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A PDE-CONSTRAINED GENERALIZED NASH EQUILIBRIUM PROBLEM WITH POINTWISE CONTROL AND STATE CONSTRAINTS

M. Hintermüller and T. Surowiec

Abstract: A generalized Nash equilibrium problem (GNEP) is formulated in which, in addition to pointwise constraints on both the control and state variables, the feasible sets are partially governed by the solutions of a linear elliptic partial differential equation. The decisions (optimal controls) of the players arise in their competitors optimization problems via the righthand side of the partial differential equation. The existence of a (pure strategy) Nash equilibrium for the GNEP is demonstrated via a relaxation argument under the presence of a constraint qualification. A numerical method based on a nonsmooth Newton iteration is presented and numerical results are provided.

Key words: *generalized Nash equilibrium problem, GNEP, PDE-constrained optimization, nonsmooth Newton methods*

Mathematics Subject Classification: *65K, 90C, 49M, 49J, 91A*

1 Introduction

Given the wealth of physical, biological, economic, and financial phenomena that can be modeled by solutions of partial differential equations, it is only natural that they should arise as constraints in many practical optimization problems. Moreover, many real world problems involve the interactions of multiple decision makers, each of whom acts according to their own preferences in a non-cooperative manner. It is for this reason that N-person, noncooperative games in which the strategy sets are partially governed by the solutions of partial differential equations are a natural subject of study. To the best of our knowledge, despite the relevance of such models, there appears to be little treatment in the literature of problems of the type considered in this paper. However, in the context of classical, i.e. non-generalized, Nash games there are some notable contributions in the context of two-player noncooperative "predator-prey"-type games, [38, **?**, 37], and multiobjective control problems relating to aerodynamic design [36, 35, 40]. The feasible sets considered in this paper were inspired by a population dynamics model discussed in [7].

During the review process for this paper, another work concerning two-player Nash equilibrium problems, [5], has been made aware to us by the co-editor of this special issue C. Kanzow. However, in that paper, there are no shared constraints, excluding the PDE. Yet after rewriting the individual problems in their reduced form, thus removing what at first appears to be a shared constraint, the resulting equilibrium problem amounts to a strongly monotone variational inequality in the classical sense. Thus, the existence result follows from

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the well-known work of Lions-Stampacchia [30]. Afterwards, the entire problem is discretized and solved in finite dimensions. In contrast, we provide an algorithmic framework in function space, which is shown to converge locally at a superlinear rate. Upon discretization, using the techniques in [25] thus allowing one to establish mesh-independent convergence.

In our work, the inclusion of an obstacle-type condition on the state of the system (solution of the partial differential equation) in each of the player's strategy sets results in a Nash equilibrium problem in which the strategies of the individuals are perturbed by the decisions of their "competitors", i.e., a so-called generalized Nash equilibrium problem (GNEP). In particular, our setting gives rise to a genuine quasivariational inequality. As will later be seen, this blocks a direct application of classical existence results for Nash equilibrium problems via either fixed-point theorems or the theory of variational inequalities.

The particular class of equilibrium problems chosen for this paper contains what we believe to be essential components of an optimization problem whose feasible set is partially governed by the solution of a partial differential equation (PDE), e.g., a PDE and control and state constraints. Though the class considered here is simple in structure from a finite dimensional perspective, the function space setting introduces additional difficulties, such as the existence of Lagrange multipliers. These aspects, along with the potentially large scale of the discretized models, add to the already nontrivial task of studying GNEPs. In this respect, this paper is meant to serve as a foundation on which future investigations of GNEPs in function space may be built.

After using a Moreau-Yosida-type penalty approach for the state constraints and reducing the resulting optimality system for the relaxed game, we propose a function space algorithm based on a nonsmooth Newton step for the numerical solution of the GNEP, see [8, 20, 43] for more on semismooth Newton methods.

The paper is structured as follows. In Section 2, we introduce the equilibrium problem and demonstrate the existence of a Nash equilibrium. In Section 3, we develop an algorithm in function space for finding generalized Nash equilibria. This method is then tested and discussed in Section 4, after a suitable discretization.

We will use a standard notation throughout the paper and we refer the reader to [1] for details about Lebesgue and Sobolev spaces, to [17, 45] for regularity theory of solutions of partial differential equations, and for any further notions of functional analysis to [46]. Much of the standard theory of PDE-constrained optimization can be found in [26] and [42].

2 A GNEP in Function Space

Throughout the text, we let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, or 3, be open and bounded. We use " $a.e.\,\Omega$ " to represent the phrase "almost everywhere on Ω". The spaces of all functions u for which $|u|^2$ is Lebesgue integrable will be denoted by $L^2(\Omega)$, whereas $W_0^{1,2}(\Omega) = H_0^1(\Omega)$ represents the Sobolev space of all $L^2(\Omega)$ functions y such that $|\nabla y|^2$ is Lebesgue integrable, where ∇y represents the weak derivative of y, and for which $y|_{\partial\Omega} = 0$ holds and is well-defined. Note that $\nabla y(x) \in \mathbb{R}^d$, in which case $|\nabla y|$ represents the pointwise Euclidean norm on $\mathbb{R}^d$. The dual space of $H_0^1(\Omega)$ will be denoted by $H^{-1}(\Omega)$. Due to the assumed boundedness of Ω, we may define the norm on $H_0^1(\Omega)$ by $||y||_{H_0^1(\Omega)} := ||\nabla y||_{L^2(\Omega)}$. The Sobolev spaces $W_0^{m,p}(\Omega)$, where $m \in \mathbb{N}$ and $1 \leq p \leq +\infty$, are defined analogously to $H_0^1(\Omega)$ with their respective dual spaces denoted by $W^{-m,s}(\Omega)$, where $s \in \mathbb{R}_+ \cup \{+\infty\}$ such that $1/p + 1/s = 1$. For a subset $A \subset \Omega$, we use the symbols $Vol(A)$ and χ_A to represent the Lebesgue measure and the characteristic function, respectively, and we let $-\Delta = -\text{div} \cdot \nabla$ be the standard Laplacian. The following data assumptions are used throughout:

- The boundary $\partial\Omega \subset \mathbb{R}^{d-1}$ is regular enough such that if $f \in L^2(\Omega)$, then the (unique) solution $u : \Omega \to \mathbb{R}$ of the Poisson equation with homogeneous Dirichlet boundary conditions and righthand side f can be continuously embedded into the Sobolev space $W_0^{1,r}(\Omega)$, with $r > \max(2, d)$, if $d > 1$. Moreover, we assume that $r < \frac{2d}{d-2}$, whenever $d \geq 3$.
- $N \geq 2, N \in \mathbb{N}$.
- $a_i, b_i \in L^2(\Omega)$ with $a_i < b_i,\ a.e.\,\Omega$, for all $i = 1, \ldots, N$.
- $\psi \in W^{1,r}(\Omega)$ with $\psi|_{\partial\Omega} < 0$.
- $y_d^i \in L^2(\Omega)$, for all $i = 1, \ldots, N$.
- $\alpha_i > 0$, for all $i = 1, \ldots, N$.
- $B_i \subset \Omega$, $|B_i| > 0$, for all $i = 1, \ldots, N$.
- $f \in L^2(\Omega)$.

As a notational convention, we define the product spaces $L^2(\Omega)^N := \Pi_{i=1}^N L^2(\Omega)$, $H_0^1(\Omega)^N := \Pi_{i=1}^N H_0^1(\Omega)$, and, for $u \in L^2(\Omega)^N$, $v \in L^2(\Omega)$, we let (v, u_{-i}) represent the vector field in $L^2(\Omega)^N$ obtained by replacing u_i in u by v.

