Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 9, No. 2, pp. 275–299, April 2013

# A PARAMETRIC ACTIVE-SET METHOD FOR QPS WITH VANISHING CONSTRAINTS ARISING IN A ROBOT MOTION PLANNING PROBLEM

Christian Kirches*, Andreas Potschka, Hans Georg Bock and Sebastian Sager

**Abstract:** Combinatorial and logic constraints arising in a number of challenging optimization applications can be formulated as *vanishing constraints*. Quadratic programs with vanishing constraints (QPVCs) then arise as subproblems during the numerical solution of such problems using algorithms of the Sequential Quadratic Programming type. QPVCs are nonconvex problems violating standard constraint qualifications. In this paper, we propose a primal–dual parametric active set method for finding strongly stationary points of QPVCs under the MPVC–LICQ regularity condition. We develop a local search strategy that allows to improve such points up to global optimality for this subclass of nonconvex QPVC subproblems. A parametric programming framework facilitates the realization of hot–starting capabilities which improves the efficiency of both the active set method and the local search. We apply the developed methods to solve several instances of a robot path–finding problem with logic communication constraints.

**Key words:** *mathematical programs with vanishing constraints, parametric quadratic programming, active set methods, robot motion planning*

**Mathematics Subject Classification:** *90C20, 90C26, 90C55*

## 1 Introduction

This paper is concerned with a sequential quadratic programming (SQP) framework and a parametric primal–dual active set method for finding locally optimal solutions of a subclass of difficult mathematical programs with so–called *vanishing constraints*, MPVCs for short. The problem class was first introduced and named in [2], and reads

$$\min_{x\in\mathbb{R}^n} \quad F(x) \tag{1.1a}$$

$$\text{s.t.} \quad 0 = C(x), \tag{1.1b}$$

$$0 \le D(x), \tag{1.1c}$$

$$0 \le H_j(x)\cdot G_j(x), \quad j \in \bar{l}, \tag{1.1d}$$

$$0 \le H_j(x), \quad j \in \bar{l}. \tag{1.1e}$$

All functions are assumed to be twice continuously differentiable with respect to the unknown $x$. The constraints $0 \le G_j(x)$, $j \in \bar{l} := \{1,\dots,l\} \subset \mathbb{N}$ in (1.1d) are taken into consideration

*Corresponding author

for a point $x \in \mathbb{R}^n$ of the feasible set if and only if the associated residual $H_j(x)$ is strictly greater than zero in (1.1e). Conversely, a constraint $0 \le G_j(x)$ is called *vanished* if $H_j(x) = 0$, giving rise to the name of the problem class.

Problem (1.1) has a nonconvex feasible set with combinatorial structure. The subclass of problems considered in this paper is assumed to satisfy a regularity condition referred to as MPVC–LICQ in the literature. This allows to retain the concept of iterating towards KKT (*Karush–Kuhn–Tucker*) based local optimality. In addition, multiplier information for a KKT point will be seen to allow for additional iterations improving the solution towards global optimality, although no guarantees are shown to hold here.

Several authors are concerned with the development of convergence results to terminal points satisfying weaker stationarity conditions under less restrictive CQs, such as [2, 15, 16] for M-stationarity and recently [10] for T-stationarity.

## 1.1 Motivation and Applications

Problem (1.1) can be interpreted as an NLP including a logic implication,

$$\min_{x \in \mathbb{R}^n} \quad F(x) \tag{1.2a}$$

$$\text{s.t.} \quad 0 = C(x), \tag{1.2b}$$

$$0 \le D(x), \tag{1.2c}$$

$$0 \le H(x), \tag{1.2d}$$

$$0 < H_j(x) \implies 0 \le G_j(x) \quad j \in \bar{l}. \tag{1.2e}$$

In the following we give two examples of challenging applications in which a vanishing constraint formulation appears in this way.

**Robot Motion Planning** One example of logic constraints in a real-world application arises in robot motion planning [1, 24, 34]. Here, a communication network of a given density needs to be maintained among a swarm of independent mobile robots. For each pair $(i, j)$ of robots, $H_{i,j}(x) > 0$ indicates that the pair is communicating. Then, $0 \le G_{i,j}(x)$ must be satisfied to ensure that the distance between robots $i$ and $j$ actually allows for communication. Conversely, this distance constraint *vanishes* for each pair $(i, j)$ of robots with $H_{i,j}(x) = 0$ which do not communicate. We return to this application in Section 5.

**Truss Topology Optimization** A prominent example of vanishing constraints arises in truss topology optimization. Here, one is interested in finding the optimal design of a truss structure using the *ground structure approach* [9]. On a grid in $\mathbb{R}^2$ or $\mathbb{R}^3$ a set of $l$ potential truss bars with cross-sectional areas $x_i \ge 0$ is considered for optimization. In order to prevent structural failure under external loads, constraints are imposed on the internal force and the stress in each truss bar. These constraints vanish for any bar with cross-sectional area $x_i = 0$, which is not implemented as a real bar. The objective may include structural weight, deformation energy, manufacturing cost, or similar performance indicators. Further details on the problem class as well as numerical results for case studies can be found e.g. in [2, 16].

## 1.2 Contributions

In this paper we follow an idea for the numerical solution of a family of structurally nonconvex NLPs that has been described as a general framework for nonconvex SQP in [32], see also Section 2.4. Its possible application to MPVC was first mentioned in passing in [2]. Subsequent works, such as [16, 18, 20], in general pursue the idea of solving regularized problems using interior-point methods. Active set approaches for solving nonconvex problems date back to [14] and subsequent works which treat piecewise linear models. In [11] an active set method with anti-cycling measures for linear programs with complementarity constraints is described.

**Sequential Quadratic Programming Approach** In contrast to [16, 18, 20] we propose an active set approach in an SQP framework to solve MPVCs under the regularity assumption of MPVC-LICQ. For the locally quadratic subproblems, we propose to refrain from linearizing the vanishing constraint, thereby carrying the nonconvexity over to the subproblems. We refer to the arising subproblems as quadratic programs with vanishing constraints (QPVCs).

**Active Set Approach** We propose a parametric primal–dual active set method for the solution of QPVCs. A related method has been described for convex quadratic programming (QP) in [7] and applied to model–predictive control in [13].

**Partitioning and Hot Starts** The parametric active set method either traces a piecewise affine linear homotopy to a locally optimal solution of the QPVC located in a certain convex subset of the problem's nonconvex feasible set, or indicates failure on the boundary of that subset. In the latter case, efficient continuation of the homotopy in an adjacent convex subset is necessary. To this end, we propose a technique for so–called *hot starts* of the parametric active set method.

**Computational Results** Reports of computational results for MPVC are still scarcely found. We are only aware of [2, 16] where numerical results for truss bar optimization problems are given. We apply the proposed algorithm to a discretized nonlinear optimal control problem, variants of which have previously been investigated in [1, 24, 34]. This problem involves a type of logic constraints for which we give a formulation as vanishing constraints. We compute locally optimal solutions to a range of problem instances for which we are not aware of previous solution reports so far.

### 1.3 Outline

The remainder of this paper is organized as follows. In Section 2 we describe an SQP framework for the class of NLPs with vanishing constraints. Carrying the structural nonconvexity of the NLP over to the SQP subproblems, we introduce the problem class of QPVCs. Constraint qualifications and stationarity concepts as found in the literature are briefly discussed as we settle on the assumption of MPVC–LICQ. In Section 3 we propose an active set approach for the solution of QPVC. It is based on an overlapping subdivision of the feasible set into convex subsets. By analyzing MPVC strong stationary conditions we develop rules for searching these subsets based on MPVC multiplier information. These rules can be extended to include progress towards global optimality for the QPVC. We describe a tree-search type algorithm and an active-set type algorithm which realize searches over the convex subsets. In Section 4 we present a primal–dual parametric active set method for convex QPs. It is efficient for solving a sequence of closely related QPs. We propose extensions to this method that allow to efficiently hot-start this algorithm during movement from one convex subset of the QPVC to another. In Section 5 a vanishing constraint formulation for the robot path-finding and communication problem is presented. Logic communication constraints are formulated as vanishing constraints. We apply the proposed primal–dual parametric active set strategy for QPVC in an SQP framework to solve a number of problem instances to optimality. We compare the obtained solutions to those known from the literature. Section 6 concludes this paper with a brief summary.

## 2 Nonlinear Programs with Vanishing Constraints

In this section we briefly collect results on the violation of commonly assumed constraint qualifications by problem (1.1) and on appropriately modified concepts of stationarity. Additionally, we indicate why conventional SQP methods are likely to fail or at least show serious deterioration of numerical convergence behavior. This establishes the need for new numerical methods for the efficient solution of problem (1.1), and we introduce the concept of MPVC strong stationarity ([17]) under the regularity assumption of MPVC–LICQ ([2]) to this end. Based on this concept we realize a nonconvex SQP framework on the basis of [32] for the case of NLPs with vanishing constraints. Therein, we choose to carry the nonconvexity of the NLP problem over to the SQP subproblems (QPVCs).

### 2.1 Constraint Qualifications

To ease the notation we consider the following NLP with vanishing constraints,

$$\min_{x\in\mathbb{R}^n} \quad F(x) \tag{2.1a}$$

$$\text{s.t.} \quad 0 \le H_j(x)\cdot G_j(x), \quad j \in \bar{l}, \tag{2.1b}$$

$$0 \le H_j(x), \qquad j \in \bar{l}. \tag{2.1c}$$

dropping standard equality and inequality constraints from problem (1.1). These are included in the presented theory and algorithms as special case $G_j(x) = 1$.

