditions for sufficiently small $h$ the finite-dimensional AVI$(K, G_{n+1}^h)$ have a (unique) solution, we are now interested in the convergence of our time-stepping scheme, i.e. the convergence of approximations $x^h$ and $u^h$ for the limit $h \to 0$. Therefore , we approximate $x(t)$ using a piecewise linear function which is based on the discrete solutions $x_n^h$

$$x^h(t) := x_n^h + \frac{t - t_n^h}{h}(x_{n+1}^h - x_n^h), \qquad \forall\, t \in [t_n^h, t_{n+1}^h], \qquad n = 0, 1, .., N_h - 1. \tag{2.5}$$

As approximation for $u(t)$ we use the piecewise constant function

$$u^h(t) := u_{n+1}^h, \qquad \forall\, t \in [t_n^h, t_{n+1}^h] \qquad n = 0, 1, .., N_h - 1. \tag{2.6}$$

In preparation for the proof of convergence, we will first give some auxiliary results.

**Lemma 2.5.** *Let each $x_{n+1}^h$ be the solution of (2.3) and $u_{n+1}^h$ be the solution of the associated AVI($K, G_{n+1}^h$) for $n = 0, 1, .., N_h - 1$. Assume that each $u_{n+1}^h$ satisfies*

$$\|u_{n+1}^h\| \leq \rho\,(1 + \|x_{n+1}^h\|) \tag{2.7}$$

*for some positive constant $\rho$. Then there exists some constant $c_u > 0$ and $\bar{h} > 0$ such that*

$$\|u_{n+1}^h\| \leq c_u(1 + \|x_n^h\|). \tag{2.8}$$

*holds for all $h \in (0, \bar{h}]$ and $n = 0, 1, .., N_h - 1$.*

*Proof.* First, by (2.3), the Lipschitz-continuity of $f$ and the boundedness of $B$ we have

$$\begin{aligned} \|u_{n+1}^h\| &\leq \rho\,(1 + \|x_{n+1}^h\|) \\ &\leq \rho\,(1 + \|x_{n+1}^h - x_n^h\| + \|x_n^h\|) \\ &\leq \rho\left[1 + h\left(\sum_i^s \omega_i\, c_f(1 + \|X_{n,h}^{(i)}\|) + C_B\|u_{n+1}^h\|\right) + \|x_n^h\|\right] \end{aligned} \tag{2.9}$$

Moreover, for each $i = 1, .., s$ we have

$$\begin{aligned} \|X_{n,h}^{(i)}\| &\leq \|x_n^h\| + h\sum_{j=1}^s |a_{ij}|(\,\|f_{n,h}^{(j)}\| + \|B_{n,h}^{(j)}u_{n+1}^h\|) \\ &\leq \|x_n^h\| + h\left(\sum_{j=1}^s |a_{ij}|[\,c_f(1 + \|X_{n,h}^{(j)}\|) + C_B\|u_{n+1}^h\|\,]\right). \end{aligned}$$

Hence, for the vector $\mathbf{x} := (\|X_{n,h}^{(1)}\|, ..., \|X_{n,h}^{(s)}\|)^T$ we obtain

$$0 \;\leq\; \mathbf{x} \;\leq\; h\,c_f\tilde{A}\mathbf{x} + [\|x_n^h\| + h\,\|A\|_\infty(c_f + C_B\|u_{n+1}^h\|)]\,\mathbf{e}$$

where $\tilde{a}_{ij} = |a_{ij}|$ and $\mathbf{e} = (1, .., 1)^T$. Therefore,

$$\|\mathbf{x}\|_\infty \leq\; h\,c_f\|A\|_\infty\|\mathbf{x}\|_\infty + [\|x_n^h\| + h\,\|A\|_\infty(c_f + C_B\|u_{n+1}^h\|)]$$

such that for all $i = 1, .., s$ and $h$ sufficiently small

$$\|X_{n,h}^{(i)}\| \;\leq\; \bar{\eta}\,(1 + \|x_n^h\| + \|u_{n+1}^h\|) \tag{2.10}$$

for suitably defined $\bar{\eta}$. Hence, if we insert this inequality in (2.9), we get

$$\|u_{n+1}^h\| \leq \rho\left[1 + h\left(c_f + c_f\,\bar{\eta}\,(1 + \|x_n^h\| + \|u_{n+1}^h\|) + C_B\|u_{n+1}^h\|\right) + \|x_n^h\|\right]\,,$$

i.e. there exists a positive constant $c_u$ and some $\bar{h} > 0$ being small enough so that (2.8) is satisfied for all $n = 0, .., N_h - 1$ and $h \in (0, \bar{h}]$. □

**Remark 2.6.** Note that condition (2.7) is satisfied, if the linear growth condition

$$\exists\,\rho > 0\;:\qquad \sup\{\|u\| : u \in \mathcal{S}(K, (r + \Gamma(u))\}\quad\leq\quad \rho(1 + \|r\|),$$

with $\Gamma(u) = Eu$ is fullfilled for any $r = d + Dx$, with $(t, x) \in \Omega$ . This condition is introduced and discussed in [23], where assumptions on the problem data are given under which the linear growth condition is proved to hold (cf. Section 6.2, in particular Theorem 6.1 in [23]). Conditions on the problem data for (2.7) to hold are for example: $K$ being a bounded set (e.g. for the box-constrained case as discussed in Section 3) or $K \subseteq \mathbb{R}^m$ being a nonempty closed convex set and $E$ being symmetric and positive definite.