The choice of boundary $\partial\Omega$ allows us to work with problems for which Ω has a non-smooth boundary and is convex as well as for cases in which $\partial\Omega$ is locally homeomorphic to the graph of a Lipschitz continuous function without the convexity requirement on Ω. In the first case, a well-known result from Kadlec, [27], shows that solutions of the Poisson equation with homogeneous Dirichlet boundary conditions are in $W^{2,2}(\Omega) \cap H_0^1(\Omega)$, whereas a famous result from Nečas, [31], shows that such a solution in the second case, i.e., $\partial\Omega$ Lipschitz and Ω non-convex, is in the fractional Sobolev space $W_0^{m,2}(\Omega)$, with $m \in [1, 3/2)$. In both cases, the Sobolev embedding theorem allows the solution to be embedded into $W_0^{1,r}(\Omega)$ with r as required (see [1] and Theorems 1.4.4.1, 2.2.2.3, 3.2.1.2 in [17]). Furthermore, the choice of r in relatio n to d allows us again to apply the Sobolev embedding theorem to show that $W_0^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ continuously.

Given these data assumptions, we consider an N-player game in which each player i has a desired state y_d^i and cost of control $\frac{\alpha_i}{2}||\cdot||^2_{L^2(\Omega)}$. Each player i is assigned a subset B_i of Ω on which their control u_i can affect the state of the system via the righthand side of a linear elliptic partial differential equation. The players seek to minimize both the distance of the equilibrium state to their respective desired states in the $L^2(\Omega)$-norm as well as their overall costs. This must all be done in such a way that the control lies pointwise almost everywhere between the prescribed bounds a_i and b_i and such that the equilibrium state satisfies the obstacle condition "$y \geq \psi,\ a.e.\,\Omega$." In other words, each player i seeks to solve the following optimization problem in which the decisions of its competitors, denoted throughout by $u_{-i} \in L^2(\Omega)^{N-1}$, arise as exogenous parameters:

$$\begin{aligned}
&\min \tfrac{1}{2}||y - y_d^i||^2_{L^2(\Omega)} + \tfrac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \text{ over } (u_i, y) \in L^2(\Omega) \times H_0^1(\Omega)\\
&\text{subject to (s.t.)}\\
&-\Delta y = \chi_{B_i} u_i + \textstyle\sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k + f, \quad a_i \leq u_i \leq b_i,\ a.e.\,\Omega, \quad y \geq \psi,\ a.e.\,\Omega.
\end{aligned} \tag{2.1}$$

We refer to a point $(u,y) \in L^2(\Omega)^N \times H_0^1(\Omega)$ such that $(u_i,y) \in L^2(\Omega) \times H_0^1(\Omega)$ is feasible for problem (2.1) for all $i = 1,\dots,N$ as a *feasible strategy*. For simplicity, we often use

$$U_i := \left\{ v \in L^2(\Omega) \,|\, a_i \le v \le b_i,\ a.e.\,\Omega \right\}.$$

We define solutions (equilibria) for this game in a standard sense.

Definition 2.1 (**Nash Equilibrium).** A feasible strategy (u,y) is referred to as a Nash equilibrium provided the following condition holds for all $i = 1\dots,N$:

$$\frac{1}{2}||y - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \le \frac{1}{2}||y' - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i'||^2_{L^2(\Omega)},$$
$$\forall u_i' \in U_i,\ \forall y' \ge \psi,\ a.e.\,\Omega : -\Delta y' = \chi_{B_i} u_i' + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k + f. \quad (2.2)$$

In other words, no player can reduce the value of their objective functional by unilaterally changing their decision.

As the constraint sets of each player i depend on the decisions of its competitors, this type of problem is often referred to as a generalized Nash equilibrium problem (GNEP). Some alternate names for this problem class are, to name only a few, pseudo-games, social equilibrium problems, and abstract economies. This category of games has been investigated since Debreu [9] and Arrow and Debreu [2] in the 1950s. A significant amount of work over the last two decades in the finite dimensional context has been completed, as can be seen in the recent survey paper by Facchinei and Kanzow [12].

GNEPs are notoriously difficult to solve numerically as they essentially require the solution of a quasi-variational inequality. To see this, recall that since the Laplace operator $-\Delta$ is an isometric isomorphism from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$, and since $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we can write y as a function linearly dependent on the righthand side of the PDE in (2.1). We denote this solution operator by

$$y(u) = y(u_i, u_{-i}) := (-\Delta)^{-1}(\chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k} u_k + f).$$

Since $L^2(\Omega)$ is compactly embedded into $H^{-1}(\Omega)$ and $(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$, y is completely continuous from $L^2(\Omega) \to H_0^1(\Omega)$. This can then be used to rewrite the GNEP as the game in which the component problems are given by

$$\begin{aligned} &\min \tfrac{1}{2}||y(u_i,u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \tfrac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} \text{ over } u_i \in L^2(\Omega) \\ &\text{s.t.} \\ &\quad a_i \le u_i \le b_i,\ a.e.\,\Omega, \quad y(u_i,u_{-i}) \ge \psi,\ a.e.\,\Omega. \end{aligned} \quad (2.3)$$

Now let $\Gamma_i : L^2(\Omega)^{N-1} \rightrightarrows L^2(\Omega)$ be the multifunction defined by

$$\Gamma_i(u_{-i}) := \left\{ v \in L^2(\Omega) \,|\, v \in U_i,\ y(v,u_{-i}) \ge \psi,\ a.e.\,\Omega. \right\}.$$

It is easy to see that Γ_i has closed convex values. Therefore, for any fixed u_{-i}, one can derive the first-order necessary and sufficient optimality condition for the i^{th} problem (in the form of a variational inequality):
Find $u_i \in \Gamma_i(u_{-i})$:

$$(\alpha_i u_i, v - u_i)_{L^2(\Omega)} + (y_{u_i}(u_i,u_{-i})^*(y(u_i,u_{-i}) - y_d^i), v - u_i)_{L^2(\Omega)} \ge 0, \forall v \in \Gamma_i(u_{-i}) \quad (2.4)$$

Here, the adjoint operator $y_{u_i}(\cdot, u_{-i})^*$ at u_i is given by $\chi_{B_i}(-\Delta)^{-1}$.

By coupling together each of the variational inequalities (2.4), one obtains a quasi-variational inequality formulation of the GNEP (2.3). Then due to convexity, we see that a feasible strategy $u \in L^2(\Omega)^N$ for (2.3) is a Nash equilibrium if and only if it solves the quasi-variational inequality.

There are two main difficulties that must be surmounted in order not only to demonstrate the existence of a generalized Nash equilibrium for the GNEP (2.1), but also, for the development of an efficient numerical method. First, the classical existence theory for N-player noncooperative games, based on the application of Kakutani's fixed point theorem, is developed in such a way that the decisions of each opposing player may only perturb their competitors' utility functions and not their strategy sets. Second, the derivation of multiplier-based necessary and sufficient optimality conditions for each nonlinear program that comprises (2.1) is significantly more difficult than in the finite dimensional setting. The ability to derive KKT-type optimality conditions is essential for the development of a numerical method as we shall see in Section 3. For these two reasons, we define a class of parameter dependent Nash equilibrium problems (NEPs) by using a smooth, convex penalty function for the pointwise constraint on the state variable y. This leads to the component problems given by

$$\begin{aligned}&\min \tfrac{1}{2}||y(u_i, u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \tfrac{\alpha_i}{2}||u_i||^2_{L^2(\Omega)} + \tfrac{\gamma}{2}||(\psi - y(u_i, u_{-i}))_+||^2_{L^2(\Omega)} \text{ over } u_i \in L^2(\Omega)\\ &\text{s.t. } a_i \le u_i \le b_i,\ a.e.\, \Omega.\end{aligned} \tag{2.5}$$

Here, $(\cdot)_+ = \max(0, \cdot)$, in the pointwise almost-everywhere sense. We refer to the γ-dependent NEPs by the notation NEP$_\gamma$ and for convenience, we refer to NEP$_\gamma$ by (2.5), despite the slight abuse of notation. Note that the idea to penalize the shared constraints in finite dimensional GNEPs was first introduced by Fukushima and Pang in [33], see also [13].

Our first result deals with the existence of Nash equilibria for NEP$_\gamma$ (2.5). We first recall a famous result of Ky Fan/Kakutani [15] as formulated in [44].