**Active Set and Index Sets** The conventional definition of sets $\mathcal{A}_{\mathrm{HG}}(\overline{x}), \mathcal{A}_{\mathrm{G}}(\overline{x})$ of active NLP constraints for a feasible point $\overline{x} \in \mathbb{R}^n$ of problem (2.1),

$$\mathcal{A}_{\mathrm{HG}}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x})\cdot G_j(\overline{x}) = 0 \right\}, \tag{2.2a}$$

$$\mathcal{A}_{\mathrm{H}}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x}) = 0 \right\}, \tag{2.2b}$$

is extended for the problem class of MPVCs as follows. According to [2] we introduce the *index sets*

$$\mathcal{I}_{0+}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x}) = 0,\ G_j(\overline{x}) > 0 \right\}, \tag{2.3a}$$

$$\mathcal{I}_{++}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x}) > 0,\ G_j(\overline{x}) > 0 \right\}, \tag{2.3b}$$

$$\mathcal{I}_{00}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x}) = 0,\ G_j(\overline{x}) = 0 \right\}, \tag{2.3c}$$

$$\mathcal{I}_{+0}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x}) > 0,\ G_j(\overline{x}) = 0 \right\}, \tag{2.3d}$$

$$\mathcal{I}_{0-}(\overline{x}) := \left\{ j \in \bar{l} \mid H_j(\overline{x}) = 0,\ G_j(\overline{x}) < 0 \right\}. \tag{2.3e}$$

which partition the set of active constraints according to signs of $G(\overline{x})$ and $H(\overline{x})$,

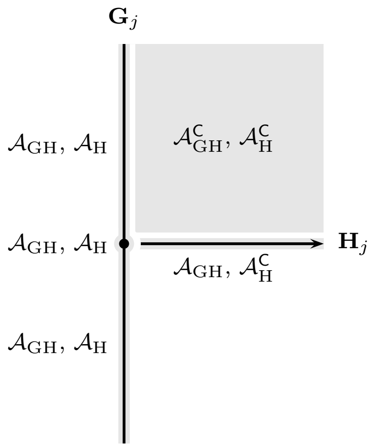
$$\mathcal{A}_{\mathrm{HG}}(\overline{x}) = \mathcal{I}_{0+}(\overline{x}) \cup \mathcal{I}_{00}(\overline{x}) \cup \mathcal{I}_{+0}(\overline{x}) \cup \mathcal{I}_{0-}(\overline{x}), \tag{2.4a}$$

$$\mathcal{A}_{\mathrm{HG}}^{\mathsf{C}}(\overline{x}) := \bar{l} \setminus \mathcal{A}_{\mathrm{HG}}(\overline{x}) = \mathcal{I}_{++}(\overline{x}), \tag{2.4b}$$

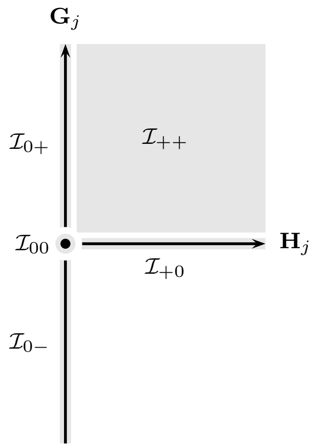
$$\mathcal{A}_{\mathrm{H}}(\overline{x}) = \mathcal{I}_{0+}(\overline{x}) \cup \mathcal{I}_{00}(\overline{x}) \cup \mathcal{I}_{0-}(\overline{x}), \tag{2.4c}$$

$$\mathcal{A}_{\mathrm{H}}^{\mathsf{C}}(\overline{x}) := \bar{l} \setminus \mathcal{A}_{\mathrm{H}}(\overline{x}) = \mathcal{I}_{++}(\overline{x}) \cup \mathcal{I}_{+0}(\overline{x}). \tag{2.4d}$$

From these relations, it already becomes clear that feasible points $\overline{x}$ with different index sets $\mathcal{I}_{0+}(\overline{x})$, $\mathcal{I}_{00}(\overline{x})$, and $\mathcal{I}_{0-}(\overline{x})$ cannot be told apart using the standard perception of an active set. Figure 1 depicts active sets and corresponding index sets.




(a) Active sets (2.2).




(b) Index sets (2.3a).

Figure 1: Active set and index sets in a neighborhood of a feasible point $\overline{x} \in \mathbb{R}^n$ of problem (2.1).

**Lower Level Strict Complementarity Condition** Clearly, if $\mathcal{I}_{00}(\overline{x}) = \emptyset$, then in a neighborhood of $\overline{x}$ problem (2.1) is a standard NLP including only those constraints $0 \leq G_j(x)$ for which $j \in \mathcal{A}_{\mathrm{H}}^{\mathsf{C}} = \mathcal{I}_{++}(\overline{x}) \cup \mathcal{I}_{+0}(\overline{x})$. This condition, referred to as LLSCC (*lower level strict complementarity condition*) in the literature, obviously is too strong to be imposed on the entire feasible set of (2.1), as this would defy the idea of capturing its combinatorial nature.

**Violation of Commonly Assumed Constraint Qualifications** If $\mathcal{I}_{00}(\overline{x}) \neq \emptyset$ and LLSCC does not hold, then in a neighborhood of $\overline{x}$ the feasible set has combinatorial structure. Both LICQ (*linear independence constraint qualification*, [26]) and MFCQ (*Mangasarian–Fromovitz constraint qualification*, [27]) are violated, as is easily verified e.g. in [16]. This poses a number of significant difficulties to KKT based descent methods, which we describe in Section 2.2.

## 2.2 Consequences for Algorithms of SQP Type

**Non-unique or Unbounded Dual Variables** As a consequence of LICQ violation, the dual solution $y^*$ is not unique. The set of dual solutions is not even bounded, as MFCQ is violated too. Update schemes for approximations of the Hessian of the NLP Lagrangian, such as BFGS updates [28], however crucially rely on secant information involving the dual solution. We may therefore expect such Hessian approximations to become ill-conditioned.

**Ill-conditioned Constraint Jacobians** Linearizations of the vanishing constraint (2.1b) in the neighborhood of points $x \in \mathbb{R}^n$ with some $H_j(x) = 0$, i.e. violating LICQ, become severely ill-conditioned. This poses a challenge to active set methods that may fail to reliably detect active sets.

**Cycling and Stalling of Active Set Methods** When applying standard active set based QP and NLP codes to problems with vanishing constraints, a consequence of ill-conditioning that can often be observed is cycling of the active set, i.e. repeated addition and removal of the same sequence of constraints without progress in the primal iterate. Hence, if the method successfully solves the QP subproblem at all, QP iteration counts and computation time for a single SQP step increase significantly.

**Suboptimal and Infeasible Steps** Linearizations of the vanishing constraint (2.1b) fail to properly represent the geometry of the feasible set in the neighborhood of points $\overline{x} \in \mathbb{R}^n$ with $H_j(\overline{x}) = 0$, $G_j(\overline{x}) = 0$. SQP methods hence perform steps that are significantly suboptimal or infeasible on the NLP level. Hence, unnecessarily many more SQP iterations may be required than would be required if the subproblem's combinatorial nature had been captured properly.

## 2.3 Modified Stationarity Concept

In view of the practical difficulties listed in Section 2.2, a modified concept of optimality under a possibly weaker constraint qualification is desirable. In order to retain the concept of iterating towards KKT based optimality, this CQ should ensure that stationary points of (2.1) are indeed KKT points.

**A Regularity Assumption** To this end we introduce the regularity assumption of MPVC–LICQ, see e.g. [2].

**Definition 2.1.** We say that MPVC–LICQ holds for a feasible point $\overline{x} \in \mathbb{R}^n$ if the gradients

$$\begin{aligned} &\nabla H_j(\overline{x}), \quad j \in \mathcal{I}_{0+} \cup \mathcal{I}_{00} \cup \mathcal{I}_{0-}, \\ &\nabla G_j(\overline{x}), \quad j \in \mathcal{I}_{+0} \cup \mathcal{I}_{00} \end{aligned} \tag{2.5}$$

are linearly independent.

While the idealizing assumption of MPVC–LICQ is sometimes held for too strict for a theoretical analysis of the full class MPVCs, see e.g. [2, 20], one frequently observes that problem instances arising from practical applications indeed comply. This in particular is the case for the vanishing constraints (5.1g, 5.1h) of the robot motion planning problem we shall investigate in Section 5. For the remainder of this paper, we will assume MPVC–LICQ to hold and refer the reader to e.g. [16] for details on weaker concepts of constraint qualification for MPVC, resulting stationarity concepts, and applicable numerical methods.

**Strong Stationarity Conditions** Under MPVC–LICQ, a KKT–like necessary condition for local optimality of a candidate point $\overline{x} \in \mathbb{R}^n$ of problem (2.1) can be given. It is based on the so-called MPVC-Lagrangian $\Lambda(x, \mu^{\mathrm{G}}, \mu^{\mathrm{H}})$ of problem (2.1),

$$\Lambda(x, \mu^{\mathrm{G}}, \mu^{\mathrm{H}}) := F(x) - (\mu^{\mathrm{G}})^T G(x) - (\mu^{\mathrm{H}})^T H(x). \tag{2.6}$$

The vectors $\mu^{\mathrm{G}}, \mu^{\mathrm{H}} \in \mathbb{R}^l$ are referred to as *MPVC multipliers*. The notion of strong stationarity for MPVC has been defined in [17] as follows:

**Definition 2.2.** A feasible point $\overline{x} \in \mathbb{R}^n$ of problem (2.1) is called *MPVC strongly stationary* if there exist MPVC multipliers $\mu^{\mathrm{G}}$, $\mu^{\mathrm{H}} \in \mathbb{R}^l$ such that it holds that

$$\Lambda_x(\overline{x}, \mu^{\mathrm{G}}, \mu^{\mathrm{H}}) = 0, \tag{2.7a}$$
$$\mu_j^{\mathrm{G}} \geq 0 \quad j \in \mathcal{I}_{+0}(\overline{x}), \tag{2.7b}$$
$$\mu_j^{\mathrm{G}} = 0 \quad j \in \mathcal{I}_{0-}(\overline{x}) \cup \mathcal{I}_{00}(\overline{x}) \cup \mathcal{I}_{0+}(\overline{x}) \cup \mathcal{I}_{++}(\overline{x}), \tag{2.7c}$$
$$\mu_j^{\mathrm{H}} \geq 0 \quad j \in \mathcal{I}_{00}(\overline{x}) \cup \mathcal{I}_{0+}(\overline{x}), \tag{2.7d}$$
$$\mu_j^{\mathrm{H}} = 0 \quad j \in \mathcal{I}_{+0}(\overline{x}) \cup \mathcal{I}_{++}(\overline{x}). \tag{2.7e}$$

In [2] it has been shown that under MPVC–LICQ strong stationarity (2.7) for MPVC is equivalent to KKT stationarity for problem (2.1). The following stronger result is due to [19] and can also be found in [20].

**Theorem 2.3.** *Let $\overline{x} \in \mathbb{R}^n$ feasible for (2.1) satisfy MPVC–LICQ. If $\overline{x}$ is a locally optimal point of (2.1), then $\overline{x}$ is an MPVC strongly stationary point. The associated MPVC multipliers $(\overline{\mu}^G, \overline{\mu}^H)$ are unique.*

## 2.4 Nonconvex Sequential Quadratic Programming

In [32] a general framework for applying SQP methods to structurally nonconvex problems has been described. Of special interest for us is the result concerning local convergence for nonconvex problems. We introduce the generic formulation

$$\min_{x \in \mathbb{R}^n} F(x) \text{ s.t. } C(x) \in \mathcal{Z} \tag{2.8}$$

wherein $F$ is the objective function of (2.1), $C : \mathbb{R}^n \to \mathbb{R}^m$ is the continuously differentiable constraint function, and the closed set $\mathcal{Z} \subset \mathbb{R}^m$ captures the nonconvex structure of the problem's feasible set.