**Lemma 2.7.** *Let $X_{n,h}^{(i)}$ $(i = 1, .., s)$ and $x_{n+1}^h$ be the solutions of (2.3) for all $n = 0, 1, .., N_h - 1$. Assume that there exists a constant $\rho$ such that the solutions $u_{n+1}^h$ of the associated AVI$(K, G_{n+1}^h)$ satisfy (2.7) for all $n = 0, 1, .., N_h - 1$. Then there exists a constant $c_X$ and $\bar{h} > 0$, such that*

$$\|X_{n,h}^{(i)}\| \leq c_X(1 + \|x_n^h\|) \tag{2.11}$$

*holds for all $n = 0, 1, .., N_h - 1$ and $h \in (0, \bar{h}]$.*

*Proof.* By assumption we know that (2.10) holds for each $i = 1, .., s$ (cf. proof of Lemma 2.5). Moreover, by Lemma 2.5 it follows that for $h > 0$ small enough $\|u_{n+1}^h\|$ satisfies (2.8). Hence,

$$\|X_{n,h}^{(i)}\| \leq \bar{\eta}\,(1 + \|x_n^h\| + c_u(1 + \|x_n^h\|)) \leq c_X\,(1 + \|x_n^h\|)$$

for $c_X := \bar{\eta}\,(1 + c_u)$. □

**Lemma 2.8.** *Let $x^h$ be defined as in (2.5) with $(x_n^h)$ $(n = 0, 1, .., N_h - 1)$ being the solutions of (2.3). Assume that the scheme (2.3) is at least of first order and suppose $u_{n+1}^h$ satisfies (2.7). Then there exists $\bar{h} > 0$, such that*

1. *there exists a constant $L_x$, such that for all $n$: $\|x_{n+1}^h - x_n^h\| \leq L_x h$*
2. *$x^h(t)$ is Lipschitz-continuous with L-constant $L_x$ (i.e. independent of $h$)*

*for all $h \in (0, \bar{h}]$.*

*Proof.* First, by (2.3), Lemma 2.5 and Lemma 2.7 we have

$$\begin{aligned}\|x_{n+1}^h - x_n^h\| &\leq h\sum_{i=1}^{s}\omega_i[c_f(1 + \|X_{n,h}^{(i)}\|) + C_B c_u(1 + \|x_n^h\|)] \\ &\leq h(c_f + c_f c_X + C_B c_u)(1 + \|x_n^h\|).\end{aligned}$$

Define $\eta_x := c_f + c_f c_X + C_B c_u$, then it follows

$$\begin{aligned}\|x_{n+1}^h\| &\leq \|x_n^h\| + h\eta_x(1 + \|x_n^h\|) = h\eta_x + (1 + h\eta_x)\|x_n^h\| \\ &\leq h\eta_x\sum_{j=0}^{n}(1 + \eta_x h)^j + (1 + h\eta_x)^{n+1}\|x^0\| \\ &= e^{h(n+1)\eta_x}\|x^0\| + (e^{h(n+1)\eta_x} - 1) \leq (e^{T\eta_x} - 1) + e^{T\eta_x}\|x^0\|\end{aligned}$$

for all $h$ sufficiently small, for all $n = 0, .., N_h - 1$. Hence,

$$\|x_{n+1}^h - x_n^h\| \leq h\,\eta_x(1 + \|x_n^h\|) \leq L_x h$$

for $L_x := \eta_x e^{T\eta_x}(1 + \|x^0\|)$. Therefore, by definition of $x^h(t)$ it follows directly that

$$\|x^h(t_1) - x^h(t_2)\| \leq L_x|t_1 - t_2| \qquad \forall\, t_1, t_2 \in [t_n^h, t_{n+1}^h]$$

for all $n = 0, 1, .., N_h - 1$. Hence,

$$\|x^h(t_1) - x^h(t_2)\| \leq L_x|t_1 - t_2| \qquad \forall\, t_1, t_2 \in [0, T].$$

□

Next, we prove the convergence of the Runge-Kutta based time-stepping scheme. It can be shown, that sequences of approximations $x^{h_k}$ and $u^{h_k}$ defined by (2.5) and (2.6) converge to limits $\bar{x}$ and $\bar{u}$ in $L^\infty(0,T)$ and $L^2(0,T)$, respectively, for the limit $h_k \searrow 0$. Moreover, these limits can be proved to satisfy the ADVI (2.1) in the weak sense.

**Theorem 2.9.** *Let $E$ be positive definite, $K$ be a closed convex set and $(h_k)$ be such that $h_k \searrow 0$ with associated aproximations $x^k(t) := x^{h_k}(t)$ and $u^k(t) := u^{h_k}(t)$. Assume that every $u^{h_k}_{n+1}$ satisfies (2.8). Then there exists a subsequence $(h_\ell) \subseteq (h_k)$, such that $x^\ell \to \bar{x}$ uniformly in $L^\infty(0,T)$ and $u^\ell \rightharpoonup \bar{u}$ in $L^2(0,T)$. Furthermore, $\bar{x}$ is Lipschitz-continuous with constant $L_x$ and $(\bar{x},\bar{u})$ is a weak solution of (2.1).*

*Proof.* The proof is organized as follows: first we will prove the existence of the limits $\bar{x}$ and $\bar{u}$. Next, we show that $\bar{x}$ satisfies the ODE of (2.1) in the weak sense and finally, that $\bar{u}$ solves (2.2) in the weak sense.

By Lemma 2.8 it follows that all approximations $x^h(t)$ are Lipschitz continuous for sufficiently small $h$ with a L-constant that is independent of $h$, i.e. the family of functions $(x^h)$ for sufficiently small $h$ is an equicontinuous family of functions. Moreover, there exists a constant $\tilde{C}$ such that for any $h$ sufficiently small $\|\bar{x}^h\|_{L^\infty(0,T)} \leq \tilde{C}$. Therefore, by the Arzelá-Ascoli theorem [16] there exists a subsequence $(h_\ell) \subseteq (h_k)$ such that the associated sequence $(x^\ell)$ converges in $L^\infty(0,T)$ to a Lipschitz-continuous function $\bar{x}$.

By assumption, it follows that

$$\|u^h\|_{L^\infty(0,T)} \leq \max_n\{\|u^h_{n+1}\|\} \leq \max_n\{c_u(1+\|x^h_n\|)\} \leq c_u e^{T\eta_x}\|x^0\| =: C_u$$

for all $h$ sufficiently small with $\eta_x$ defined as in Lemma 2.8. Hence, $\|u^h\|_{L^2(0,T)} \leq \sqrt{C_u T}$. Since $L^2(0,T)$ is a Hilbert space, there exists a subsequence $(u^{k_j}) \subseteq (u^\ell)$ that satisfies $u^{k_j} \rightharpoonup \bar{u}$ in $L^2(0,T)$ (wlog we assume that $(u^{k_j}) = (u^\ell)$, i.e. $u^\ell \rightharpoonup \bar{u}$ ) .