Theorem 2.2. *Let S be a compact convex set in a real locally convex topological space X and let $\psi : S \rightrightarrows S$ such that $\psi(x) \subset S$ is nonempty, convex, and compact for all $x \in S$. If $x_n \to_X x$ and $y_n \in \psi(x_n)$ such that $y_n \to_X y$ implies $y \in \psi(x)$, then there exists an $x^* \in S$ such that $x^* \in \psi(x^*)$.*

Theorem 2.3 **(Existence of a Nash Equilibrium for NEP$_\gamma$).** *For all $\gamma > 0$, the associated NEP$_\gamma$ (2.5) has a Nash equilibrium.*

Proof. We need to adapt (2.5) to the setting of Theorem 2.2. To begin, we define the locally convex topological vector spaces X_i for $i = 1, \dots, N$ by $X_i := (L^2(\Omega), \tau_{weak})$, i.e., X_i is $L^2(\Omega)$ endowed with the weak topology τ_{weak}. We then let $X := \Pi_{i=1}^N X_i$ be the real locally convex topological space required in Theorem 2.2 and set $S_i := \text{cl}\,\{U_i\}_{X_i}$. Due to the equivalence of weak and strong closure for convex sets in locally convex topological vectors spaces, $S_i = U_i$. Accordingly, we define $S \subset X$ by $S := \Pi_{i=1}^N S_i$. The weak compactness of closed convex bounded subsets in reflexive Banach spaces implies that S is convex and compact in X. Using these spaces and subsets, we define the best response functions $\psi_i^\gamma : X \rightrightarrows X_i$,

$i = 1, \ldots, N$:

$$\psi_i^\gamma(u) := \Big\{ v_i \in S_i \Big| \frac{1}{2}||y(v_i, u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(v_i, u_{-i}))_+||^2_{L^2(\Omega)} = \inf_{w_i \in S_i} \frac{1}{2}||y(w_i, u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(w_i, u_{-i}))_+||^2_{L^2(\Omega)} \Big\}$$

along with the multifunction $\psi^\gamma : S \rightrightarrows S$ given by $\psi^\gamma(u) := \psi_1^\gamma(u) \times \cdots \times \psi_N^\gamma(u),\ u \in S$. Clearly, $\psi^\gamma(u)$ is single-valued for all $u \in S$ and therefore, it satisfies the hypotheses of Theorem 2.3.

Now let $u^n \to u$ in X and $v^n \in \psi^\gamma(u^n)$ such that $v^n \to v$ in X. By definition this means $u_i^n \to u_i$, $v_i^n \to v_i$ weakly in $L^2(\Omega)$ for each $i = 1, \ldots, N$. Moreover, $v^n \in \psi^\gamma(u^n)$ implies that for each $i = 1, \ldots, N$ the following holds

$$\frac{1}{2}||y(v_i^n, u_{-i}^n) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i^n||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(v_i^n, u_{-i}^n))_+||^2_{L^2(\Omega)} \leq \frac{1}{2}||y(w_i, u_{-i}^n) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(w_i, u_{-i}^n))_+||^2_{L^2(\Omega)},\ \forall w_i \in S_i.$$

Since $y(\cdot)$ is completely continuous from $L^2(\Omega)^N$ to $H_0^1(\Omega)$ and the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, $y(\cdot)$ is continuous from X to $H_0^1(\Omega)$, and therefore, to $L^2(\Omega)$. Since the mapping $(\psi - \cdot)_+^2$ is pointwise convex, its integral is a convex functional and hence weakly lower semicontinous on $L^2(\Omega)$, i.e. lower-semicontinous on X_i. In fact, the previous observation makes this functional weakly continuous. Passing to the limit inferior in the previous inequality, we obtain

$$\frac{1}{2}||y(v_i, u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(v_i, u_{-i}))_+||^2_{L^2(\Omega)} \leq \frac{1}{2}||y(w_i, u_{-i}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(w_i, u_{-i}))_+||^2_{L^2(\Omega)},\ \forall w_i \in S_i.$$

It follows that $v \in \psi^\gamma(u)$. Then by Theorem 2.2, there exists some $u^* \in S$ such that $u^* \in \psi^\gamma(u^*)$. In other words, there exists a $u^* \in S$ such that for all $i = 1, \ldots, N$

$$\frac{1}{2}||y(u_i^*, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^*||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(u_i^*, u_{-i}^*))_+||^2_{L^2(\Omega)} \leq \frac{1}{2}||y(w_i, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||w_i||^2_{L^2(\Omega)} + \frac{\gamma}{2}||(\psi - y(w_i, u_{-i}^*))_+||^2_{L^2(\Omega)},\ \forall w_i \in S_i,$$

This concludes the proof. □

In order to demonstrate that the GNEP (2.3) has a Nash equilibrium, we will require the fulfillment of a constraint qualification.

Definition 2.4 (**Strict Uniform Feasible Responses**)**.** We will say that the GNEP satisfies the strict uniform feasible response constraint qualification (SUFR), if there exists an $\varepsilon > 0$, for all $i = 1, \ldots, N$:

$$\forall u_{-i} \in U_{-i},\ \exists u_i \in U_i\ :\ y(u_i, u_{-i}) \geq \psi + \varepsilon,\ a.e.\,\Omega.$$

Remark 2.5. Since we could easily transform the GNEP by redefining $u_i := u_i - a_i$, $b_i := b_i - a_i$, $f := f - \sum_{i=1}^N \chi_{B_i} a_i$, we can assume without loss of generality that $a_i \equiv 0$. Consider first the i^{th} player's optimization problem with $u_{-i} = 0$. Then if there exists $\varepsilon > 0$ such that $\chi_{B_i} b_i \geq -\Delta\psi - \min\{0, f\} + \varepsilon$ and $-\Delta\psi \in L^2(\Omega)$, it follows from the weak maximum principle, by setting $u_i = b_i$, that $y(b_i, 0) \geq \psi + \varepsilon$. Thus, for any other choice of $u_{-i} \in U_{-i}$, which incidentally implies $u_k \geq 0$, $a.e.\,\Omega$ for all $k \neq i$ $k = 1, \ldots, N$, it follows that $y(b_i, u_{-i}) \geq \psi + \varepsilon$. Hence, SUFR holds.

In effect, the SUFR condition allows us to show that the mappings Γ_i are continuous with respect to the topology induced by Mosco convergence of closed convex sets, c.f. [4]. Considering that our GNEP is equivalent to a quasivariational inequality of the type (2.4), for which, amongst other things, Mosco convergence of the sets $\Gamma(u^n) := \Gamma_1(u^n_{-1}) \times \cdots \times \Gamma_N(u^n_{-N})$, along a feasible sequence u^n, is generally required for the proof of existence of a solution, the SUFR condition is a natural requirement, see e.g. [29].

In the first part of the proof of Theorem 2.6, we will see that for the convergence of a sequence of equilibria $\{u^\gamma\}_\gamma$ to a feasible strategy of GNEP it suffices for SUFR to hold with $\varepsilon = 0$. However, for the convergence to a Nash equilibrium, $\varepsilon > 0$ is required.

Theorem 2.6 (**Consistency of the Relaxed Problems**). *If the GNEP (2.1) satisfies the SUFR, then there exists a sequence of penalty parameters $\gamma_n \to +\infty$ and an associated sequence of Nash equilibria $\{u^{\gamma_n}\}$ for the NEP$_{\gamma_n}$'s (2.5) such that for all $i = 1, \ldots, N$, $u_i^{\gamma_n} \rightharpoonup_{L^2(\Omega)} u_i^*$ as $\gamma_n \to +\infty$, where u^* is a Nash equilibrium for the GNEP.*

Proof. Let $U := \Pi_{i=1}^N U_i$ and fix an arbitrary sequence $\gamma_n \to +\infty$. According to Theorem 2.3, each NEP$_{\gamma_n}$ has a Nash equilibrium $u^{\gamma_n} \in U$. By definition, $a_i \leq u_i^{\gamma_n} \leq b_i$, $a.e.\,\Omega$. Therefore, the sequence of equilibria $\{u^{\gamma_n}\}$ is uniformly bounded in $L^2(\Omega)^N$. As U is weakly closed and $L^2(\Omega)^N$ a Hilbert space, there exists a subsequence, denoted by γ_n', and some element $u^* \in U$ such that $u^{\gamma_n'} \rightharpoonup u^*$ in $L^2(\Omega)^N$.