For the MPVC (2.1), we have in particular $C(x) := (H(x), G(x)) \in \mathbb{R}^{2l}$, i.e. $m = 2l$, and the nonconvex structure of the problem is captured by defining $\mathcal{Z}$ as

$$\mathcal{Z} := \left\{ (z', z'') \in \mathbb{R}^{2l} \;\middle|\; \forall j \in \bar{l} : \; (z'_j \geq 0 \;\wedge\; z''_j \geq 0) \;\vee\; (z'_j = 0) \right\} \subset \mathbb{R}^{2l}. \tag{2.9}$$

**Definition 2.4.** A constraint function $C_j$ of problem (2.8) is called *inactive* in a feasible point $\overline{x}$ if there exists $\varepsilon > 0$ such that $C(\overline{x}) + \delta e_j \in \mathcal{Z}$ holds for all $|\delta| < \varepsilon$, where $e_j$ denotes the $j$-th unit vector. In other words, the validity of the statement "$C(x) \in \mathcal{Z}$" is locally independent of the value of $C_j(x)$ in a neighborhood of $\overline{x}$. The constraint function is called *active* otherwise.

**Definition 2.5.** A feasible point $\overline{x} \in \mathbb{R}^n$ of problem (2.8) is called *regular* if the gradients of all active constraint functions $C_j$ are linearly independent.

**Definition 2.6.** A feasible point $x^* \in \mathbb{R}^n$ of problem (2.8) is called *critical* if $d = 0$ is a local minimizer of the auxiliary problem

$$\min_d \nabla F(x^*)^T d \ s.t.\ C(x^*) + \nabla C(x^*)^T d \in \mathcal{Z}. \tag{2.10}$$

This subproblem gives rise to the existence of multipliers $\mu^* \in \mathbb{R}^m$ associated with the constraints.

**Definition 2.7.** A regular critical point $(x^*, \mu^*) \in \mathbb{R}^{n+m}$ of problem (2.8) is called *stationary* if $v = 0$ is a local minimizer of the auxiliary problem

$$\max_v {\mu^*}^T v \ s.t.\ C(x^*) + v \in \mathcal{Z}. \tag{2.11}$$

**Definition 2.8.** A stationary point $(x^*, \mu^*) \in \mathbb{R}^{n+m}$ of problem (2.8) is said to satisfy *strict complementarity* if there exists $\varepsilon > 0$ such that $v = 0$ remains the solution of (2.11) for all $\mu \in U_\varepsilon(\mu^*)$. In other words, small local changes to the multiplier $\mu_i^*$ do not affect stationarity of $x^*$.

Based on this notion of stationarity, the following convergence result for the exact Hessian SQP method holds. The sequential QPVC algorithm proposed in this paper is a special case of this method.

**Theorem 2.9.** *Let $(x^*, \mu^*)$ be a stationary point of (2.8), and let the set $\mathcal{Z}$ be locally star-shaped in $z^* = C(x^*)$. Let strict complementarity hold in $(x^*, \mu^*)$ and let the exact Hessian be positive definite on the nullspace of the active constraints. Then exact Hessian SQP converges locally quadratically to the stationary point $x^*$.*

*Proof.* See [32]. □

This result can now be used for the special case of the MPVC (2.1).

**Theorem 2.10.** *Let $(x^*, \mu^*)$ be an MPVC strongly stationary point of (2.1) and satisfy MPVC–LICQ. Let strict complementarity hold in $(x^*, \mu^*)$ and let the exact Hessian be positive definite on the nullspace of the active constraints. Then exact Hessian SQP converges locally quadratically to the MPVC strongly stationary point $x^*$.*

*Proof.* Under MPVC-LICQ, any MPVC strongly stationary point is stationary in the sense of [32] and Definition 2.7. We define $C(x) := (H(x), G(x))$ and the set $\mathcal{Z}$ as mentioned above. Being a finite union of non-disjoint convex sets, it is locally star shaped, c.f. [32]. Observe now that for any point $x \in \mathbb{R}^n$ in a neighborhood of a feasible point $\overline{x}$ of problem (2.8), the functions $G_j(x)$ are inactive iff $j \in \mathcal{I}_{0-}(\overline{x}) \cup \mathcal{I}_{0+}(\overline{x}) \cup \mathcal{I}_{++}(\overline{x})$, whereas the functions $H_j(x)$ are inactive iff $j \in \mathcal{I}_{+0} \cup \mathcal{I}_{++}$. Hence by Definition 2.1 regular points are exactly the points satisfying MPVC–LICQ and Theorem 2.9 is applicable. □

Theorem 2.9 assumes strict complementarity to hold in the KKT point in order to establish a locally quadratical rate of convergence. Strict complementarity is not assumed to hold in the remainder of this paper. We may in general expect SQP to still converge, although with a suboptimal rate of convergence. Moreover, the locally quadratical rate of convergence will also be lost if the exact Hessian is replaced by an approximation such as e.g. BFGS.

### 2.5 Convex Quadratic Programs with Vanishing Constraints

In the proposed SQP framework for vanishing constraint problems, the subproblems that arise from a locally quadratic model of the MPVC-Lagrangian are convex quadratic programs extended by affine linear vanishing constraints,

$$0 \le (h_j^T x - \alpha_j) \cdot (g_j^T x - \beta_j), \quad j \in \bar{l}, \tag{2.12a}$$

$$\alpha_j \le h_j^T x, \qquad j \in \bar{l}. \tag{2.12b}$$

We denote by $H = (h_1, \ldots, h_n)^T$, $G = (g_1, \ldots, g_n)^T \in \mathbb{R}^{l \times n}$ the vanishing constraint Jacobians and their row vectors, and by $\alpha$, $\beta \in \mathbb{R}^l$ the vanishing constraint vectors of lower bounds. Again, the feasible set of (2.12) is structurally combinatorial, hence nonconvex. In the interest of a simplified notation we restrict ourselves to the more specific vanishing constraint formulation

$$0 \le x_j \cdot (g_j^T x - \beta_j), \quad 0 \le x_j, \qquad j \in \bar{l} \tag{2.13}$$

in place of (2.12). This problem structure can always be obtained by introduction of $l$ additional variables $\tilde{x}_j := h_j^T x - \alpha_j$ and suitable arrangement of the constraint rows in $G$. In addition, we restrict ourselves to vanishing constraints having lower constraint bounds only. We are hence interested in the QPVC

$$\min_{x \in \mathbb{R}^n} \quad \tfrac{1}{2} x^T B x + x^T b \tag{2.14a}$$

$$\text{s.t.} \quad 0 \le \quad x_j \cdot (g_j^T x - \beta_j), \quad 0 \le x_j, \qquad j \in \bar{l}. \tag{2.14b}$$

Different from standard convex QP notation, in problem (2.14) $B \in \mathbb{R}^{n \times n}$ denotes the Hessian of the MPVC Lagrangian, or a suitable positive definite approximation thereof.

## 3 Partitioning and Continuation for the QPVC Subproblems

In this section we show how the nonconvex feasible set of a QPVC can be partitioned into multiple, mutually overlapping convex subsets by introduction of an additional constraint. We compare KKT conditions for QP subproblems QP($\mathcal{S}$) on convex subsets with MPVC strong stationarity conditions for the QPVC, in order to obtain MPVC multiplier information that allows for an efficient iteration over the convex subproblems defined by particular choices of $\mathcal{S}$. To this end we describe a tree–search type algorithm and an active set type algorithm.

### 3.1 Convex Quadratic Programs on Subsets

Fixing sets $\mathcal{S} := \{j \mid x_j = 0\}$ of constraints $g_j^T x \ge \beta_j$ that have vanished, and $\mathcal{S}^{\mathsf{C}}$ of constraints that may not vanish, restricts the feasible set of (2.14) to a convex subset. In the neighborhood of a feasible point $\overline{x} \in \mathbb{R}^n$ of the QPVC (2.14) we then consider the following convex problem QP($\mathcal{S}$) with a smaller but convex feasible set,

$$\min_{x \in \mathbb{R}^n} \quad \tfrac{1}{2} x^T B x + x^T b \tag{3.1a}$$

$$\text{s.t.} \quad \beta_j \le g_j^T x, \quad j \notin \mathcal{S}, \tag{3.1b}$$

$$0 \le x_j, \qquad j \notin \mathcal{S}, \tag{3.1c}$$

$$0 = x_j, \qquad j \in \mathcal{S}. \tag{3.1d}$$

We assume problem (3.1) to have a positive definite Hessian $B \in \mathbb{R}^{n\times n}$ of the MPVC Lagrangian, and denote the objective's linear part by $b \in \mathbb{R}^n$, the vanishing constraints Jacobian $G \in \mathbb{R}^{l\times n}$, and the constraint bounds vectors by $\beta \in \mathbb{R}^l$.

Based on usual KKT optimality ([28]) for every solution $x^* \in \mathbb{R}^n$ of problem (3.1) there exists a (unique) vector of MPVC multipliers $\mu^{\mathrm{G}^*}, \mu^{\mathrm{H}^*} \in \mathbb{R}^l$ such that the following system of optimality conditions for subproblem (3.1) is satisfied,

$$
\begin{aligned}
0 &= Bx^* + b - \sum\nolimits_{j\notin\mathcal{S}} (G^T)_j \mu_j^{\mathrm{G}^*} - E\mu^{\mathrm{H}^*}, && && (3.2a)\\
0 &\le g_j^T x^* - \beta_j, && j \notin \mathcal{S}, && (3.2b)\\
0 &\le x_j^*, \quad 0 = x_k^*, && j \notin \mathcal{S},\ k \in \mathcal{S}, && (3.2c)\\
(g_j^T x^* - \beta_j)\mu_j^{\mathrm{G}^*} &= 0, \quad \mu_j^{\mathrm{G}^*} \ge 0 && j \notin \mathcal{S}, && (3.2d)\\
x_j^* \mu_j^{\mathrm{H}^*} &= 0, \quad \mu_j^{\mathrm{H}^*} \ge 0 && j \notin \mathcal{S}. && (3.2e)
\end{aligned}
$$

In (3.2a) $(G^T)_j$ denotes the $j$-th column of the Jacobian $G$, and $E$ denotes the $l \times l$ identity matrix with $n-l$ zero rows appended. Moreover, let $\mu_j^{\mathrm{G}} := 0$ for $j \in \mathcal{S}$, i.e. those vanishing constraints that have vanished in problem (3.1). By positive definiteness of $B$ the solution $x^*$ is unique, and it is a global solution of (3.1). We obtain the following result.