Next, we show that

$$\bar{x}(t_1) - \bar{x}(t_2) = \int_{t_1}^{t_2} f(\tau,\bar{x}(\tau)) + B(\tau,\bar{x}(\tau))\,\bar{u}(\tau)\,d\tau\,. \tag{2.12}$$

By assumption and Lemma 2.7, for all $\tau \in [t^h_n, t^h_{n+1}]$ it holds with $c_{RK} = \max_i\{\sum_{j=1}^s |a_{ij}|\}$

$$\begin{aligned}\|f^{(i)}_{n,h} - f(\tau,x^h(\tau))\| &\leq L_f(h + \|X^{(i)}_{n,h} - x^h(\tau)\|)\\ &\leq L_f h[1 + c_{RK}(c_f + (c_f c_X + C_B c_u)(1+\|x^h_n\|)) + L_f L_x h]\\ &\leq \eta_f h \end{aligned} \tag{2.13}$$

for some suitably defined $\eta_f > 0$. Similarly we obtain for $\tau \in [t^h_n, t^h_{n+1}]$

$$\begin{aligned}\|B^{(i)}_{n,h}u^h_{n+1} - B(\tau,x^h(\tau))u^h(\tau)\| &\leq \|(B^{(i)}_{n,h} - B(\tau,x^h(\tau)))u^h_{n+1}\|\\ &\leq \eta_f \frac{L_B}{L_f} C_u\,h\,. \end{aligned} \tag{2.14}$$

Now, it follows by (2.13) and (2.14)

$$x^{h_k}_{n+1} - x^{h_k}_n = \int_{t^{h_k}_n}^{t^{h_k}_{n+1}} f(\tau,x^k(\tau)) + B(\tau,x^k(\tau))\,u^k(\tau)\,d\tau + O((h_k)^2)$$

such that for any $0 \le t_1 \le t_2 \le T$:

$$x^k(t_1) - x^k(t_2) = \int_{t_1}^{t_2} f(\tau, x^k(\tau)) + B(\tau, x^k(\tau))\, u^k(\tau)\, d\tau + O(h_k)\,. \tag{2.15}$$

Since $x^\ell \to \bar{x}$ uniformly in $L^\infty(0,T)$ and by the continuity of $f$ we have

$$\lim_{\ell\to\infty} \int_{t_1}^{t_2} f(\tau, x^\ell(\tau))d\tau = \int_{t_1}^{t_2} f(\tau, \bar{x}(\tau))d\tau.$$

and furthermore, by the continuity of $B$ on $\Omega$ and $u^\ell \rightharpoonup \bar{u}$ in $L^2(0,T)$

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} B(\tau, x^\ell(\tau))\, u^\ell(\tau)\, d\tau - \int_{t_1}^{t_2} B(\tau, \bar{x}(\tau))\, \bar{u}(\tau)\, d\tau \right\| \\ &\quad \le \int_{t_1}^{t_2} \|B(\tau, x^\ell(\tau)) - B(\tau, \bar{x}(\tau))\|\, \|u^\ell(\tau)\|\, d\tau \\ &\quad\quad + \left\| \int_{t_1}^{t_2} B(\tau, \bar{x}(\tau))\, (u^\ell(\tau) - \bar{u}(\tau))\, d\tau \right\| \\ &\quad\quad \longrightarrow \quad 0 \qquad\qquad (\ell \to \infty) \end{aligned}$$

Hence, by (2.15) we obtain (2.12).

In order to prove that $\bar{u}$ satisfies the AVI (in the weak sense), we first prove that $\bar{u}(t) \in K$ for almost all $t \in [0,T]$. First, it follows by Mazur's theorem [16], that we can build a sequence $\tilde{u}^m$ of convex combinations of $u^\ell$ ($\ell \in \mathbb{N}$) such that $\tilde{u}^m \to \bar{u}$ in $L^2(0,T)$. Since $K$ is convex and $u^\ell(t) \in K$ for all $\ell \in \mathbb{N}$ and almost all $t$, we have that $\tilde{u}^m(t) \in K$ for almost all $t \in [0,T]$. Furthermore, as there exists a subsequence of $(\tilde{u}^m)$ that converges pointwise almost everywhere in $(0,T)$ and since $K$ is closed, $\bar{u}(t) \in K$ for almost all $t \in [0,T]$. Hence, it remains to show that

$$\int_0^T (d + D\bar{x}(t) + E\bar{u}(t))^T (u(t) - \bar{u}(t))\, dt \ge 0 \tag{2.16}$$

holds for all continuous functions $u : [0,T] \to K$. By definition of $u^h$, $u^h_{n+1} \in \mathcal{S}(K, G_{n+1})$ for all $h$ and $n \in \mathbb{N}$, Lemma 2.8 and the convexity of $K$, we have

$$\begin{aligned} &\int_0^T (d + Dx^h(t) + Eu^h(t))^T (u(t) - u^h(t))\, dt \\ &\quad = \sum_{n=0}^{N_h-1} \int_{t^h_n}^{t^h_{n+1}} (d + Dx^h(t) + Eu^h_{n+1})^T (u(t) - u^h_{n+1})\, dt \\ &\quad \ge h \sum_{n=0}^{N_h-1} (d + Dx^h_{n+1} + Eu^h_{n+1})^T \left( \frac{1}{h} \int_{t^h_n}^{t^h_{n+1}} u(t)\, dt - u^h_{n+1} \right) \\ &\quad\quad -h^2 N_h (\|u\|_{L^\infty(0,T)} + C_u) \|D\| L_x \\ &\quad \ge \quad -h^2 N_h (\|u\|_{L^\infty(0,T)} + C_u) \|D\| L_x \end{aligned}$$

Hence,

$$\limsup_{\ell\to\infty} \int_0^T (d + Dx^\ell(t) + Eu^\ell(t))^T (u(t) - u^\ell(t))\, dt \ge 0, \tag{2.17}$$

i.e. it remains to show that

$$\limsup_{\ell\to\infty} \int_0^T (d + Dx^\ell(t) + Eu^\ell(t))^T (u(t) - u^\ell(t))\, dt \le \int_0^T (d + D\bar{x}(t) + E\bar{u}(t))^T (u(t) - \bar{u}(t))\, dt. \tag{2.18}$$

The convergence properties of $x^\ell$ and $u^\ell$ and the uniform boundedness of $u^h$ imply

$$\begin{aligned} \left| \int_0^T (Dx^\ell)^T u^\ell - (D\bar{x})^T \bar{u}\, dt \right| &= \left| \int_0^T (D(x^\ell - \bar{x}))^T u^\ell + (D\bar{x})^T (u^\ell - \bar{u})\, dt \right| \\ &\le C_u \int_0^T \|D(x^\ell - \bar{x})\|_1 dt + \left| \int_0^T (D\bar{x})^T (u^\ell - \bar{u})\, dt \right| \\ &\longrightarrow 0 \qquad (\ell \to \infty). \end{aligned}$$