According to the SUFR, there a sequence $\left\{v^{\gamma_n'}\right\} \subset U$ such that $y(v_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) \geq \psi$, $a.e.\,\Omega$. As in the previous argument, we can deduce the uniform boundedness of $\left\{v^{\gamma_n'}\right\}$ in $L^2(\Omega)^N$. Thus, there exists a constant $M \geq 0$, independent of γ_n', such that

$$\frac{1}{2}||y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma_n'}||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma_n'}||^2_{L^2(\Omega)} + \frac{\gamma_n'}{2}||(\psi - y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}))_+||^2_{L^2(\Omega)} \leq$$
$$\frac{1}{2}||y(v_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i^{\gamma_n'}||^2_{L^2(\Omega)} \leq M.$$

Using the weak lower semicontinuity of the $L^2(\Omega)$-norm, it follows that

$$\frac{1}{2}||y(u_i^*, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^*||^2_{L^2(\Omega)} \leq \liminf_{n\to+\infty}\left[\frac{1}{2}||y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma_n'}||^2_{L^2(\Omega)}\right].$$

Therefore, $\frac{\gamma_n'}{2}||(\psi - y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}))_+||^2_{L^2(\Omega)}$ is bounded as $\gamma_n' \to +\infty$. But this can only hold if $||(\psi - y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}))_+||^2_{L^2(\Omega)} \to 0$. Due to the complete continuity of the solution operator y from $L^2(\Omega)^N$ to $H_0^1(\Omega)$ and the continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we also have $||(\psi - y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}))_+||^2_{L^2(\Omega)} \to ||(\psi - y(u_i^*, u_{-i}^*))_+||^2_{L^2(\Omega)}$. Thus, $u^* \in U$ such that

$y(u_i^*, u_{-i}^*) \geq \psi$, $a.e.\,\Omega$. In other words, there exists a subsequence of equilibria of the NEP_{γ_n} that converges weakly to a feasible strategy for the GNEP (2.1). Our next step is to demonstrate that u^* is also a generalized Nash equilibrium.

Define $X_i := \left\{ v_i \in U_i \,\middle|\, y(v_i, u_{-i}^*) \geq \psi,\ a.e.\,\Omega \right\}$. Note that X_i is non-empty due to the SUFR condition. Since for all such γ_n', $u^{\gamma_n'}$ is a Nash equilibrium for $\text{NEP}_{\gamma_n'}$, it holds that

$$\begin{aligned}\frac{1}{2}||y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^{\gamma_n'}||^2_{L^2(\Omega)} + \frac{\gamma_n'}{2}||(\psi - y(u_i^{\gamma_n'}, u_{-i}^{\gamma_n'}))_+||^2_{L^2(\Omega)} \leq \\ \frac{1}{2}||y(v_i, u_{-i}^{\gamma_n'}) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)} + \frac{\gamma_n'}{2}||(\psi - y(v_i, u_{-i}^{\gamma_n'}))_+||^2_{L^2(\Omega)},\ \forall v_i \in X_i, \quad (2.6)\end{aligned}$$

Now, for any $v_i \in X_i$, we construct a strongly convergent sequence $v_i^{\gamma_n'}$ such that $v_i^{\gamma_n'} \to_{L^2(\Omega)} v_i$ and $y(v_i^{\gamma_n'}, u_{-i}^{\gamma_n'}) \geq \psi$.

According to the SUFR condition, there exists a constant $\varepsilon > 0$ and, for each n, a point $v_i^n \in U_i$ such that $y(v_i^n, u_{-i}^{\gamma_n'}) \geq \psi + \varepsilon$, $a.e.\,\Omega$. Clearly,$\{v_i^n\}$ is bounded in $L^2(\Omega)$. Since U_i is convex, the points $v_i^n(\lambda) = \lambda v_i^n + (1-\lambda)v_i \in U_i$ for all $\lambda \in (0,1)$. Due to the linearity of the solution operator y, it holds for each $\lambda \in (0,1)$ that

$$\begin{aligned} y(v_i^n(\lambda), u_{-i}^{\gamma_n'}) &= y(\lambda v_i^n + (1-\lambda)v_i, u_{-i}^{\gamma_n'}) \\ &= \lambda y(v_i^n, u_{-i}^{\gamma_n'}) + (1-\lambda) y(v_i, u_{-i}^{\gamma_n'}) \\ &\geq \lambda(\psi + \varepsilon) + (1-\lambda) y(v_i, u_{-i}^{\gamma_n'}). \end{aligned}$$

As discussed at the beginning of this section, the assumed regularity of $\partial\Omega$, with $r > d$, $d \in \{2,3\}$, yields $y(v_i, \cdot) : L^2(\Omega)^{N-1} \to W_0^{1,r}(\Omega)$. Thus, by the Sobolev and Rellich-Kondrachov theorems, we can continuously and compactly embed solutions of the state equation into the space of continuous functions over $\overline{\Omega}$. This renders the solution operator $y(v_i, \cdot)$ completely continuous from $L^2(\Omega)^{N-1} \to C(\overline{\Omega})$. It follows from the convergence of $y(v_i, u_{-i}^{\gamma_n'}) \to y(v_i, u_{-i}^*)$ in $C(\overline{\Omega})$ that there exists a subsequence $k(n) \in \mathbb{N}$ such that $y(v_i, u_{-i}^{k(n)}) \geq \psi - 1/2^n$ on Ω for all $n \geq 1$. By defining $\lambda_n := (1/2^n)/(\varepsilon + 1/2^n)$, we obtain a null sequence, whose elements all lie in the interval $(0,1)$, and for which $y(v_i^{k(n)}(\lambda_n), u_{-i}^{k(n)}) \geq \psi$, $a.e.\,\Omega$. Then since

$$\begin{aligned} ||v_i^{k(n)}(\lambda_n) - v_i||_{L^2(\Omega)} &= ||\lambda_n v_i^{k(n)} + (1-\lambda_n)v_i - v_i||_{L^2(\Omega)} \\ &= |\lambda_n| ||v_i^{k(n)} - v_i||_{L^2(\Omega)} \\ &\leq |\lambda_n| (||v_i^{k(n)}||_{L^2(\Omega)} + ||v_i||_{L^2(\Omega)}), \end{aligned}$$

it follows that $v_i^{k(n)}(\lambda_n) \to v_i$, and therefore in $L^2(\Omega)$. This implies then that any element $v_i \in U_i$ such that $y(v_i, u_{-i}^*) \geq \psi$, $a.e.\,\Omega$ can be obtained by such a sequence $\left\{ v_i^{k(n)}(\lambda_n) \right\}_{n=1}^{\infty}$.

Upon substitution of this sequence into (2.6), passing to the limit inferior over γ_n' yields the following inequality for all $i = 1, \ldots, N$:

$$\frac{1}{2}||y(u_i^*, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||u_i^*||^2_{L^2(\Omega)} \leq \frac{1}{2}||y(v_i, u_{-i}^*) - y_d^i||^2_{L^2(\Omega)} + \frac{\alpha_i}{2}||v_i||^2_{L^2(\Omega)},\ \forall v_i \in X_i$$

as was to be shown. □

Remark 2.7. Note that by relaxing the SUFR condition to only hold for those $u_{-i} \in U_{-i}$ such that $\Gamma_i(u_{-i}) \neq \emptyset$, we cannot guarantee that $\Gamma_i(u^n_{-i}) \neq \emptyset$. Hence, there may exist $v_i \in X_i$ for some i that cannot be "reached" be a sequence of feasible v_i^n as constructed in the proof.

Now that we have shown the existence of a Nash equilibrium for the GNEP (2.3), we derive first order optimality conditions, which will be needed in the coming sections for the development of an implementable solution method.