**Theorem 3.1.** *Let $\mathcal{S} \subseteq \bar{l}$ be given and let $(x^*, \mu^{G^*}, \mu^{H^*})$ be a KKT point of the subproblem QP($\mathcal{S}$) associated with the choice $\mathcal{S}$. Then the KKT point is MPVC strongly stationary for the QPVC (2.14) if and only if $\mu_j^{G^*} = 0$ for all $j \in \mathcal{I}_{00}(x^*)$.*

*Proof.* We observe that the set of KKT conditions (3.2) is almost identical to the set of MPVC strong stationarity conditions (2.7) plus constraint (3.1d): MPVC strong stationarity is defined for primary feasible points which is also given by (3.2b, 3.2c, 3.1d) and the gradient of the Lagrangian vanishes in (2.7a) due to (3.2a). Furthermore, the inequalities in (3.2d) and (3.2e) imply (2.7b) and (2.7d), respectively. Finally, the equality conditions in (2.7c) and (2.7e) are implied by the equalities in (3.2d) and (3.2e) for all indices $j$ with $g_j^T x^* - \beta_j \neq 0$ and $x_j \neq 0$. This leaves condition (2.7c) for $j \in \mathcal{I}_{00}(x^*)$.

The requirement $\mu_j^{\mathrm{G}^*} = 0$ for $j \in \mathcal{I}_{00}(x^*)$ (2.7c) is relaxed to $\mu_j^{\mathrm{G}^*} \ge 0$ in (3.2d), giving rise to the if and only if condition in the claim. □

It should be stressed that the choice of the set $\mathcal{S}$ is an algorithmic one, and solving the QPVC effectively means identifying the optimal choice of $\mathcal{S} \in \mathbb{P}(\bar{l})$ among the power set of all $2^l$ possible choices. In contrast, the index sets (2.3) denote active and inactive constraints in solutions of the QPVC and of QP($\mathcal{S}$). Exploiting the multiplier information found in $\mu_j^{\mathrm{G}^*}$, $j \in \mathcal{I}_{00}(x^*)$, and additionally in $\mu_j^{\mathrm{H}^*}$, $j \in \mathcal{S}$ for the imposed subset constraint (3.1d), turns out to be crucial for the development of an efficient continuation method that iterates over $\mathbb{P}(\bar{l})$.

### 3.2 Continuation in Adjacent Subsets

For any choice of $\mathcal{S}$ the solution of the subproblem QP($\mathcal{S}$) (3.1) associated with $\mathcal{S}$ must fall into one of three categories.

**KKT Point with $\mu_j^{\mathrm{G}^*} > 0$ for some $j \in \mathcal{I}_{00}(x^*)$** For a KKT Point with $\mu_j^{\mathrm{G}^*} > 0$ for some $j \in \mathcal{I}_{00}(x^*)$ we know that this point violates MPVC strong stationarity which requires $\mu_j^{\mathrm{G}^*} = 0$. Consequently, the convex subset of problem (2.14) selected by the current choice of $\mathcal{S} \subseteq \bar{l}$ does not contain an MPVC strongly stationary point, as otherwise this point would have been found as the unique KKT point. We may continue the solution of the QPVC in any convex subset selected by an index set $\mathcal{S} \cup \{j\}$.

**Any Other KKT Point** A KKT point that does not fall into the above category is said to lie "in the interior" ([32]) of problem (3.1). The chosen subset $\mathcal{S}$ of vanishing constraints that must have vanished is locally optimal. The point $(x^*, \mu^{\mathrm{G}^*}, \mu^{\mathrm{H}^*})$ is an MPVC strongly stationary point of the original problem (2.14), albeit not necessarily a globally optimal one as will be addressed in Section 3.3.

**Infeasible Subproblem** Subproblem QP($\mathcal{S}$) (3.1) may have an empty feasible set for certain choices of $\mathcal{S}$. Initially, it is not clear whether starting with either of the obvious choices $\mathcal{S} = \emptyset$ (all vanishing constraints must be satisfiable at once) or $\mathcal{S} = \bar{l}$ (all variables $x_j$ are fixed to zero) will lead to a feasible subproblem. In a sequential QPVC algorithm, an obvious initial candidate for $\mathcal{S}$ is the set $\mathcal{I}_{0-}(\overline{x})$ defined by the feasible point $\overline{x}$ in which the QPVC is set up. We return to this issue in Section 4.4.

## 3.3 Improvement towards Global Optimality

MPVC strongly stationary points of (2.14) are not necessarily globally optimal. Stationarity of the solution of all SQP subproblems is sufficient to reach stationarity of the NLP solution, see e.g. [32]. Still, in QPVCs derived from applications, local solutions might be missing some more or less obvious features modeled by vanishing constraints. By reduction to MIQP, finding a globally optimal solution of a QPVC is an NP-hard problem. Hence, what we propose in this section is a heuristics based on a sufficient condition, that allows the continuation of the QPVC solution process in order to improve the solutions beyond the first strongly stationary point. A sufficient condition for global optimality is given in ([16], Corollary 6.2.5) which applies in particular to the QPVC subproblems (2.14) with $B$ positive definite. We state this condition in the following, more restrictive form for problem (2.1):

**Theorem 3.2.** *Let the objective function $F$ be convex, and the constraint functions $G$, $H$ be concave. Further, let $x^* \in \mathbb{R}^n$ be an MPVC strongly stationary point of problem (2.1). If $\mu_j^{G^*} = 0$ for all $j \in \mathcal{I}_{+0}(x^*)$ and $\mu_j^{H^*} \geq 0$ for all $j \in \mathcal{I}_{0-}(x^*)$ then $x^*$ is a globally optimal solution of problem (2.1).*

From this theorem we derive two further continuation rules in addition to those found in Section 3.2 and comment on the benefits of memorizing stationary points.

**KKT Point with $\mu_j^{\mathbf{G}^*} > 0$ for some $j \in \mathcal{I}_{+0}(x^*)$** For a KKT point $(x^*, \mu^{\mathrm{G}^*}, \mu^{\mathrm{H}^*})$ with $\mu_j^{\mathrm{G}^*} > 0$ for some $j \in \mathcal{I}_{+0}(x^*)$, i.e. a vanishing constraint active at its lower bound, but with inactive associated variable $x_j > 0$, we may continue the solution in the adjacent convex subset of problem (2.14) with $\mathcal{S} \cup \{j\}$, now including the vanishing constraint $j$ indicating possible improvement towards global optimality. This effectively means extending the first rule of Section 3.2 from the set $\mathcal{I}_{00}(x^*)$ also to the set $\mathcal{I}_{+0}(x^*)$.

**KKT Point with $\mu_j^{\mathbf{H}^*} < 0$ for some $j \in \mathcal{S}$** For a KKT Point with $\mu_j^{\mathrm{H}^*} < 0$ for some $j \in \mathcal{S}$ we know that the additionally introduced equality constraint (3.1d) would be inactive if it was an inequality constraint as in (2.13). Consequently, improvement of the objective may be possible if $x_j > 0$. We may continue the solution of the QPVC in any convex subset selected by an index set $\mathcal{S} \setminus \{j\}$.

**Memorizing Stationary Points** In an actual implementation of this strategy of improvement towards global optimality, we need to be aware of the nature of Theorem 3.2. As it is a sufficient condition only, there may well exist stationary points being globally optimal solutions but violating the conditions of Theorem 3.2. Hence a memory of MPVC strongly stationary points found so far needs to be maintained, including the associated objective function value of problem (2.14) in these points. In addition, the proposed heuristics does not guarantee that all possible convex subsets are actually visited, and hence does not provide a certificate of global optimality for the solution found. Upon exhaustion of the proposed continuation procedure, the best point that has been found is returned.

## 3.4 Tree Search Algorithms for Selection of Convex Subsets

The described process of subdivision into convex subsets lends itself to treatment in a branching type algorithm on the power set $\mathbb{P}(\bar{l})$ of vanishing constraint indices. Starting with an initial choice $\mathcal{S} \in \mathbb{P}(\bar{l})$ we solve the corresponding subproblem QP($\mathcal{S}$) for $(x^*, \mu^{\mathrm{G}^*}, \mu^{\mathrm{H}^*})$. Analysis of the multiplier information on the index sets $\mathcal{I}_{00}(x^*)$, $\mathcal{I}_{0+}(x^*)$, and $\mathcal{S}$ as proposed in Sections 3.2, 3.3 yields a list of candidate subproblems to continue with. These can be evaluated in a recursive depth-first search, or alternatively in a list-based breadth-first search.

Several challenges remain with this approach, though. As mentioned, the initial choice of $\mathcal{S}$ is not obvious. Moreover, a choice associated with an infeasible subproblem does not yield sufficient multiplier information that would allow for continuation in a feasible one. Second, the convex subsets do not form a proper partition of the feasible set of (2.14) but are mutually overlapping. Hence solving a convex QP afresh on each convex subset comes with a significant computational effort as identical subsequences of active set exchanges have to be repeatedly carried out for each QP.

## 3.5 Active Set Algorithm for Selection of Convex Subsets

To address these issues, we propose an active set type framework for the selection of convex subsets that blends with the QP active set method used for solving the subset QPs. In the following, we describe in more detail those active set exchange moves between index sets that are different from a standard active set method. Figure 2 depicts the discussed active set exchange moves. Index sets are always understood to refer to the current iterate $x$.

**A vanishing constraint enters $\mathcal{I}_{00}$ from $\mathcal{I}_{+0}$** If for an active vanishing constraint $\beta_j \leq g_j^T x$ the controlling variable $x_j$ becomes zero, the associated MPVC multiplier $\mu_j^{\mathrm{G}}$ may remain positive and is then in violation of MPVC strong stationarity conditions (2.7). We immediately let the constraint's index $j$ enter the set $\mathcal{S}$ of constraints that have vanished. Thereby, a move to a neighboring convex subset problem (3.1) of the QPVC (2.14) is accomplished. In Figure 2(a), two arcs have to be traversed.