Hence (2.18) holds, if

$$\liminf_{\ell\to\infty} \int_0^T (Eu^\ell(t))^T u^\ell(t)\, dt \ge \int_0^T (E\bar{u}(t))^T \bar{u}(t)\, dt\,.$$

Since $E$ is positive definite, we have

$$\begin{aligned} u^\ell(t)^T E u^\ell(t) &= \bar{u}(t)^T E\bar{u}(t) + (u^\ell - \bar{u})(t)^T (E + E^T)\bar{u}(t) \\ &\quad + (u^\ell - \bar{u})(t)^T E (u^\ell - \bar{u})(t) \\ &\ge \bar{u}(t)^T E\bar{u}(t) + (u^\ell - \bar{u})(t)^T (E + E^T)\bar{u}(t) \end{aligned}$$

for almost all $t \in [0, T]$, such that

$$\begin{aligned} \liminf_{\ell\to\infty} \int_0^T (Eu^\ell(t))^T u^\ell(t)\, dt &\ge \int_0^T (E\bar{u}(t))^T \bar{u}(t)\, dt \\ &\quad + \liminf_{\ell\to\infty} \int_0^T (u^\ell - \bar{u})(t)^T (E + E^T)\bar{u}(t)\, dt \\ &= \int_0^T (E\bar{u}(t))^T \bar{u}(t)\, dt\,, \end{aligned}$$

which proves (2.16). □

## 3 Semismooth Newton Method

Applying the Runge-Kutta time discretization as presented in the previous section, we obtain finite-dimensional problem consisting of the nonlinear Runge-Kutta system and the finite-dimensional AVIs (2.4). Moreover, the $AVI(K, G^h_{n+1})$ can be reformulated by a nonlinear, nonsmooth equation using the so-called natural map [8] associated with the $AVI(K, G^h_{n+1})$. Define $z = (z_1, z_2, z_3) := (X^h_n, x^h_{n+1}, u^h_{n+1}) \in \mathbb{R}^{sn} \times \mathbb{R}^n \times \mathbb{R}^m$, then $z_3 = u^h_{n+1}$ is a solution of $AVI(K, G^h_{n+1})$ if and only if [8]

$$F_{VI}(z) := z_3 - \Pi_K(z_3 - (d + Dz_2 + Ez_3)) = 0,$$

where $\Pi_K$ denotes the projection operator onto the convex set $K$.

Hence, in oder to solve the discretized affine differential variational inequality, for each time step $n = 0, 1, .., N - 1$, we have to solve the nonlinear, nonsmooth system

$$\begin{pmatrix} F_{RK}(z) \\ F_{VI}(z) \end{pmatrix} = 0. \tag{3.1}$$

In the following we assume that $K \subseteq \mathbb{R}^m$ is a rectangular box, i.e. $K = \prod_{j=1}^{m}[\alpha_j, \beta_j]$. By asssumption, $F_{RK}(z)$ is continuously differentiable. Since $F_{VI}$ concatenates linear functions with the nonsmooth function $\Pi_K$, it is itself nonsmooth. However, it is known [8], that the natural map (i.e. $F_{VI}$) is semismooth, i.e. $F$ itself is semismooth. In the following we review some definitions and relations of nonsmooth analysis that we will refer to later on, taken from [4, 5] and [25, 26], respectively.

**Definition 3.1.**

1. The directional derivative of $F$ is given by the limit

$$F'(z; d) = \lim_{t \downarrow 0} \frac{F(z + td) - F(z)}{t}.$$

2. Let $F$ be locally Lipschitz-continuous, then the generalized Jacobian of F is given by

$$\partial F(z) = \text{conv}\{ V \in \mathbb{R}^{(sn+n+m)\times(sn+n+m)} \;:\; \exists z^i \to z, F'(z^i) \to V \}$$

3. $F$ is said to be semismooth at $z$, if it is locally Lipschitz-continuous at $z$ and

$$\lim_{\substack{V \in \partial F(z+td) \\ d \to s, t\downarrow 0}} Vd$$

exists for any $s$.

**Proposition 3.2.** *Suppose that $F$ is a locally Lipschitz-continuous function. Then the following statements are equivalent:*

1. *$F$ is semismooth at $z$.*
2. *for any $V \in \partial F(z + s)$ , $s \to 0$, $Vs - F'(z; s) = o(\|s\|)$.*

**Proposition 3.3.** *Suppose that $F$ is semismooth. Then for any $s \to 0$:*

$$F(z + s) - F(z) - F'(z; s) = o(\|s\|).$$

The semismooth Newton method for (3.1) is based on the semismooth Newton step $s^k$ that is obtained by the solution of the nonsmooth Newton equation

$$V_k\, s^k = -F(z^k) \qquad \text{with} \qquad V_k \in \partial F(z^k). \tag{3.2}$$

**Algorithm 1**: Semismooth Newton Method

*Choose an initial vector $z^0 \in \mathcal{N}(z^*)$.*
**for** $k = 0, 1, 2, ...$ **do**
  *If $\|F(z^k)\| = 0$ then STOP.*
  *Choose some $V_k \in \partial F(z^k)$ and determine the semismooth Newton step $s^k$ that solves (3.2).*
  *Set $z^{k+1} := z^k + s^k$ and $k \leftarrow k + 1$.*

Proposition 2.4 implies that under appropriate conditions $F(z) = 0$ has a (unique) solution. We next show that under suitable assumptions the step computation in Algorithm 1 is well-defined and we can expect locally superlinear convergence of the generated iterates $z^k$.