Proposition 2.8 (Necessary and Sufficient Optimality Conditions NEP$_\gamma$). *For any $\gamma > 0$, a feasible strategy u^γ is a Nash equilibrium for NEP$_\gamma$ (2.5) if and only if there exists a $y^\gamma \in H_0^1(\Omega)$ and for all $i = 1, \dots, N$, a $p_i^\gamma \in H_0^1(\Omega)$ such that*

$$u_i^\gamma = \frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma - (\frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma - b_i)_+ + (-(\frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma - a_i))_+, \tag{2.7}$$

$$-\Delta y^\gamma = \chi_{B_i} u_i^\gamma + \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k^\gamma + f, \tag{2.8}$$

$$-\Delta p_i^\gamma = y_d^i - y^\gamma + \gamma(\psi - y^\gamma)_+. \tag{2.9}$$

Proof. By applying the argument used for (2.4) to the current setting, we can derive first-order necessary and sufficient optimality conditions for a Nash equilibrium u^γ of the form: Find $u^\gamma \in U$ such that for all $i = 1, \dots, N$

$$(\alpha_i u_i^\gamma, v - u_i^\gamma)_{L^2(\Omega)} + (y_{u_i}(u_i^\gamma, u_{-i}^\gamma)^*(y(u_i^\gamma, u_{-i}^\gamma) - y_d^i - \gamma(\psi - y(u_i^\gamma, u_{-i}^\gamma))_+), v - u_i^\gamma)_{L^2(\Omega)} \geq 0, \forall v \in U_i.$$

By letting $-p_i^\gamma = (-\Delta)^{-1}(y(u_i^\gamma, u_{-i}^\gamma) - y_d^i - \gamma(\psi - y(u_i^\gamma, u_{-i}^\gamma))_+)$, we obtain the equivalent coupled system for each $i = 1, \dots, N$:

$$(\alpha_i u_i^\gamma - \chi_{B_i} p_i^\gamma, v - u_i^\gamma)_{L^2(\Omega)} \geq 0, \forall v \in U_i$$

The nonsmooth equation (2.7) arises from the equivalence between the variational inequality and the projection of $\frac{1}{\alpha_i}\chi_{B_i} p_i^\gamma$ onto U_i, (cf. [28, 42]), whereas (2.8) and (2.9) follow from the definitions of p^γ and y^γ. □

The following constraint qualification is based on one developed in [41], see also [24, 32].

Definition 2.9 (A Uniform Range Space Condition). We say that the GNEP satisfies the uniform range space constraint qualification (URS) with respect to the control and state spaces $L^2(\Omega), W_0^{1,r}(\Omega)$, respectively, with $1 \leq r \leq +\infty$, if the following holds for all $i = 1, \dots, N$: There exists a $\delta_i > 0$ and a bounded set

$$M_i \subset \left\{ (v, z) \in L^2(\Omega) \times W_0^{1,r}(\Omega) \,|\, v \in U_i, z \geq \psi, \ a.e.\, \Omega \right\}$$

such that for all $u_{-i} \in U_{-i}$

$$\mathbb{B}_{\delta_i}(0) \subset \left\{ -\Delta y - \chi_{B_i} u_i - \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k - f \,\middle|\, (u_i, y) \in M_i \right\}$$

where $\mathbb{B}_{\delta_i}(0)$ is the open ball of radius δ_i in $W^{-1,r}(\Omega)$.

The URS condition is needed to ensure the existence of an adjoint state for the GNEP (2.3). Together the constraint qualifications will be needed to guarantee the convergence of stationary points which satisfy (2.7)-(2.9). Nevertheless, we show in the following lemma that the SUFR condition in fact implies the URS condition. Note that $\mathcal{M}(\overline{\Omega})$ represents the space of all regular bounded Borel measures, i.e. the topological dual of $C(\overline{\Omega})$.

Lemma 2.10 (**SUFR $\Rightarrow$ URS**). *Under the standing data assumptions, suppose that the GNEP (2.3) satisfies the SUFR condition. Then the URS condition holds with respect to the control and state spaces $L^2(\Omega)$ and $W_0^{1,r}(\Omega)$, where $r > d$, if $d > 1$.*

Proof. Based on the data assumptions, there exists a constant $C > 0$ such that for all $y \in W_0^{1,r}(\Omega)$, $||y||_{W_0^{1,r}(\Omega)} \geq C||y||_{C(\overline{\Omega})}$. In addition, we know that the inverse operator $(-\Delta)^{-1}$ is bounded in the operator norm $||\cdot||_{op}$ from $W^{-1,r}(\Omega)$ to $W_0^{1,r}(\Omega)$. Suppose then that $\delta > 0$ with

$$\delta \leq \frac{C\varepsilon}{2||(-\Delta)^{-1}||_{op}},$$

where the positive constant ε is taken from the definition of the SUFR condition.

Fix an arbitrary feasible strategy u for the GNEP (2.3). By the SUFR condition and regularity assumptions on $\partial\Omega$, there exists a $y \in W_0^{1,r}(\Omega)$ and $u_i^\delta \in U_i$ such that

$$-\Delta y - \chi_{B_i} u_i^\delta - \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k - f = 0 \text{ and } y \geq \psi + \varepsilon,\ a.e.\, \Omega.$$

Now let $w_\delta \in W^{-1,r}(\Omega)$ such that $||w_\delta||_{W^{-1,r}(\Omega)} < \delta$ and $y_\delta \in W_0^{1,r}(\Omega)$ such that $-\Delta y_\delta = w_\delta$. Then

$$C||y_\delta||_{C(\overline{\Omega})} \leq ||y_\delta||_{W_0^{1,r}(\Omega)} = ||(-\Delta)^{-1}(w_\delta)||_{W_0^{1,r}(\Omega)} \leq ||(-\Delta)^{-1}||_{op}||w_\delta||_{W^{-1,r}(\Omega)} < \frac{C\varepsilon}{2}.$$

Therefore, $-\varepsilon/2 \leq y_\delta \leq \varepsilon/2$ for all $x \in \overline{\Omega}$, from which it follows that $y + y_\delta \geq \psi + \varepsilon/2 \geq \psi,\ a.e.\, \Omega$. Finally, we observe that

$$-\Delta(y + y_\delta) - \chi_{B_i} u_i^\delta - \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k - f = -\Delta y_\delta = w_\delta$$

and

$$||y_\delta + y||_{W_0^{1,r}(\Omega)} \leq C\varepsilon/2 + ||y||_{W_0^{1,r}(\Omega)} = C\varepsilon/2 + ||(-\Delta)^{-1}(\chi_{B_i} u_i^\delta + \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k + f)||_{W_0^{1,r}(\Omega)} \leq$$

$$C\varepsilon/2 + C'||(-\Delta)^{-1}||_{op}(Vol(B_i)^{1/2}||u_i^\delta||_{L^2(\Omega)} + \sum_{\substack{k=1 \\ k\neq i}}^{N} Vol(B_k)^{1/2}||u_k||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}),$$

where $C' > 0$ is the constant arising from the (continuous) embedding $L^2(\Omega) \hookrightarrow W^{-1,r}(\Omega)$. In light of the boundedness of the sets U_i, $i = 1, \ldots, N$, the assertion follows. $\square$

Theorem 2.11 (Convergence of Stationary Points). *Suppose the GNEP (2.3) satisfies the SUFR condition. Then there exist sequences $\gamma_n \to +\infty$, $\{u^n\} \subset L^2(\Omega)^N$, $\{y^n\} \subset W_0^{1,r}(\Omega)$, and $\{p^n\} \subset W_0^{1,s}(\Omega)^N$ along with $u^* \in L^2(\Omega)^N$, $y^* \in W_0^{1,r}(\Omega)$, $p^* \in W_0^{1,s}(\Omega)^N$, and $\lambda^* \in \mathcal{M}(\overline{\Omega})$ where, for all $i = 1, \ldots, N$:*

$$u_i^n \to_{L^2(\Omega)} u_i^*, \quad y^n \to_{W_0^{1,r}(\Omega)} y^*, \quad p_i^n \rightharpoonup_{W_0^{1,s}(\Omega)} p_i^*, \quad \gamma_n(\psi - y^n)_+ \rightharpoonup_{\mathcal{M}(\overline{\Omega})} \lambda^*$$

such that (u_i^n, y^n, p_i^n) satisfies (2.7)-(2.9) and

$$u_i^* = \frac{1}{\alpha_i}\chi_{B_i}p_i^* - (\frac{1}{\alpha_i}\chi_{B_i}p_i^* - b_i)_+ + (-(\frac{1}{\alpha_i}\chi_{B_i}p_i^* - a_i))_+ \tag{2.10}$$

$$-\Delta y^* = \chi_{B_i}u_i^* + \sum_{\substack{k=1\\k\neq i}}^{N} \chi_{B_k}u_k^* + f \tag{2.11}$$

$$-\Delta p_i^* = y_d^i - y^* + \lambda^*, \tag{2.12}$$

$$\langle \lambda^*, \varphi \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \geq 0, \forall \varphi \in C(\overline{\Omega}) : \varphi \geq 0, \quad y^* \geq \psi, \ a.e.\,\Omega, \ \langle \lambda^*, y^* - \psi \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} = 0. \tag{2.13}$$