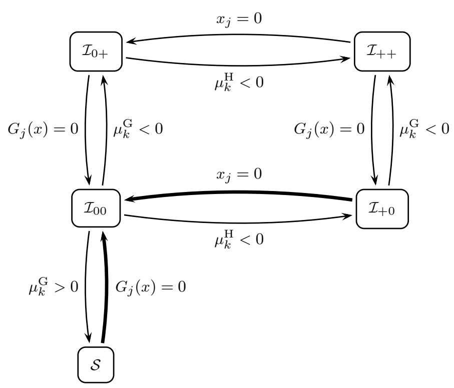
**A vanishing constraint leaves $\mathcal{S}$ for $\mathcal{I}_{00}$** If an infeasible and hence vanished constraint $\beta_j \leq g_j^T x$ becomes feasible, we remove the constraint's index $j$ from the set $\mathcal{S}$. The index will enter the set $\mathcal{I}_{00}$ if $\mu_j^{\mathrm{H}} > 0$, or the set $\mathcal{I}_{+0}$ if $\mu_j^{\mathrm{H}} \leq 0$. In the latter case, we must set $\mu_j^{\mathrm{H}} := 0$ and restore stationarity as detailed in Section 4.4. Again, a move to a neighboring convex subset problem (3.1) of the QPVC (2.14) is accomplished. In Figure 2(a), one or two arcs are traversed.

If we choose to include the global optimality criterion in the active set strategy, further active set exchange moves need to be considered. As the criterion is of sufficient nature only, these moves are not binding.

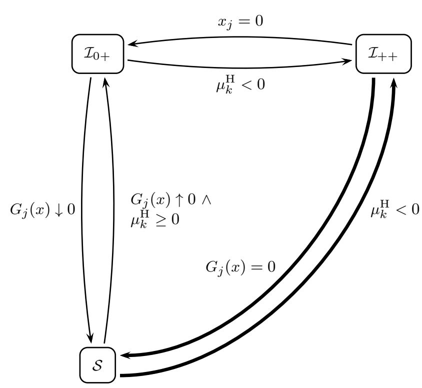
**A vanishing constraint enters $\mathcal{I}_{+0}$ from $\mathcal{I}_{++}$** If a vanishing constraint $\beta_j \le g_j^T x$ becomes active with $x_j > 0$, the associated MPVC multiplier $\mu_j^{\mathrm{G}}$ may become positive. This is in accordance with MPVC strong stationarity, but in violation of the sufficient condition for global optimality (Theorem 3.2). We may choose to let the constraint's index $j$ enter the set $\mathcal{S}$ of constraints that have vanished. If we do so, we must set $x_j := 0$ and make this simple lower bound active. Primal and/or dual feasibility are restored as detailed in Section 4.4. This move is shown in Figure 2(b) together with its counterpart move. The previously addressed move from $\mathcal{I}_{+0}$ to $\mathcal{I}_{00}$ can then not occur anymore.

**Linear dependence caused by vanishing constraints** Adding the simple lower bound $x_j = 0$ to the active set, required when moving a constraint index $j$ to $\mathcal{S}$ from either $\mathcal{I}_{+0}$ or $\mathcal{I}_{++}$ as just described, may cause linear dependence of the active constraint Jacobian rows. In Section 4 we give references to a fast and efficient resolution procedure that indicates a constraint $k \ne j$ to be removed from the active set in order to restore linear independence.

It may happen that for this constraint $k \in \mathcal{S}$ holds, i.e. the simple bound $x_k = 0$ is to be removed even though the associated vanishing constraint $g_k \le G_k x$ would be violated. In this case, linear dependence cannot be resolved inside the convex subset selected by the current choice of $\mathcal{S}$. We remove $k$ from the set $\mathcal{S}$, thus moving to an adjacent convex subset, and restore feasibility of the vanishing constraint by modifying a homotopy between quadratic problems in a suitable manner. Details are given in Section 4.4.




(a) Schematic of the active set algorithm. Emphasized moves may violate MPVC strong stationarity (but not KKT conditions for the subset QP) and trigger a second move as detailed in Section 3.5.




(b) Schematic of the proposed active set algorithm including the global optimality criterion. Emphasized moves require appropriate modification of the QP to restore feasibility and/or stationarity as detailed in Section 4.4.

Figure 2: Schematics of the proposed active set algorithms for the selection of convex subsets.

# 4 A Parametric Primal-Dual Active Set Strategy for Hot Starting

In this section we make a recourse to the more familiar problem class of convex QPs. We describe a parametric primal–dual active set method for the numerical solution of such programs. The method is due to [7], and has been proposed for application in an online optimization in [13] and related works. In [6], it has been used for sensitivity analysis of convex QP solutions. Here we propose to use the described method to realize a hot starting facility for the subsequent solution of multiple QPs in convex feasible subsets of a QPVC as described in Section 3. Hot starting procedures are presented for each of the active set exchange moves described in Section 3.5 that move to neighboring convex subsets of the QPVC.

## 4.1 Parametric Convex Quadratic Programs

A convex QP becomes a *convex parametric quadratic program* if the gradient vector and all constraint bound vectors are affine linear vector valued functions, depending on a scalar homotopy parameter $\tau \in [0,1] \subset \mathbb{R}$,

$$\min_{x\in\mathbb{R}^n} \quad \tfrac{1}{2}x^T Bx + x^T b(\tau) \tag{4.1a}$$

$$\text{s.t.} \quad c(\tau) \leq Cx, \tag{4.1b}$$

with $b \in \mathcal{H}^n$, $c \in \mathcal{H}^m$, and $\mathcal{H}^k$ denoting a set of affine linear functions

$$\mathcal{H}^k := \left\{ f : [0,1] \to \mathbb{R}^k \mid \forall \tau \in (0,1): \ f(\tau) = (1-\tau)f(0) + \tau f(1) \right\}, \ k \geq 1. \tag{4.2}$$

The restriction to homotopies of the gradient and constraint vectors is not a real one. Changes of the Hessian $B$ or the constraint matrices $C, D$ can also be rewritten as vector-valued ones by virtue of a simple transformation of the system of KKT conditions [13].

In problem (4.1) we seek a solution of the QP in $\tau = 1$, assuming a–priori knowledge of a solution in $\tau = 0$. Most often, these two QPs will be closely related in a certain way. Progressing along the homotopy path then constitutes a highly efficient way of accomplishing *hot starts*. This situation arises naturally in a number of application cases, e.g. Sequential Quadratic Programming (SQP) methods, model predictive control algorithms [13], or in algorithms of the branching type with QP subproblems on the branch tree's nodes, cf. Section 3.

The QPVC subproblem (3.1) assumes the shape of (4.1) if for notational convenience we collect vanishing constraints $j \notin \mathcal{I}_{0-}$ and simple lower bounds and equality constraints in the common matrix $C$.

**No Phase One Necessary** Note that an optimal solution in $\tau = 0$ for the "trivial QP" with $b(0) = 0$, $c(0) = 0$ is always available with $(x^*(0), \mu^*(0)) = (0, 0)$, such that the homotopy also makes a phase one strategy unnecessary. Such a strategy might otherwise be required to find an initial feasible guess if none is available.

## 4.2 The Parametric Active Set Method

For a fixed value $\tau \in [0,1] \subset \mathbb{R}$ and a given active set $\mathcal{A} \subseteq \overline{m}$ the system of optimality conditions (3.2a–3.2c) for problem (4.1) reads in matrix form

$$\begin{pmatrix} B & C_{\mathcal{A}}{}^T \\ C_{\mathcal{A}} & 0 \end{pmatrix} \begin{pmatrix} x^*(\tau) \\ -\mu^*_{\mathcal{A}}(\tau) \end{pmatrix} = \begin{pmatrix} -b(\tau) \\ c_{\mathcal{A}}(\tau) \end{pmatrix}, \tag{4.3}$$

with $(x^*(\tau), \mu^*_{\mathcal{A}}(\tau))$ denoting the primal–dual optimal solution in $\tau$. Based on the fact that affine linearity of the right hand side of (4.3) in $\tau$ necessarily leads to piecewise affine linearity of the solution set $(x^*(\tau), \mu^*_{\mathcal{A}}(\tau))$, $\tau \in [0, 1]$, the underlying idea of the primal–dual parametric active set strategy now is to proceed as follows.

Iteration $k = 0$ starts in $\tau^{(0)} = 0$ with the known optimal solution $(x^*(0), \mu^*_{\mathcal{W}}(0))$ and a maximal linear independent subset $\mathcal{W} \subseteq \mathcal{A}(x^*(0))$ of the active set, referred to as the *working set.*

In each iteration $k$, the step direction $(\Delta x^{(k)}, \Delta\mu^{(k)})$ is determined by solving the system of optimality conditions

$$\begin{pmatrix} B & C_{\mathcal{W}}^T \\ C_{\mathcal{W}} & 0 \end{pmatrix} \begin{pmatrix} \Delta x^{(k)} \\ -\Delta\mu_{\mathcal{W}}^{(k)} \end{pmatrix} = \begin{pmatrix} -\Delta b(\tau^{(k)}) \\ \Delta c_{\mathcal{W}}(\tau^{(k)}) \end{pmatrix}. \tag{4.4}$$

Herein, the vectors $\Delta b(\tau^{(k)})$ and $\Delta c_{\mathcal{W}}(\tau^{(k)})$ denote the gradient and constraint vector steps from $\tau^{(k)}$ to end $\tau = 1$ of the homotopy. Let further $\Delta\mu_j^{(k)} = 0$ for $j \in \overline{m} \setminus \mathcal{W}$.

The step length $\alpha^{(k)} \in [0, 1]$ is determined as the maximum advance in the homotopy parameter $\tau$ that satisfies both (3.2b, 3.2c) and positivity of the duals $\mu^*(\tau)$, i.e. that keeps the working set $\mathcal{W}$ both primal and dual feasible, given the computed primal–dual step direction.

In the obtained solution for $\tau^{(k+1)} := \tau^{(k)} + \alpha^{(k)}$,

$$(x^*(\tau^{k+1}), \mu^*(\tau^{(k+1)})) = (x^*(\tau^k), \mu^*(\tau^{(k)})) + \alpha^{(k)}(\Delta x^{(k)}, \Delta\mu^{(k)}) \tag{4.5}$$

the primal or dual blocking constraint is added to or removed from the working set $\mathcal{W}$. The homotopy advances by letting

$$b(\tau^{(k+1)}) := b(\tau^{(k)}) + \alpha^{(k)}\Delta b(\tau^k), \tag{4.6a}$$

$$c(\tau^{(k+1)}) := c(\tau^{(k)}) + \alpha^{(k)}\Delta c(\tau^{(k)}), \tag{4.6b}$$

and the procedure continues with iteration $k + 1$. Once the homotopy end point $\tau^{(k)} = 1$ has been reached, the procedure terminates and a piecewise affine linear solution trajectory for problem (4.1) has been determined. Finite termination of this procedure in a KKT point of the QP can be shown under the usual nondegeneracy assumptions, e.g. [7].