The generalized Jacobian $\partial F(z^k)$ is given by the family of matrices

$$V = \begin{pmatrix} I - hA_1 & 0 & -hB_1 \\ -hA_2 & I & -hB_2 \\ 0 & \Lambda D & I - \Lambda(I-E) \end{pmatrix}, \tag{3.3}$$

where $\Lambda$ denotes the generalized derivative of $\Pi_K$, which equals the convex hull of the B-(Bouligand) subdifferential $\partial_B$ of $\Pi_K$ (cf. Definition 3.1), which is given by $\Lambda = \operatorname{diag}(\lambda_j)$ with

$$\lambda_j \in \begin{cases} 0, & \text{if } \; w_j \notin [\alpha_j, \beta_j] \\ 1, & \text{if } \; w_j \in (\alpha_j, \beta_j) \\ \{0,1\}, & \text{if } \; w_j \in \{\alpha_j, \beta_j\} \end{cases} \tag{3.4}$$

where $w_j := (-d - Dz_2 + (I-E)z_3)_j$. Next, we show that if $E$ is strictly diagonally dominant with positive diagonal entries, then for sufficiently small $h > 0$ all elements $V \in \partial F(z^k)$ are regular. Moreover, since $\partial_B F(z^k) \subseteq \partial F(z^k)$, we not only get the regularity of all elements of the generalized Jacobian, but also of the B-subdifferential, i.e. we also get the BD-regularity of $z^k$.

**Lemma 3.4.** *Let $E$ be strictly diagonally dominant with positive diagonal entries $e_{ii}$. Then there exists $\bar{h} > 0$ such that for every $h \in (0, \bar{h}]$ all elements $V \in \partial F(z)$ are regular for all $z$.*

*Proof.* First, we will prove that $R := I - \Lambda(I - E)$ is regular for any $\Lambda = \operatorname{diag}(\lambda_j)$ with $\lambda_j \in [0,1]$. Since $E$ is assumed to be diagonally dominant with $e_{ii} \geq 0$,

$$\sum_{\substack{j=1 \\ j \neq i}}^{m} |r_{ij}| = \sum_{\substack{j=1 \\ j \neq i}}^{m} |\lambda_i e_{ij}| < |\lambda_i e_{ii}| \leq |\lambda_i e_{ii} + (1 - \lambda_i)| = |r_{ii}|$$

for all $\lambda_i \in (0,1]$. However, if $\lambda_i = 0$, then $r_{ij} = 0$ for all $j \neq i$ such that $\sum_{j=1, j\neq i}^{m} |r_{ij}| = 0 < 1 = r_{ii}$. Thus, $R$ is again strictly diagonally dominant and therefore regular.

Now, consider the homogeneous linear system $Vs = 0$, where $s = (s_1, s_2, s_3) \in \mathbb{R}^{sn} \times \mathbb{R}^n \times \mathbb{R}^m$ and $V \in \partial F(z)$ for any $z$. Then by (3.3) and the regularity of $R$, we obtain

$$s_3 = -R^{-1} \Lambda D s_2.$$

Next, we substitute $s_3$ by the right-hand side in $Vs = 0$, this yields the homogeneous linear system

$$\begin{pmatrix} I - hA_1 & hB_1(R^{-1}\Lambda D) \\ -hA_2 & I + hB_2(R^{-1}\Lambda D) \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $h > 0$ is sufficiently small, then corresponding system matrix is again strictly diagonally dominant. This implies $(s_1, s_2) = 0$ and furthermore $s_3 = 0$. ☐

**Lemma 3.5.** *Assume that $z^*$ is a solution to (3.1) with $h > 0$ sufficiently small. Then there exists a constant $C_V > 0$ such that all elements $V$ of the generalized Jacobian $\partial F(z)$ satisfy $\|V^{-1}\| \leq C_V$ for all $z$ in some neighbourhood $\mathcal{N}(z^*)$ of $z^*$.*

*Proof.* First note that by the Lipschitz-continuity of $f, B$ and the projection operator $\Pi_K$ the function $F$ is also Lipschitz. Hence $\partial F(z)$ is bounded in a neighbourhood of $z^*$. Now assume that there is no such neighbourhood $\mathcal{N}(z^*)$, then there exists a sequence $z^k \to z^*$ and $V_k \in \partial F(z^k)$ with $\|V_k^{-1}\| > C_V$. However, by the boundedness of $\partial F(z)$ in a neighbourhood of $z^*$, it follows that $V_k$ contains a subsequence that converges to some limit $\bar{V} \in \partial F(z^*)$ that has to be singular. This, however, contradicts Lemma 3.4. □

**Theorem 3.6.** *Let $z^*$ be a solution to (3.1). Assume that $E \in \mathbb{R}^{m\times m}$ is strictly diagonally dominant with positive diagonal entries $e_{ii}$. Then there exists a neighbourhood $\mathcal{N}(z^*)$ such that for all $z^0 \in \mathcal{N}(z^*)$ Algorithm 1 either terminates after finitely many iterations with $z^k = z^*$ or it produces a sequence $(z^k)$ that converges superlinearly to $z^*$, i.e. $\|z^{k+1} - z^*\| = o(\|z^k - z^*\|)$.*

*Proof.* By assumption and Lemma 3.5 and Lemma 3.5, there exists a neighbourhood $\mathcal{N}(z^*)$ such that for all $z^0 \in \mathcal{N}(z^*)$, all elements $V$ of the generalized Jacobian $\partial F(z^0)$ are nonsingular and satisfy $\|V^{-1}\| \leq C_V$ for some constant $C_V$. The semismoothness of $F$ therefore implies (for $k = 0$)

$$\begin{aligned}
\|z^{k+1} - z^*\| &= \|(z^k - z^*) - V_k^{-1} F(z^k)\| \\
&\leq \|V^{-1}\| \big(\|V_k(z^k - z^*) - F'(z^*; z^k - z^*)\| \\
&\qquad + \|F(z^k) - F(z^*) - F'(z^*; z^k - z^*)\|\big) \\
&= o(\|z^k - z^*\|)
\end{aligned}$$

for $\|z^0 - z^*\|$ sufficiently small. By induction the implications can be transferred to all $k \in \mathbb{N}$. □

**Remark 3.7.** Note that instead of using the generalized Jacobian in Algorithm 1, we could also use the B-subdifferential of F given by (3.3) and (3.4). In that case one needs to guarantee the weaker BD-regularity of each $z^k$ as a condition to prove a similiar convergence result as Theorem 3.6 (see [25]). However, our preliminary numerical tests indicated that for some examples it might be advantageous to use the generalized Jacobian. Therefore, as a start we remained with the generalized Jacobian.

## 4 Numerical Tests

In the following, we present some numerical results that we obtained for the previously discussed semismooth Newton method applied to the Runge-Kutta discretization scheme analysed in Section 2.