Proof. According the Theorem 2.6, there exists a sequence $\gamma_n \to +\infty$ along with a sequence $\{u^n\} \subset L^2(\Omega)^N$ of Nash equilibria for the NEP$_{\gamma_n}$ that converges weakly to a Nash equilibrium u^* for the GNEP (2.3) in the sense that for each i, $u_i^n \rightharpoonup_{L^2(\Omega)} u_i^*$. It follows then from Proposition 2.8 that there exists a $y^n \in H_0^1(\Omega)$ and, for each $i \in \{1, \ldots, N\}$, a $p_i^n \in H_0^1(\Omega)$ such that the relations (2.7)-(2.9) hold at (u_i^n, y^n, p_i^n) with $\gamma = \gamma_n$. Clearly, $\{y^n\}$ is bounded in $L^2(\Omega)$. Given the assumptions of $\partial\Omega$, the sequences $\{y^n\}$ and $\{p_i^n\}_n$ are contained in $W_0^{1,r}(\Omega)$, for all $i = 1, \ldots, N$. We begin by demonstrating the assertions on the sequences of adjoint states $\{p_i^n\}$.

Let $(u_i, y) \in M_i$, where M_i is the bounded subset of $L^2(\Omega) \times W_0^{1,r}(\Omega)$ given by the URS condition. Multiplying (2.9) by $y^n - y$, we obtain

$$\begin{aligned}
&\langle -\Delta p_i^n, y^n - y \rangle_{H^{-1},H_0^1} - \gamma_n \int_\Omega (\psi - y^n)_+(y^n - y)dx = \int_\Omega (y_d^i - y^n)(y^n - y)dx \\
&\Leftrightarrow \langle -\Delta p_i^n, y^n - y \rangle_{H^{-1},H_0^1} - \gamma_n \int_\Omega (\psi - y^n)_+(y^n - \psi + \psi - y)dx = \int_\Omega (y_d^i - y^n)(y^n - y)dx \\
&\Leftrightarrow \langle -\Delta p_i^n, y^n - y \rangle_{H^{-1},H_0^1} + \gamma_n \int_\Omega (\psi - y^n)_+^2 = \gamma_n \int_\Omega (\psi - y^n)_+(\psi - y)dx \\
&+ \int_\Omega (y_d^i - y^n)(y^n - y)dx
\end{aligned}$$

Then since $\psi - y \leq 0$, $a.e.\,\Omega$, it must hold that

$$\langle -\Delta p_i^n, y^n - y \rangle_{H^{-1},H_0^1} \leq \langle -\Delta p_i^n, y^n - y \rangle_{H^{-1},H_0^1} + \gamma_n \int_\omega (\psi - y^n)_+^2 \leq \int_\Omega (y_d^i - y^n)(y^n - y)dx. \tag{2.14}$$

Clearly, $y^n - y \in W_0^{1,r}(\Omega)$, which in turn implies that $-\Delta(y^n - y) \in W^{-1,r}(\Omega)$. It follows then that $\langle -\Delta p_i^n, y^n - y \rangle_{H^{-1},H_0^1} = \langle p_i^n, -\Delta(y^n - y) \rangle_{W_0^{1,s},W^{-1,r}}$. Since $r > 2$, $W_0^{1,r}(\Omega) \hookrightarrow L^2(\Omega)$ continuously. Thus, every $L^2(\Omega)$-function φ defines a bounded linear functional on $W_0^{1,r}(\Omega)$

via $(\varphi, \cdot)_{L^2(\Omega)}$. This allows us to make the following calculation

$$\begin{aligned}
&\langle p_i^n, -\Delta(y^n - y)\rangle_{W_0^{1,s},W^{-1,r}} = \\
&\quad \langle p_i^n, -\Delta y^n - \chi_{B_i} u_i - \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n - f + \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}} = \\
&\quad \langle p_i^n, \chi_{B_i}(u_i^n - u_i) + \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}} = \\
&\quad (p_i^n, \chi_{B_i}(u_i^n - u_i))_{L^2(\Omega)} + \langle p_i^n, \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}}.
\end{aligned} \tag{2.15}$$

Next, we show that the term $(p_i^n, \chi_{B_i}(u_i^n - u_i))_{L^2(\Omega)}$ is bounded. Using (2.7), we deduce the existence of multipliers $\overline{\lambda}_i^n, \underline{\lambda}_i^n \in L^2(\Omega)$ such that

$$\overline{\lambda}_i^n \geq 0,\ a.e.\,\Omega,\ (\overline{\lambda}_i^n, u_i^n - b_i)_{L^2(\Omega)} = 0, \quad \underline{\lambda}_i^n \geq 0,\ a.e.\,\Omega,\ (\underline{\lambda}_i^n, a_i - u_i)_{L^2(\Omega)} = 0.$$

and $\chi_{B_i} p_i^n = \alpha_i u_i^n + \overline{\lambda}_i^n - \underline{\lambda}_i^n$. But then

$$\begin{aligned}
(p_i^n, \chi_{B_i}(u_i^n - u_i))_{L^2(\Omega)} &= (\chi_{B_i} p_i^n, u_i^n - u_i)_{L^2(\Omega)} \\
&= (\alpha_i u_i^n + \overline{\lambda}_i^n - \underline{\lambda}_i^n, u_i^n - u_i)_{L^2(\Omega)} \\
&= (\alpha_i u_i^n, u_i^n - u_i)_{L^2(\Omega)} + (\overline{\lambda}_i^n, u_i^n - b_i + b_i - u_i)_{L^2(\Omega)} \\
&\qquad - (\underline{\lambda}_i^n, u_i^n - a_i + a_i - u_i)_{L^2(\Omega)} \\
&= (\alpha_i u_i^n, u_i^n - u_i)_{L^2(\Omega)} + (\overline{\lambda}_i^n, b_i - u_i)_{L^2(\Omega)} - (\underline{\lambda}_i^n, a_i - u_i)_{L^2(\Omega)} \\
&\geq (\alpha_i u_i^n, u_i^n - u_i)_{L^2(\Omega)}.
\end{aligned} \tag{2.16}$$

Combining (2.14)-(2.16), we obtain the inequality

$$\begin{aligned}
\langle p_i^n, \Delta y + \chi_{B_i} u_i + \sum_{\substack{k=1\\k\neq i}}^N \chi_{B_k} u_k^n + f\rangle_{W_0^{1,s},W^{-1,r}} &\leq (\alpha_i u_i^n, u_i - u_i^n)_{L^2(\Omega)} + \int_\Omega (y_d^i - y^n)(y^n - y)dx \\
&\leq |\alpha_i| ||u_i^n||_{L^2(\Omega)} ||u_i - u_i^n||_{L^2(\Omega)} + ||y_d^i - y^n||_{L^2(\Omega)} ||y^n - y||_{L^2(\Omega)}.
\end{aligned} \tag{2.17}$$

Since $(u_i, y) \in M_i$ was arbitrarily chosen and M_i is bounded, taking the supremum over both sides of (2.17) implies there exists a constant $C > 0$ such that

$$\sup_{\substack{\varphi \in W^{-1,r}(\Omega)\\ ||\varphi||_{W^{-1,r}(\Omega)}=1}} \langle p_i^n, \varphi\rangle_{W_0^{1,s},W^{-1,r}} \leq \delta C.$$