### 4.3 Algorithmic Details

Several details of the described algorithm merit further discussion and need to be addressed in an efficient implementation of the parametric active set strategy. We mention them briefly and give appropriate references. Concerning publicly available implementations of the parametric active set strategy, we are only aware of the code `qpOASES` [13].

**Solution of the Saddle Point Problem** Finding the step direction $(\Delta x, \Delta\mu)$ requires the solution of the linear system (4.4). The numerically stable and efficient solution of this saddle-point problem in $n + m$ unknowns requires exploitation of the problem structures, a topic outside the scope of this paper. We refer the reader to e.g. [5, 28] for surveys of applicable linear algebra. Block structured linear algebra techniques applicable to optimal control problem structures can be found e.g. in [21, 35]. Matrix update procedures are used to recover KKT system factorizations after a constraint entered or left the active set. We refer to [28] for updates in the dense nullspace method, to [3] for Schur complement updates, and to [22] for updates to optimal control problem block structures.

**Regularity of the Working Set** Addition of a primal blocking constraint to the working set may cause singularity of the constraints matrix $C_{\mathcal{W}}$, i.e. the working set may become *degenerate*. In [7, 13] a cheap and efficient strategy for degeneracy resolution is described that determines a constraint to be removed from the working set $\mathcal{W}$, allowing the primal blocking one to be added without loss of regularity.

**Primal and Dual Ties** Neither the primal nor the dual blocking constraint found when determining the step length are necessarily unique. The situation of non-uniqueness is referred to as a *tie*. The authors are not aware of implementations that systematically resolve ties. A costly procedure to this end that requires the solution of a larger auxiliary QP is proposed in [36]. In [31] a fast heuristic is described that avoids a tie in $\tau$ by applying a suitable perturbation of the homotopy.

## 4.4 Parametric Hot Starting for QPs with Vanishing Constraints

The described parametric active set method can be efficiently used to facilitate hot starts if the solution of a QP on an adjacent convex subset of a QPVC's feasible set, once an initial QP has been solved.

**Hot Starting if a Constraint Vanishes** If in a point $\tau^{(k)} \in (0,1)$ on the homotopy path a vanishing constraint $j \in \bar{l} \setminus \mathcal{I}_{0-}$ vanishes, one of two situations arises as derived in Section 3.5. If $j \in \mathcal{I}_{00}$, the active simple bound $x_j^*(\tau^{(k)}) = 0$ becomes an equality constraint, and the active vanishing constraint is removed from the QP. If $j \in \mathcal{I}_{+0}$, the variable $x_j^*(\tau^{(k)}) \neq 0$ must be set to zero in addition. In the latter case, both feasibility and stationarity of the perturbed solution $(\tilde{x}(\tau^{(k)}), \mu^*(\tau^{(k)}))$ are lost. We compute a suitable perturbation of the problem's right hand side in $\tau^{(k)}$,

$$\tilde{b}(\tau^{(k)}) := C^T \mu^*(\tau^{(k)}) - B\tilde{x}(\tau^{(k)}), \tag{4.7a}$$

$$\tilde{c}(\tau^{(k)}) := C\tilde{x}(\tau^{(k)}), \tag{4.7b}$$

This approach can be viewed as determining the QP in $\tau^{(k)}$ for which the perturbed point $(\tilde{x}(\tau^{(k)}), \mu^*(\tau^{(k)}))$ with $\tilde{x}_j(\tau^{(k)}) = 0$ is optimal. This is done without affecting the QP in $\tau = 1$, which is the QP we are interested in. We continue by progressing along the new homotopy towards $\tau = 1$.

**Hot Starting if a Constraint Appears** If in a point $\tau^{(k)} \in (0,1)$ on the homotopy path a vanishing constraint $j \in \mathcal{I}_{0-}$ appears, again one of two situations arises as derived in Section 3.5. If $j \in \mathcal{I}_{0-}$ enters the index set $\mathcal{I}_{00}$, the equality constraint on $x_j$ is lifted and becomes a simple lower bound. If $j$ enters $\mathcal{I}_{+0}$ and $\mu_j^{\mathrm{H}*} < 0$ must be set to zero, stationarity of the perturbed solution $(x^*(\tau^{(k)}), \tilde{\mu}(\tau^{(k)}))$ is lost. We again compute a suitable perturbation of the problem's right hand side in $\tau^{(k)}$,

$$\tilde{b}(\tau^{(k)}) := C^T \tilde{\mu}(\tau^{(k)}) - Bx^*(\tau^{(k)}), \tag{4.8}$$

and continue by progressing along the homotopy path towards the unaffected QP to be solved in $\tau = 1$.

**The Initial QP Subproblem in the First SQP Iteration** For the first QPVC of the first SQP iteration $k = 0$, we initially do not have an optimal solution to a related QPVC at hand. We start with the "trivial" QPVC

$$\min_{x \in \mathbb{R}^n} \quad \tfrac{1}{2} x^T B^{(0)} x \tag{4.9a}$$

$$\text{s.t.} \quad 0 \le C_j^{(0)} x \qquad j \notin \mathcal{I}_{0-}, \tag{4.9b}$$

$$0 = x_j \qquad j \in \mathcal{I}_{0-}, \tag{4.9c}$$

which allows to use the choice $\mathcal{I}_{0-} = \bar{l}$. The associated optimal solution is $(x^*, \mu^*) = (0, 0)$. The first parametric QPVC solution then progresses along the new homotopy

$$b(\tau): \ [0,1] \longrightarrow \mathbb{R}^n: \quad \tau \mapsto 0 + \tau b^{(0)}(1), \tag{4.10a}$$

$$c(\tau): \ [0,1] \longrightarrow \mathbb{R}^l: \quad \tau \mapsto 0 + \tau c^{(0)}(1), \tag{4.10b}$$

where $b^{(0)}(1)$, $c^{(0)}(1)$ denote the gradient and constraint bound vector of the QPVC for SQP iteration $k = 0$. This has been noted in [12] for convex QPs. As the initial problem (4.9) turns out to have $m$ ties in $\tau = 0$, an alternative initialization is proposed in [31] that relies on a homotopy perturbation concept.

**Initial QP Subproblem in Subsequent Iterations** For all subsequent SQP iterations $k > 0$, we have an MPVC strongly stationary point $(x^*, \mu^*)$ of the previously solved QP subproblem at hand. Denoting the old and new Hessians by $B^{(k-1)}$ and $B^{(k)}$, and the old and new constraint Jacobians by $C^{(k-1)}$ and $C^{(k)}$, we start the solution of the parametric QP

$$\min_{x \in \mathbb{R}^n} \quad \tfrac{1}{2} x^T B^{(k)} x + b(\tau)^T x \tag{4.11a}$$

$$\text{s.t.} \quad c(\tau) \le C_j^{(k)} x \qquad j \notin \mathcal{I}_{0-}, \tag{4.11b}$$

$$0 = x_j \qquad j \in \mathcal{I}_{0-}, \tag{4.11c}$$

with homotopy

$$b(\tau): \ [0,1] \longrightarrow \mathbb{R}^n: \quad \tau \mapsto (1-\tau) b^{(k)}(0) + \tau b^{(k)}(1), \tag{4.12a}$$

$$c(\tau): \ [0,1] \longrightarrow \mathbb{R}^l: \quad \tau \mapsto (1-\tau) c^{(k)}(0) + \tau c^{(k)}(1). \tag{4.12b}$$

starting in $\tau = 0$ with the modified initial right hand side

$$b^{(k)}(0) := b^{(k-1)}(1) - (B^{(k)} - B^{(k-1)}) x^* + (C^{(k)} - C^{(k-1)})^T \mu^*, \tag{4.13a}$$

$$c^{(k)}(0) := c^{(k-1)}(1) + (C^{(k)} - C^{(k-1)}) x^*. \tag{4.13b}$$

This choice maintains optimality of the known previous solution $(x^*, \mu^*)$ for $\tau = 0$. In [31] an alternative initialization is proposed that does not require evaluation of the matrix differences (4.13).

## 5 A Robot Pathfinding and Communication Problem

In this section, we demonstrate the applicability of the described parametric active set method for QPVCs by computing a family of MPVC strongly stationary points to a robot motion planning problem with logic communication constraints that can be cast as vanishing constraints.

### 5.1 Problem Formulation

Robot motion planning problems are frequently studied, see e.g. [24] for an introduction and [1] for details on modeling questions and a variant of the problem we investigate here. We consider a swarm of $N$ two-wheeled mobile robots indexed by $i = 1, \ldots, N$ moving on prescribed fixed paths $s : [0,1] \to (x_i(s), y_i(s)) \in \mathbb{R}^2$ on the cartesian plane $(x, y)$ according to tangential accelerations $a$ and velocities $v$. Starting in the given initial positions $(x(0), y(0)) \in \mathbb{R}^{2N}$ on their respective paths, the robots shall complete their paths to the given final positions $(x(1), y(1)) \in \mathbb{R}^{2N}$ in the minimum possible time. Each robot is able to communicate at any point in time with any other robot of the swarm that satisfies a communication constraint, e.g. that is within a prescribed distance $T$. While the swarm of robots proceeds along the paths, a communication network needs to be maintained among the swarm: each robot is required to be in communication with at least $K$ other robots.

**Optimal Control Problem** The resulting nonlinear optimal control problem can be formulated as follows: We minimize a time transformation parameter $h$,

$$\min_{a,c,s,v,h} \quad h \tag{5.1a}$$

subject to the dynamic equations of movement on the time horizon $[0, h] \subset \mathbb{R}$ for the swarm of robots on the fixed paths $(x(s), y(s))$ on the cartesian plane,

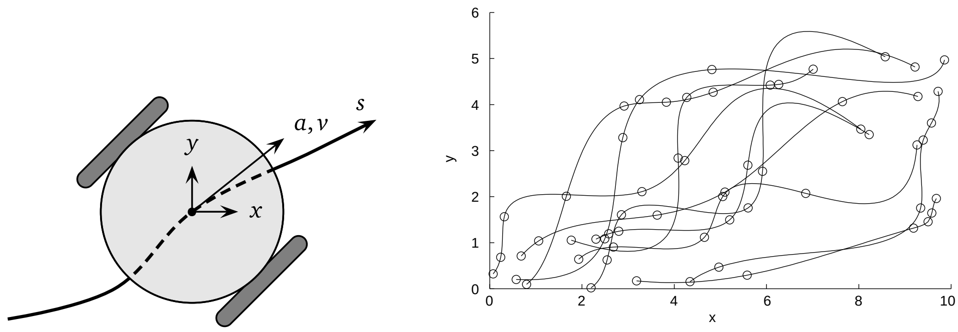
$$\frac{\mathrm{d}s_i}{\mathrm{d}t}(t) = h \cdot v_i(t) \qquad t \in [0,1],\ i \in \overline{N}, \tag{5.1b}$$

$$\frac{\mathrm{d}v_i}{\mathrm{d}t}(t) = h \cdot a_i(t) \qquad t \in [0,1],\ i \in \overline{N}. \tag{5.1c}$$

At $t = 0$ all robots are located at the their prescribed initial positions,

$$0 = s_i(0), \qquad i \in \overline{N}, \tag{5.1d}$$

$$0 = v_i(0), \qquad i \in \overline{N}, \tag{5.1e}$$




(a) Model of a two-wheeled mobile robot.