We discuss two problems taken from [6]. Both of them are examples for a differential Nash game as presented in Section 1. As discussed there, these problems can be solved by the solution of the corresponding set of Hamiltonian systems. Due to the boundary conditions for the adjoint equations at time $t = T$ the resulting DVIs for these problems are, however, not initial value problems but boundary value problems as (1.1) (in contrast to the DVIs discussed in Section 2 and 3). Therefore, instead of solving (3.1) for each time step seperately, we concatenate the systems (3.1) into one large nonlinear system and solve the concatenated system for all time steps all at once.

### 4.1 Example 1

The first problem that we test is a differential game with two players each of them facing an optimal control problem. The optimal control problems of player 1 and 2 are given by

$$\max_{u_i} J_i(x,u) = \int_0^T \theta_i(x,u)\,dt \qquad \text{subject to} \quad u_i \in U_i = [0,1], \qquad i=1,2,$$

respectively, with $u_1$ and $u_2$ being the controls for player 1 and player 2, respectively,

$$\begin{aligned} \theta_1(x,u_1,u_2) &= x - u_1 u_2 - \alpha(u_1)^2, \\ \theta_2(x,u_1,u_2) &= (1+u_1)\,x - u_2 - \beta(u_2)^2 \end{aligned}$$

and $x$ satisfying the state equation

$$\dot{x}(t) = u_1(t) + u_2(t) \qquad t \in [0,T]$$

where $T=2$. The initial vector $z^0 = (x^0, p_1^0, p_2^0, u_1^0, u_2^0)$ is given by the associated discretizations of $x^0(t) \equiv 1$, $p_1^0(t) = T-t$, $p_2^0(t) = 2(T-t)$, $u_1^0(t) \equiv 1$ and $u_2^0(t) \equiv 0$ for $t \in [0,T]$. As the right-hand side of the state equation in this example does not depend on the state itself (i.e. $f(t,x)=0$) in this case the particular choice of the Runge-Kutta method is insignificant.

If $\alpha, \beta > 0$ are set equal to zero, we obtain the example given in [6]. However in that case, the associated matrices $E$ are not positive definite and strictly diagonally dominant. We, therefore solved the problem for various strictly positive values of the parameters $\alpha, \beta$, (i.e. positive definite, diagonally dominant matrices $E$). It can be observed, that for small values of $\alpha, \beta$, the original result of the Nash game given in [6]:

$$u_1(t) = \begin{cases} 1 & \text{if} \quad t \in [0, T-1) \\ 0 & \text{if} \quad t \in [T-1, T-0.5) \\ 1 & \text{if} \quad t \in [T-0.5, T], \end{cases} \qquad u_2(t) = \begin{cases} 1 & \text{if} \quad t \in [0, T-0.5) \\ 0 & \text{if} \quad t \in [T-0.5, T], \end{cases}$$

is approximately recovered. In Figure 1, we display the result for $\alpha, \beta = 0.0625$, the constant value $\lambda = 0.5$ for the subdifferential of $\Pi_K$ (cf. (3.4)) and grid size $N_h = 640$. The state $x$ is given by the dashed line, the two associated adjoint states are given by the dashed-dotted line and the two controls are given by the solid lines. Next, the results given in Table 1, indicate that in this case the semismooth Newton method is independent of the gridsize as the number of Newton iterations remains almost constant for various gridsizes.

Moreover, we display the evolution of the error $\|z^{k+1} - z^*\| / \|z^k - z^*\|$ and $\log_{10}(\|F(z^k)\|)$ for various gridsizes and two parameter settings for Example 1 in Figure 2.

### 4.2 Example 2

The second example corresponds to a linear quadratic two-player differential game model taken from [6]. The optimal control problems of the players are given by

$$\max_{u_i} J_i(x,u) = \int_0^T e^{-rt}\theta_i(x,u_1,u_2)\,dt \qquad \text{subject to} \quad u_i \in U_i = [-5,5], \qquad i=1,2,$$

with $r = 1.0$,

$$\begin{aligned} \theta_1(x,u_1,u_2) &= \frac{5}{2}x^2 - \frac{3}{4}(u_1)^2, \\ \theta_2(x,u_1,u_2) &= 3\,x^2 - \frac{7}{4}(u_2)^2 \end{aligned}$$

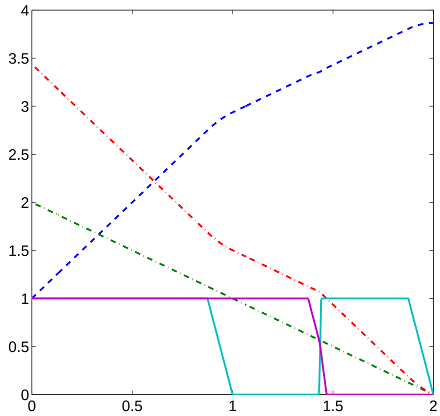



Figure 1: Plot of the state, the adjoint states and the controls of the Nash equilibrium of Example 1 for $\alpha, \beta = 0.0625$, $\lambda = 0.5$ and grid size $N_h = 640$.

| | $\alpha, \beta = 0.0625$ | | $\alpha, \beta = 0.75$ | |
|---|---|---|---|---|
| $N_h$ | It. | CPU time | It. | CPU time |
| 20 | 93 | 0.04 | 17 | 0.02 |
| 40 | 92 | 0.14 | 18 | 0.04 |
| 80 | 91 | 0.81 | 18 | 0.15 |
| 160 | 97 | 2.77 | 18 | 0.51 |
| 320 | 95 | 6.79 | 18 | 1.11 |
| 640 | 91 | 24.76 | 19 | 5.00 |

Table 1: Number of iterations (It.) and CPU times (in sec.) for Example 1 with $\lambda = 0.5$ and two parameter settings for $(\alpha, \beta)$.