It follows that $\{p_i^n\}_n \subset W_0^{1,s}(\Omega)$ is bounded. Given $1 < r < +\infty$, $W_0^{1,s}(\Omega)$ is a reflexive Banach space. Therefore, there exists a subsequence of $\{p_i^n\}_n$, denoted still by n, and an element $p_i^* \in W_0^{1,s}(\Omega)$ such that $p_i^n \rightharpoonup_{W_0^{1,s}(\Omega)} p_i^*$. By the Rellich-Kondrachov theorem, the embedding $W_0^{1,s}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, in which case, there exists a further subsequence of $\{p_i^n\}_n$, denoted still by n, such that $p_i^n \to_{L^2(\Omega)} p_i^*$. As the $\max(0,\cdot)$-operator is Lipschitz continuous from $L^2(\Omega)$ to $L^2(\Omega)$, the strong convergence of p_i^n to p_i^* implies that $u_i^n \to_{L^2(\Omega)} u_i^*$ for each $i = 1, \ldots, N$. Furthermore, we know that the solution operator $y(\cdot)$ of the state equation is (completely) continuous from $L^2(\Omega)$ to $W_0^{1,r}(\Omega)$, from which can deduce the strong convergence of the sequence $y^n := y(u^n)$ in $W_0^{1,r}(\Omega)$ to y^*. These implications lead to the equations (2.10) and (2.11).

Next, we turn our attention to the sequence $\lambda_n := \gamma_n(\psi - y^n)_+$. By the previous arguments, in particular due to the uniform boundedness of Δp_i^n in $W^{-1,s}(\Omega)$, we have from (2.9) that $\{\lambda_n\}$ is bounded in $W^{-1,s}(\Omega)$. Moreover, the SUFR condition yields the existence of a constant $\varepsilon > 0$ and a (bounded) sequence of controls $\{\tilde{u}_i^n\} \subset U_i$ such that the sequence $\{\tilde{y}^n\}$ defined by $\tilde{y}^n := y(\tilde{u}_i^n, u_{-i}^n)$ satisfies $\tilde{y}^n - \psi \geq \varepsilon$, *a.e.* Ω for all $n \geq 1$. Since $\tilde{y}^n$ must also solve the state equation, it enjoys the increased regularity of y^n, i.e., $\tilde{y}^n \in W_0^{1,r}(\Omega)$. Multiplying (2.9) by $\tilde{y}^n$ yields the relation

$$\int_\Omega \lambda_n \tilde{y}^n = \langle -\Delta p_i^n, \tilde{y}^n \rangle_{W^{-1,s}, W_0^{1,r}} + (y^n - y_d^i, \tilde{y}^n)_{L^2(\Omega)}.$$

Using the continuity of the embedding $W_0^{1,r}(\Omega) \hookrightarrow L^2(\Omega)$, the boundedness of the sequences $\{\tilde{u}_i^n\}$ and $\{u_{-i}^n\}$, and the definition of the solution operator $y(\cdot)$, we can deduce the existence of constants $C, C' > 0$ such that

$$||\tilde{y}^n||_{L^2(\Omega)} \leq C||\tilde{y}^n||_{W_0^{1,r}} = C||(-\Delta)^{-1}(\chi_{B_i}\tilde{u}_i^n + \sum_{\substack{k=1 \\ k\neq i}}^{N} \chi_{B_k} u_k^n + f)||_{W_0^{1,r}} \leq$$

$$C||(-\Delta)^{-1}||_{\mathcal{L}(L^2(\Omega), W_0^{1,r})}((\mathrm{Vol}(B_i))^{1/2}||\tilde{u}_i^n||_{L^2(\Omega)} + \sum_{\substack{k=1 \\ k\neq i}}^{N} (\mathrm{Vol}(B_k))^{1/2}||u_k^n||_{L^2(\Omega)} + ||f||_{L^2(\Omega)}) \leq C'.$$

Therefore, there exists a constant $C'' > 0$, independent of n, such that

$$\int_\Omega \lambda_n \tilde{y}^n dx \leq |\int_\Omega \lambda_n \tilde{y}^n dx| = |\langle \lambda^n, \tilde{y}^n \rangle_{W^{-1,s}, W_0^{1,r}}| \leq ||\lambda^n||_{W^{-1,s}} ||\tilde{y}^n||_{W_0^{1,r}} \leq C''. \tag{2.18}$$

Moreover, we have

$$\int_\Omega \lambda_n \tilde{y}^n dx = \int_\Omega \lambda_n(\tilde{y}^n - \psi + \psi)dx = \int_\Omega \lambda_n(\tilde{y}^n - \psi)dx + \int_\Omega \lambda_n \psi dx.$$

By substitution into (2.18), it follows that

$$\int_\Omega \lambda_n(\tilde{y}^n - \psi)dx \leq C'' - \int_\Omega \lambda_n \psi dx$$

Defining the subset $\mathcal{A}^n := \{x \in \Omega \,|\, \psi - y^n > 0\}$, we then deduce

$$\int_\Omega \lambda_n \psi dx = \int_{\mathcal{A}^n} \lambda_n \psi dx \geq \int_{\mathcal{A}^n} \lambda_n y^n dx = \int_\Omega \lambda_n y^n dx = \langle \lambda_n, y^n \rangle_{W^{-1,s}, W_0^{1,r}}.$$

Thus,

$$\begin{aligned} 0 \leq \varepsilon \int_\Omega \lambda_n dx \leq \int_\Omega \lambda_n(\tilde{y}^n - \psi)dx \leq C'' - \int_\Omega \lambda_n \psi dx < C'' - \langle \lambda_n, y^n \rangle_{W^{-1,s}, W_0^{1,r}} \\ \leq C'' + |\langle \lambda_n, y^n \rangle_{W^{-1,s}, W_0^{1,r}}| \leq 2C''. \end{aligned}$$

Using the SUFR condition and the pointwise almost-everywhere non-negativity of λ_n, it holds that

$$0 \leq \int_\Omega \lambda_n dx = ||\lambda_n||_{L^1} \leq 2\varepsilon^{-1}C'',$$

from which it follows that the sequence $\{\lambda_n\}$ is bounded in $L^1(\Omega)$. Therefore, there exists a subsequence of λ_n, denoted still by n and an element $\lambda^* \in \mathcal{M}(\overline{\Omega})$ such that $\{\lambda_n\}$ converges in the weak topology $\sigma(\mathcal{M}(\overline{\Omega}), C(\overline{\Omega}))$ to λ^* (see e.g., Theorem IV.6.2 in [11] or Corollary 2.4.3 in [3]). The limiting adjoint equation (2.12) thus follows.

It remains to verify the complementarity relations (2.13). The feasibility of y^* follows from the fact that $y^n \to_{W_0^{1,r}(\Omega)} y^*$ implies $y^n - \psi \to_{L^1(\Omega)} y^* - \psi$. Hence, there exists a subsequence of $\{y^n - \psi\}$ that converges pointwise almost everywhere to $y^* - \psi$, in which case, $y^* \geq \psi$, *a.e.* Ω. Now let $\varphi \in C(\overline{\Omega})$ such that $\varphi \geq 0$ on Ω. Then $0 \leq (\lambda_n, \varphi)_{L^2(\Omega)} = \langle \lambda_n, \varphi \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}$. Passing to the limit in n, yields $\langle \lambda^*, \varphi \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \geq 0$. Since $W_0^{1,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ is continuous and $0 \geq (\lambda_n, y^n - \psi)_{L^2(\Omega)} = \langle \lambda_n, y^n - \psi \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}$, $\langle \lambda^*, y^* - \psi \rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} \leq 0$ holds. By the feasibility of y^*, the latter holds as an equality. □

Remark 2.12. Due to the convexity and differentiability properties of the individual problems, any feasible strategy u^*, for which there exist y^* and multipliers p^* and λ^* as in Theorem 2.11 such that (2.10)-(2.13) hold for all i, is a generalized Nash equilibrium. Thus, these conditions are both necessary and sufficient for a generalized Nash equilibrium.