(b) Predefined paths for a swarm of ten robots.

Figure 3: Model of a two-wheele mobile robot, and predefined paths for a swarm of ten robots on the cartesian plane. Nodes delimit piecewise cubic spline segments. Initial positions are found in the lower left corner, and final ones in the upper right corner.

and at $t = 1$ arrival of all robots at the end of their prescribed paths is required,

$$0 = s_i(1) - s_{\mathrm{max,i}}, \qquad i \in \overline{N}. \tag{5.1f}$$

We introduce a communication function $c_{i,j}(t) \geq 0$ for each pair $(i, j)$ of robots. This function may assume a positive value if and only if the associated pair of robots is within communication distance,

$$0 \leq c_{i,j}(t) \cdot (T - D^2_{i,j}(t)) \quad t \in [0, 1],\ (i, j) \in \overline{N} \times \overline{N}, \tag{5.1g}$$
$$0 \leq c_{i,j}(t) \leq 1 \qquad t \in [0, 1],\ (i, j) \in \overline{N} \times \overline{N}. \tag{5.1h}$$

In (5.1g) the Euclidean distance $D^2_{i,j}(t)$ between any pair $(i, j)$ of two robots is defined as

$$D^2_{i,j}(t) := [x_i(s_i(t)) - x_j(s_j(t))]^2 + [y_i(s_i(t)) - y_j(s_j(t))]^2 . \tag{5.1i}$$

The communication network is maintained by imposing a communication constraint for each robot that counts the number of swarm members that are within reach,

$$K + 1 \leq \sum_{j \in \overline{N}} c_{i,j}(t), \qquad t \in [0, 1],\ i \in \overline{N}.$$

Hence, any optimal solution will assume $c_{i,j}(t) = 1$ for a pair $(i, j)$ within reach, if constraint (5.1j) is active. Finally, simple bounds apply to the positions $s$ on the prescribed paths, to the path tangential velocities, and to the acceleration of each robot,

$$0 \leq s_i(t) \leq s_{\mathrm{max},i}, \qquad t \in [0, 1],\ i \in \overline{N}, \tag{5.1j}$$
$$0 \leq v_i(t) \leq 0.5, \qquad t \in [0, 1],\ i \in \overline{N}, \tag{5.1k}$$
$$-1 \leq a_i(t) \leq 0.5, \qquad t \in [0, 1],\ i \in \overline{N}. \tag{5.1l}$$

To complete problem (5.1), piecewise cubic spline representations for the paths $(x_i(s_i), y_i(s_i))$ on $s_i \in [0, s_{\mathrm{max},i}]$ according to Figure 3 are required. By courtesy of Hande Y. Benson, the same scenario as in [1] could be used. Note that problem formulation (5.1) leaves ample freedom for implementation of a more detailed communication range model, taking e.g. frequency, noise, fading, or crosstalk into account, and also allows for asymmetric communication conditions. We refer the reader to [1, 24] and the references found therein. For the purpose of this paper, we are interested in the combinatorial structure introduced into problem (5.1) by the communication variables $c_{i,j}$ in (5.1g, 5.1h) and the imposed constraint (5.1j) only.

**Discretized Problem** Problem (5.1) is transformed into a time-discrete NLP by introducing a discretization

$$0 = t_0 < t_1 < \ldots < t_{M-1} < t_M = 1 \tag{5.2}$$

of the time horizon $[0, 1]$, and replacing (5.1b, 5.1c) by a fixed-step integration scheme using e.g. a higher-order Runge-Kutta method. For details on more elaborate adaptive schemes for the numerical solution of ODE initial value problems, and for numerically stable and efficient methods for sensitivity generation, we refer to e.g. [4, 29]. Constraints are enforced on the grid $\{t_i\}$ (5.2) only. This approach may in general yield solutions that are slightly

Table 1: Initial values for $h$ used for the computations presented in Tables 2 and 3. Empty fields denote known infeasible choices of $T$ and $K$.

| | $T = 2.0$ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|---|
| $K = 1$ | 10 | 10 | 10 | 10 | 10 | 10 | 10 |
| 2 | | 10 | 10 | 10 | 10 | 10 | 10 |
| 3 | | 35 | 9 | 10 | 10 | 10 | 10 |
| 4 | | FAIL | 11 | 10 | 10 | 10 | 10 |
| 5 | | | 20 | 10 | 10 | 10 | 10 |
| 6 | | | | | 11 | 12 | 10 |
| 7 | | | | | 10 | 12 | 10 |
| 8 | | | | | 50 | 12 | 10 |

infeasible with respect to the original problem, but serves the purpose. We refer to [30] for semi-infinite programming techniques that handle this issue in an exact way, and to [1] for more detailed evaluations of the effects of discretized constraints on solutions for the class of robot motion planning problems.

## 5.2 Numerical Results

We chose $M = 10$ time intervals, and a swarm of $N = 10$ robots. The obtained NLP has 836 unknowns, 550 constraints, and 450 additional vanishing constraints. NLP unknowns introduced in the $M + 1$ points $\{t_k\}$ of the grid (5.2) were initialized to

$$s_i(t_k) = \frac{k}{M} s_{\max,i}, \qquad i \in \overline{N},\ 0 \le k \le M, \tag{5.3a}$$

$$v_i(t_k) = 0, \qquad i \in \overline{N},\ 0 \le k \le M, \tag{5.3b}$$

$$a_i(t_k) = 0, \qquad i \in \overline{N},\ 0 \le k \le M, \tag{5.3c}$$

$$c_{i,j}(t_k) = 0, \qquad (i,j) \in \overline{N} \times \overline{N},\ 0 \le k \le M, \tag{5.3d}$$

The initial values for the total time $h$ required until all robots have reached their destinations are listed in Table 1.

The SQP algorithm proceeds as detailed in Section 2.4, where we did not implement a globalization strategy but always performed full steps. The Hessian matrix of the MPVC-Lagrangian was approximated using BFGS updates on the space of primal variables and MPVC multipliers. For the ODE system solution, a $4^{\text{th}}$ order Runge-Kutta method with fixed step size was used. All QPVCs and all NLPs were solved up to a KKT tolerance (see [25]) of $10^{-8}$. For all computations we used a single core of an *Intel Core i7 940 at 2.67 GHz*, running *Ubuntu Linux 9.10 (64 bit)*.

**Computational Solutions Found** The objective functions (minimal path completion times) of the solutions we found for the presented robot motion planning problem are listed in Table 2. We evaluated 56 problem instances, with communication radii $T$ ranging from 2.0 to 5.0 in steps of 0.5, and with communication network constraints $K$ ranging from 1 to 8 other robots within reach.

The number of SQP iterations and the total accumulated number of QPVC iterations for all solved problem instances is listed in Table 3.

Table 2: Computational solutions found for the robot path-finding and communication problem of Section 5 for various values of the communication radius $R$ and the minimum number $K$ of robots required to be in communication. Empty fields denote known infeasible choices of $T$ and $K$.

| | $T = 2.0$ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|---|
| $K = 1$ | 7.99575 | 7.99575 | 7.99575 | 7.99575 | 7.99575 | 7.99575 | 7.99575 |
| 2 | | 7.99575 | 7.99575 | 7.99575 | 7.99575 | 7.99575 | 7.99575 |
| 3 | | 12.7161 | 8.58713 | 7.99575 | 7.99575 | 7.99575 | 7.99575 |
| 4 | | FAIL | 9.54828 | 8.64826 | 7.99575 | 7.99575 | 7.99575 |
| 5 | | | 14.3232 | 10.2715 | 7.99575 | 7.99575 | 7.99575 |
| 6 | | | | | 10.1474 | 7.99575 | 7.99575 |
| 7 | | | | | 13.7652 | 7.99575 | 7.99575 |
| 8 | | | | | 21.3840 | 11.1614 | 7.99575 |

Table 3: Number of SQP steps and total number of QPVC iterations required to compute the solutions reported in Table 2. Empty fields denote known infeasible choices of $T$ and $K$.

| | $T = 2.0$ | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 |
|---|---|---|---|---|---|---|---|
| $K = 1$ | 16/440 | 16/ 501 | 16/ 440 | 16/ 387 | 16/ 336 | 15/317 | 15/285 |
| 2 | | 17/ 578 | 17/ 524 | 16/ 391 | 16/ 342 | 16/319 | 16/289 |
| 3 | | 32/15845 | 25/ 875 | 16/ 445 | 16/ 355 | 16/337 | 16/295 |
| 4 | | FAIL | 25/ 2920 | 27/ 539 | 16/ 388 | 16/347 | 16/296 |
| 5 | | | 33/22295 | 44/3247 | 16/ 427 | 16/398 | 16/334 |
| 6 | | | | | 28/ 541 | 28/657 | 17/355 |
| 7 | | | | | 16/1195 | 26/962 | 17/463 |
| 8 | | | | | 21/2454 | 13/737 | 21/551 |

**Discussion** We evaluated 56 problem instances, of which 17 turned out to be infeasible. In addition, all problem instances with $K = 9$, i.e. requiring all robots to be in communication with all other robots in all points of the time grid, are infeasible. For 27 of the remaining 39 feasible problems, a minimum time for completion of 7.99575 seconds was determined. This solution corresponds to the isolated time optimal solution for robot number 8, subject only to acceleration and velocity constraints. Hence, this solution is a globally optimal one. For 11 problem instances we determined minimum times for completion that are larger, depending on the restrictiveness of the choice of $K$ and $T$. Verification of global optimality is not easily possible in the proposed framework, though. The proposed approach failed to solve only the single instance $K = 4$, $T = 2.5$ due to divergence of the SQP method. We conjecture that the use of a suitable globalization procedure for the SQP method, e.g. transferring the works of [8, 23, 33] to MPVC, may lead to improvements here.