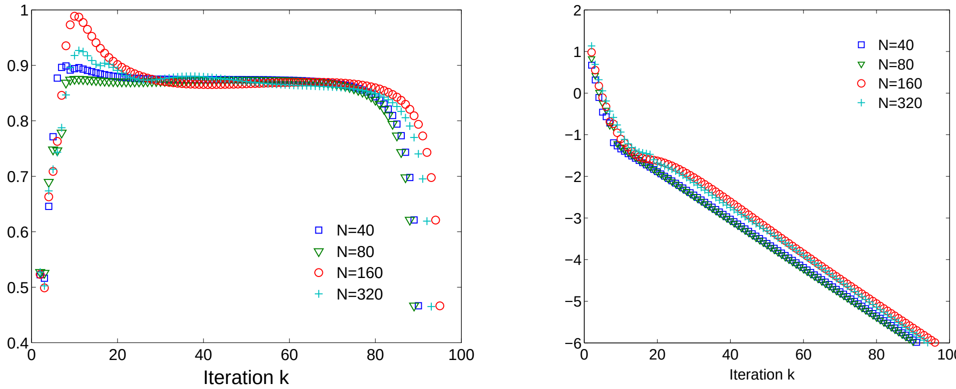
and $x$ satisfying the state equation

$$\dot{x}(t) = -\frac{1}{2}\, x + \frac{3}{10}\, u_1(t) + \frac{1}{2}\, u_2(t), \qquad x(0) = 15.0, \qquad t \in [0, T]$$

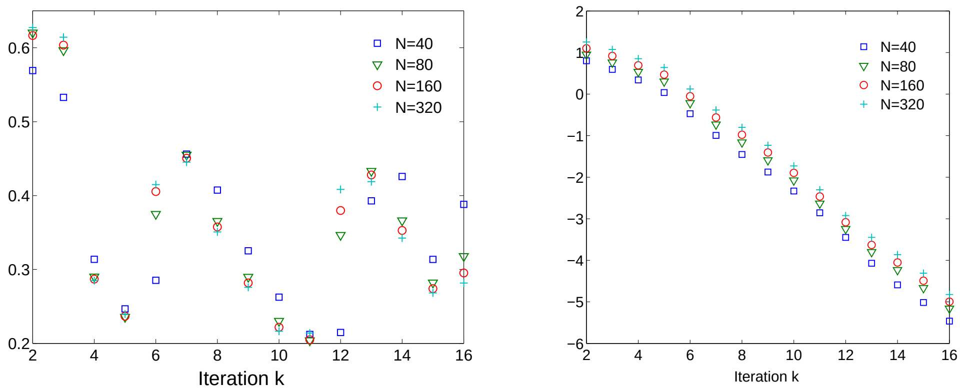
where $T = 5$. The initial vector $z^0 = (x^0, p_1^0, p_2^0, u_1^0, u_2^0)$ for the optimization is given by the discretizations of $x^0(t) \equiv 1$, $p_1^0(t) \equiv 1$, $p_2^0(t) \equiv 1$, $u_1^0(t) \equiv 0$ and $u_2^0(t) \equiv 0$ for $t \in [0, T]$. Furthermore, we used again $\lambda = 0.5$ for the generalized Jacobian. We test two Runge-Kutta schemes a first-order 1-stage method and a second- order 2-stage method. Moreover, as discussed e.g. in the papers [3, 11, 14, 15], we used partitioned Runge-Kutta methods [12] such that the optimization and the discretization process commute [3, 15]. As first-order method, we apply an explicit Euler method to the state equation with the corresponding implicit method for the adjoint equations. As second-order method for the state equation we choose the trapezoidal rule.

In Figure 3 we display the solution and the evolution of the error for $N_h = 640$ and the second-order RK-method. We display the state, the adjoints $p_1$ and $p_2$ for player 1 and 2 and the controls $u_1$ and $u_2$ of the players.

Furthermore, in Table 2 we display again the numbers of iterations for various gridsizes




(a) $\|z^{k+1} - z^*\|/\|z^k - z^*\|$ (left) and $\log_{10}(\|F(z^k)\|)$ (right) for $\alpha, \beta = 0.0625$




(b) $\|z^{k+1} - z^*\|/\|z^k - z^*\|$ (left) and $\log_{10}(\|F(z^k)\|)$ (right) for $\alpha, \beta = 0.75$

Figure 2: Evolution of the errors $\|z^{k+1} - z^*\|/\|z^k - z^*\|$ and $\log_{10}(\|F(z^k)\|)$ for Example 1 with $\alpha, \beta \in \{0.0625, 0.75\}$, $\lambda = 0.5$ and $N_h \in \{40, 80, 160, 320\}$.

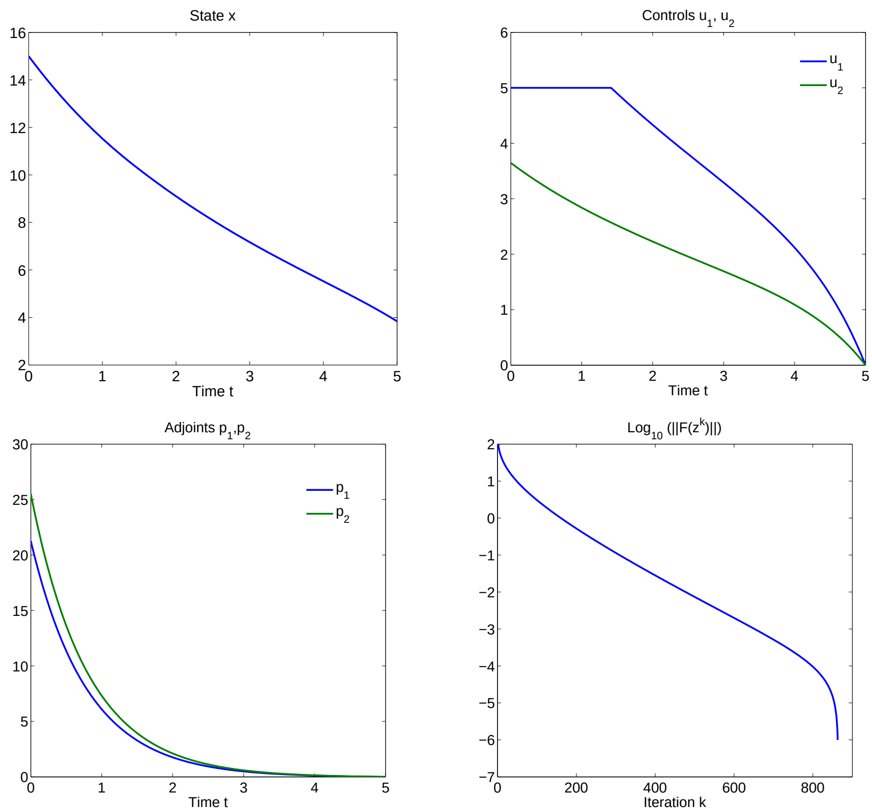



Figure 3: State $x$, controls $u_1, u_2$, adjoints $p_1, p_2$ and evolution of the error for Example 2 obtained for the second-order RK-method and $N_h = 640$.

| first-order | | second-order | |
|---|---|---|---|
| $N_h$ | It. | $N_h$ | It. |
| 20 | 691 | 20 | 708 |
| 40 | 732 | 40 | 742 |
| 80 | 765 | 80 | 772 |
| 160 | 799 | 160 | 801 |
| 320 | 831 | 320 | 832 |

Table 2: Number of iterations (It.) for Example 2 with $\lambda = 0.5$ and a first- and second-oder RK method.

| | first-order Runge-Kutta scheme | | |
|---|---|---|---|
| $N_h$ | $\|x_N^{opt} - x^*\|_2$ | $\|u_{N,1}^{opt} - u_1^*\|_2$ | $\|u_{N,2}^{opt} - u_2^*\|_2$ |
| 20 | 5.22 E+0 (0.00) | 3.01 E+0 (0.00) | 1.87 E+0 (0.00) |
| 40 | 3.54 E+0 (1.46) | 2.11 E+0 (1.43) | 1.31 E+0 (1.43) |
| 80 | 2.37 E+0 (1.49) | 1.44 E+0 (1.46) | 8.95 E-1 (1.46) |
| 160 | 1.53 E+0 (1.54) | 9.48 E-1 (1.52) | 5.91 E-1 (1.51) |
| 320 | 8.93 E-1 (1.71) | 5.72 E-1 (1.66) | 5.72 E-1 (1.03) |

Table 3: Error in the state and the controls (with the corresponding ratio in brackets) for Example 2 for the first-oder Runge-Kutta scheme.

$N_h$ and both RK schemes. In this case, it can be observed that the number of iterations are alomst identical for both methods, however the independence of gridsize is lost as the number of iterations are slightly increasing with the gridsize.

Finally, in Table 3 and 4 we report the evolution of the error $\|x_N^{opt} - x^*\|_2$ in the state $x$ and $\|u_{N,i}^{opt} - u_i^*\|_2$ in the controls $u_1$ and $u_2$ for the two RK schemes and several gridsizes $N_h$, where $x_N^{opt}$ and $u_{N,i}^{opt}$ denote the optimal state or control, respectively, obtained for the gridsize $N_h$. The optimal solutions $x^*$ and $u_i^*$ correspond to the solution obtained for the second-order Runge-Kutta scheme for $N_h = 640$. The values in brackets correspond to the ratio of the error divided by the previous one.

Since the solution of the ODE is coupled with the solution of the VI and we cannot expect the order of convergence for the ODE to transfer directly to the solution of the VI, we will not obtain the complete order of convergence of the corresponding Runge-Kutta

| | second-order Runge-Kutta scheme | | |
|---|---|---|---|
| $N_h$ | $\|x_N^{opt} - x^*\|_2$ | $\|u_{N,1}^{opt} - u_1^*\|_2$ | $\|u_{N,2}^{opt} - u_2^*\|_2$ |
| 20 | 3.13 E+0 (0.00) | 1.69 E+0 (0.00) | 1.02 E+0 (0.00) |
| 40 | 2.10 E+0 (1.49) | 1.14 E+0 (1.48) | 6.81 E-1 (1.49) |
| 80 | 1.37 E+0 (1.53) | 7.48 E-1 (1.52) | 4.44 E-1 (1.53) |
| 160 | 8.29 E-1 (1.66) | 4.52 E-1 (1.66) | 2.68 E-1 (1.66) |
| 320 | 3.90 E-1 (2.13) | 2.12 E-1 (2.13) | 2.12 E-1 (1.26) |

Table 4: Error in the state and the controls (with the corresponding ratio in brackets) for Example 2 for the second-oder Runge-Kutta scheme.

| $\lambda$ | 0.0 | 0.25 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|
| $\alpha, \beta = 0.75$ | 55 | 29 | 18 | 13 | 6 |
| $\alpha, \beta = 0.0625$ | 160 | 124 | 95 | 91 | 90 |
| $\alpha, \beta = 0.01$ | 606 | 521 | 497 | 493 | 493 |

Table 5: Number of iterations for Example 1 for various $\lambda \in [0,1]$, $N_h = 320$ and $\alpha, \beta \in \{0.01, 0.0625, 0.75\}$.

method. However, as can be observed, by using a higher-order method for the ODE we might anyhow slightly improve the convergence for the DVI using a higher-order method.

We also tested the sensitivity of our method with respect to the choice of $\lambda \in [0,1]$. Note, that choosing $\lambda \in \{0,1\}$, refers to the choice $V_k \in \partial_B F(z_k)$. As for Example 1, we fixed $N_h = 320$ and $\alpha, \beta \in \{0.01, 0.0625, 0.75\}$ and varied the parameter $\lambda \in \{0, 0.25, 0.5, 0.75\}$. The results are displayed in Table 5. It seems that unfortunately the efficiency of the method is somehow sensitive to the choice of the elements of the subdifferential. In particular for $\lambda = 0$ we obtain a less efficient method. Concerning Example 2, we have made similiar observations. The lack of efficiency (and robustness in the case of Example 2) might be due to the missing information, since $\lambda = 0$ disregards information that originates from the directional derivatives associated with $\lambda = 1$. However, these observations deserve some closer attention and the question which elements of the subdifferential to choose will be investigated amongst others in subsequent research.

## Summary and Outlook

In this paper, we analysed a numerical scheme to solve differential variational inequalities as they appear e.g. as necessary conditions of differential Nash game models. The method we presented is based on a Runge-Kutta time discretization scheme and a semismooth Newton method. The method was proved to be convergent under suitable assumptions. Moreover, some preliminary numerical tests indicated a good performance of the method though still being at a preliminary stage. Future research topics hence not only include the extension of the theoretical analysis to nonlinear DVIs as well as the interaction of the convergence properties for the ODE with the convergence of solutions of the VI but also the numerical improvement of the method that involves globalization techniques as e.g. line-search [25], trust-region [29] or nonsmooth filter methods, improvement of the order of convergence and the investigation of the correct or best choice of elements of the subdifferential.

Furthermore, in our theory, we considered initial value problems though in practice (in particular in the case of differential Nash games) one often deals with boundary value problems. Hence, another desirable extension of the theory and numeric presented here concerns the treatment of boundary value problems (where for the proof of convergence possibly similar techniques as in [7, 19] can be applied).

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