Table 3 shows that all problems that could be solved were solved within 15 to 44 SQP iterations. Increases in the number of QPVC iterations can generally be observed for the more difficult instances with larger values of $K$, respectively with smaller values of $T$. The increased number of SQP iterations for the solution of the instances $(K, T) = (3, 2.5)$ and $(5, 3.0)$ could possibly also be improved upon using a suitable globalization procedure.

We conclude our discussion with the remark that, using a standard SQP method based on linearizations of the multiplicative vanishing constraint (5.1g, 5.1h), hence ignoring the combinatorial nature of the problem and its implications for the validity of constraint qualifications, we have not been able to solve even a single instance of this robot motion planning problem with $N = 10$ robots. This observation again demonstrates the necessity of exploiting the combinatorial problem structure explicitly.

## 6 Summary and Conclusions

In this paper we have considered the challenging class of NLPs with vanishing constraints. Problems that fall into this class violate commonly assumed constraint qualifications, and we have given a number of detrimental consequences arising when standard SQP type methods are applied to such problems. To address this issue, we have presented a nonconvex SQP framework for the subclass of MPVCs that satisfy the regularity condition of MPVC–LICQ. We have described a search procedure for the solution of QPVCs that has been derived by partitioning the problem's nonconvex feasible set into overlapping feasible convex subsets and comparing MPVC strong stationarity conditions to KKT conditions. We have shown how multiplier information can be exploited to efficiently move between the introduced convex subsets in an active set method. In addition, iterations towards global optimality of the QPVC subproblem solution can be made. We have embedded the proposed approach in a parametric primal-dual active set method for convex QPs and have used the parametric framework of this method to facilitate hot starts when moving between the convex subsets. Within an SQP framework, we have applied the derived QPVC active set method to a robot path-finding and communication problem. Here, communication constraints on a swarm of robots have been formulated as vanishing constraints. We have considered 39 feasible problem instances of varying combinatorial difficulty. Using the proposed algorithm we have solved 27 of them to global optimality and have found solutions to a further 11 instances whose global optimality cannot be verified easily. One problem instance failed to solve and we have conjectured that the development of a suitable globalization procedure for the proposed SQP framework could yield improved convergence behavior here.

### Acknowledgements

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement n$^o$ FP7-ICT-2009-4 248940. The second author was supported by the German Research Foundation (DFG) within the priority program SPP1253 under grant BO864/12-1. The Heidelberg Graduate School of Mathematical and Computational Methods for the Sciences (HGS MathComp) supported this work by providing travel grants. We thank Hande Y. Benson for providing details of the numerical example. The remarks and comments by two anonymous referees have helped to considerably improve the presentation.

## References

[1] P. Abichandani, H.Y. Benson and M. Kam, Multi-vehicle path coordination under communication constraints, in *Proceedings of the American Control Conference*, 2008, pp. 650–656.

[2] W. Achtziger and C. Kanzow, Mathematical programs with vanishing constraints: optimality conditions and constraint qualifications, *Mathematical Programming Series A* 114 (2008) 69–99.

[3] R.A. Bartlett and L.T. Biegler, QPSchur: A dual, active set, Schur complement method for large-scale and structured convex quadratic programming algorithm, *Optimization and Engineering* 7 (2006) 5–32.

[4] I. Bauer, H.G. Bock, S. Körkel and J.P. Schlöder, Numerical methods for initial value problems and derivative generation for DAE models with application to optimum experimental design of chemical processes, in *Scientific Computing in Chemical Engineering II*, Springer, Berlin- Heidelberg, 1999, pp. 282–289.

[5] M. Benzi, G.H. Golub and J. Liesen, Numerical solution of saddle–point problems, *Acta Numerica* 14 (2005) 1–137.

[6] A.B. Berkelaar, K. Roos and T. Terlaky, *Recent Advances in Sensitivity Analysis and Parametric Programming*, Kluwer Publishers, Dordrecht, 1997, ch. 6: The Optimal Set and Optimal Partition Approach to Linear and Quadratic Programming, pp. 6–1–6–45.

[7] M.J. Best, *An Algorithm for the Solution of the Parametric Quadratic Programming Problem*, Applied Mathematics and Parallel Computing, Physica-Verlag, Heidelberg, 1996, ch. 3, pp. 57–76.

[8] A.R. Conn, N.I.M. Gould and P.L. Toint, *Trust–Region Methods*, SIAM, Philadelphia, PA, 2000.

[9] W. Dorn, R. Gomory, and M. Greenberg, *Automatic design of optimal structures*, *Journal de Méchanique* 3 (1964) 25–52.

[10] D. Dorsch, V. Shikhman and O. Stein, *Mathematical programs with vanishing constraints: critical point theory*, *Journal of Global Optimization* 52 (2012) 591–605.

[11] H. Fang, S. Leyffer and T.S. Munson, A pivoting algorithm for linear programs with complementarity constraints, *Optimization Methods and Software* 27 (2012) 89–114.

[12] H.J. Ferreau, An online active set strategy to overcome the limitations of explicit MPC, Diplomarbeit, Ruprecht–Karls–Universität Heidelberg, 2006.

[13] H.J. Ferreau, H.G. Bock and M. Diehl, An online active set strategy to overcome the limitations of explicit MPC, *International Journal of Robust and Nonlinear Control* 18 (2008) 816–830.

[14] R. Fourer, A simplex algorithm for piecewise–linear programming I: Derivation and proof, *Mathematical Programming* 33 (1985) 204–233.

[15] T. Hoheisel, C. Kanzow and A. Schwartz, Convergence of a local regularization approach for mathematical programs with complementarity or vanishing constraints, *Optimization Methods and Software* 27 (2012) 483–512.

[16] T. Hoheisel, Mathematical Programs with Vanishing Constraints, PhD thesis, Julius–Maximilians–Universität Würzburg, July 2009.

[17] T. Hoheisel and C. Kanzow, First- and second-order optimality conditions for mathematical programs with vanishing constraints, *Applications of Mathematics* 52 (2007) 459–514.

[18] T. Hoheisel and C. Kanzow, Stationary conditions for mathematical programs with vanishing constraints using weak constraint qualifications, *J. Math. Anal. Appl.* 337 (2008) 292–310.

[19] T. Hoheisel and C. Kanzow, On the Abadie and Guignard constraint qualifications for mathematical programs with vanishing constraints, *Optimization* 58 (2009) 431–448.

[20] A.F. Izmailov and M.V. Solodov, Mathematical programs with vanishing constraints: Optimality conditions, sensitivity, and a relaxation method, *Journal of Optimization Theory and Applications* 142 (2009) 501–532.

[21] C. Kirches, H.G. Bock, J.P. Schlöder and S. Sager, Block structured quadratic programming for the Direct Multiple Shooting method for optimal control, *Optimization Methods and Software* 26 (2011) 239–257.

[22] C. Kirches, H.G. Bock, J.P. Schlöder and S. Sager, *A factorization with update procedures for a KKT matrix arising in direct optimal control*, *Mathematical Programming Computation* 3 (2011) 319–348.

[23] D. Klatte and B. Kummer, Constrained minima and Lipschitzian penalties in metric spaces, *SIAM J. Optim.* 13 (2002) 613–633.

[24] J.-C. Latombe, *Robot Motion Planning*, Kluwer Academic Publishers, Norwell, MA, 1991.

[25] D.B. Leineweber, I. Bauer, A. Schäfer, H.G. Bock and J.P. Schlöder, *An Efficient Multiple Shooting Based Reduced SQP Strategy for Large-Scale Dynamic Process Optimization (Parts I and II)*, *Computers & Chemical Engineering* 27 (2003) 157–174.

[26] D.G. Luenberger, *Optimization by vector space methods*, Wiley Professional Paperback Series, John Wiley & Sons, Inc., New York, NY, 1969.

[27] O.L. Mangasarian and S. Fromovitz, Fritz John necessary optimality conditions in the presence of equality and inequality constraints, *Journal of Mathematical Analysis and Applications* 17 (1967) 37–47.

[28] J. Nocedal and S.J. Wright, *Numerical Optimization*, 2nd ed., Springer Verlag, Berlin Heidelberg New York, 2006.

[29] L. Petzold, S. Li, Y. Cao and R. Serban, Sensitivity analysis of differential-algebraic equations and partial differential equations, *Computers and Chemical Engineering*, 30 (2006) 1553–1559.

[30] A. Potschka, H.G. Bock and J.P. Schlöder, A minima tracking variant of semi-infinite programming for the treatment of path constraints within direct solution of optimal control problems, *Optimization Methods and Software* 24 (2009) 237–252.

[31] A. Potschka, C. Kirches, H.G. Bock and J.P. Schlöder, Reliable solution of convex quadratic programs with parametric active set methods, Technical Report, Interdisciplinary Center for Scientific Computing, Heidelberg University, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany, November 2010. Available online: http://www.optimization-online.org/DB_HTML/2010/11/2828.html.

[32] S. Scholtes, Nonconvex structures in nonlinear programming, *Operations Research* 52 (2004) 368–383.

[33] S. Scholtes and M. Stöhr, How stringent is the linear independence assumption for mathematical programs with stationarity constraints?, *Math. Oper. Res.* 26 (2001) 851–863.

[34] T. Siméon, S. Leroy, and J. Laumond, Path coordination for multiple mobile robots: a resolution-complete algorithm, *IEEE Transactions on Robotics and Automation* 18 (2002) 42–49.

[35] M.C. Steinbach, Structured interior point SQP methods in optimal control, *Zeitschrift für Angewandte Mathematik und Mechanik* 76 (1996) 59–62.

[36] X. Wang, Resolution of ties in parametric quadratic programming, Master's thesis, University of Waterloo, Ontario, Canada, 2004.

---

*Manuscript received 28 March 2012*
*revised 19 August 2012*
*accepted for publication 22 October 2012*

Christian Kirches
Interdisciplinary Center for Scientific Computing (IWR)
Heidelberg University, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany
E-mail address: christian.kirches@iwr.uni-heidelberg.de

Andreas Potschka
Interdisciplinary Center for Scientific Computing (IWR)
Heidelberg University, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany
E-mail address: potschka@iwr.uni-heidelberg.de

Hans Georg Bock
Interdisciplinary Center for Scientific Computing (IWR)
Heidelberg University, Im Neuenheimer Feld 368, 69120 Heidelberg, Germany.
E-mail address: bock@iwr.uni-heidelberg.de

Sebastian Sager
Institute of Mathematical Optimization, Otto-von-Guericke-Universität Magdeburg
Universitätsplatz 2, 39106 Magdeburg, Germany
E-mail address: sager@ovgu.de