This concludes our theoretical study of the GNEP. Before continuing, we note that one could easily extend some of these arguments to include bilateral constraints on the state and/or more general (linear) differential operators than $-\Delta$, provided the solutions are regular enough. One could also consider more control constraints, assuming they remain convex and bounded, however the simple reformulation of the variational inequality used in the previous proposition may no longer be available.

3 The Algorithm

Due to the constructive nature of Theorem 2.6 and 2.11, we can develop an infinite dimensional solution algorithm for the GNEP (2.3). The algorithm works by approximating a generalized Nash equilibrium for (2.3) by Nash equilibria obtained for each NEP$_\gamma$ along some sequence of constants $\{\gamma\}$ with $\gamma \to +\infty$. We describe this outer loop in Algorithm 1. It follows from Theorem 2.11 that the iterates $(u^\gamma, y^\gamma, p^\gamma)$ of Algorithm 1 converge (along a subsequence of γ) to a point (u^*, y^*, p^*) in $L^2(\Omega)^N \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)^N$ as $\gamma \to +\infty$ where u^* is a Nash equilibrium for (2.3) and for which there exists a $\lambda^* \in \mathcal{M}(\overline{\Omega})$ such that $(u^*, y^*, p^*, \lambda^*)$ satisfies (2.10)-(2.13). Note that the data needed for Algorithm 1 also includes the model data, e.g., α_i, y_d^i, f, a_i, b_i, etc.

Algorithm 1

Data: $\gamma_0 > 0$ $N \in \mathbb{N}$.

1: Choose $(u^0, y^0, p^0) \in L^2(\Omega)^N \times H_0^1(\Omega) \times H_0^1(\Omega)^N$ and set $k := 0$.
2: **repeat**.
3: Solve the coupled optimality conditions derived from (2.7)-(2.9) with $\gamma = \gamma_k$ to obtain $(u^{k+1}, y^{k+1}, p^{k+1})$ using initial values (u^k, y^k, p^k) for the solution algorithm.
4: Choose $\gamma_{k+1} > \gamma_k$.
5: Set $k := k + 1$.
6: **until** some stopping criterion is fulfilled.

In order to find an equilibrium for each γ_k in Algorithm 1, we propose that one solves the coupled system of necessary and sufficient first-order optimality conditions (2.7)-(2.9) derived in Proposition 2.8.

Consider the conditions (2.7)-(2.9) for all $i = 1, \cdots, N$ as one comprehensive system and let $q_i := (-\Delta)^{-1}(y_d^i)$, $\tilde{\alpha}_i = 1/\alpha_i$, $\tilde{f} := \sum_{i=1}^N \chi_{B_i} \tilde{\alpha}_i q_i + f$, $\tilde{a}_i := a_i - \tilde{\alpha}_i \chi_{B_i} q_i$, and $\tilde{b}_i := b_i - \tilde{\alpha}_i \chi_{B_i} q_i$. Then we can condense the system of nonsmooth equations to a problem in two variables:

$$-\Delta y = \sum_{i=1}^{N} \chi_{B_i} \left(\tilde{\alpha}_i p - (\tilde{\alpha}_i \chi_{B_i} p - \tilde{b}_i)_+ + (\tilde{a}_i - \tilde{\alpha}_i \chi_{B_i} p)_+ \right) + \tilde{f} \tag{3.1}$$

$$-\Delta p = \gamma_k (\psi - y)_+ - y \tag{3.2}$$

Once this system is solved, then the controls u_i can be easily obtained by reverse substitution.

By letting $F_{\gamma_k} : H_0^1(\Omega) \times H_0^1(\Omega) \to H^{-1}(\Omega) \times H^{-1}(\Omega)$ be defined by

$$F_{\gamma_k}(y,p) := \begin{bmatrix} -\Delta y - \sum_{i=1}^N \chi_{B_i} \left(\tilde{\alpha}_i p - (\tilde{\alpha}_i \chi_{B_i} p - \tilde{b}_i)_+ + (\tilde{a}_i - \tilde{\alpha}_i \chi_{B_i} p)_+ \right) - \tilde{f} \\ -\Delta p - \gamma_k (\psi - y)_+ + y \end{bmatrix},$$

we seek a solution to the equation $F_{\gamma_k}(y,p) = 0$. Since F_{γ_k} is nonsmooth, we will need a generalized derivative concept in order to design an algorithm based on a (nonsmooth) Newton step.

In the following definition, taken from [8] and [20], let X, Y be Banach spaces, $D \subset X$ an open subset of X, and $F : D \to Y$.

Definition 3.1. The mapping $F : D \subset X \to Y$ is said to be Newton-differentiable on the open subset $U \subset D$, if there exists a family of mappings $G : U \to \mathcal{L}(X,Y)$ such that

$$\lim_{h \to 0} \frac{1}{||h||_X} ||F(x+h) - F(x) - G(x+h)h||_Y = 0,$$

for every $x \in U$.

One typically refers to G as the Newton derivative for F on U. A well-known result from [20], shows that

$$G_\delta(y)(x) = \begin{cases} 1 & \text{if} \quad y(x) > 0 \\ 0 & \text{if} \quad y(x) < 0 \\ \delta & \text{if} \quad y(x) = 0 \end{cases} \tag{3.3}$$

for every $y \in X$ and $\delta \in \mathbb{R}$ is a Newton-derivative of the $\max(0,\cdot)$, provided one has $\max(0,\cdot) : L^p(\Omega) \to L^q(\Omega)$ with $1 \le q < p \le \infty$. Suppose now that we wish to solve the equation $F(x) = 0$. If a Newton-derivative of F is available, then a generalized Newton step can be derived. The following result is well known, see e.g., [8, 20].

Theorem 3.2. *Suppose that $F(x^*) = 0$ and that F is Newton-differentiable on an open neighborhood U of x^* with Newton derivative G. If $G(x)$ is nonsingular for all $x \in U$ and the set $\left\{||G(x)^{-1}||_{\mathcal{L}(Y,X)} : x \in U\right\}$ is bounded, then the semismooth Newton iteration*

$$x_{l+1} = x_l - G(x_l)^{-1} F(x_l), \quad l = 0, 1, 2, ... \tag{3.4}$$

converges superlinearly to x^*, provided $||x_0 - x^*||_X$ is sufficiently small.

At this point, we have enough tools to develop a semismooth Newton algorithm. Suppose $(y,p) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is the current iterate. We define the following approximations of the active sets:

$$\begin{aligned} A_i^a &:= \{x \in \Omega \,|\, \tilde{a}_i(x) - \tilde{\alpha}_i p(x) > 0\},\, A_i^b := \left\{x \in \Omega \,\middle|\, \tilde{\alpha}_i p(x) - \tilde{b}_i(x) > 0\right\}, \\ A_y &:= \{x \in \Omega \,|\, \psi(x) - y(x) > 0\}. \end{aligned}$$

Additionally, we define approximations for the inactive sets by

$$I_i := \Omega \setminus (A_i^a \cup A_i^b), \quad J_i := \Omega \setminus A_y.$$

In the following, we let $(\delta y, \delta p)$ denote the difference between the new iterate and previous iterate (y,p) in the Newton step. Then by using G_0 in (3.3) as the Newton-derivative of the $\max(0,\cdot)$ operator, one can easily show that (3.4) (applied to $F_{\gamma_k}(y,p) = 0$) is equivalent to solving the following system in $(\delta y, \delta p)$:

$$\begin{bmatrix} -\Delta & \sum_{i=1}^N \tilde{\alpha}_i \chi_{B_i} \chi_{I_i} \\ I + \gamma_k \chi_{A_y} & -\Delta \end{bmatrix} \begin{bmatrix} \delta y \\ \delta p \end{bmatrix} = -F(y,p), \tag{3.5}$$

Clearly, for every fixed $(y,p) \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $\gamma_k > 0$, the operator

$$G_{\gamma_k}(y,p) = \begin{bmatrix} -\Delta & \sum_{i=1}^N \tilde{\alpha}_i \chi_{B_i} \chi_{I_i} \\ I + \gamma_k \chi_{A_y} & -\Delta \end{bmatrix}$$

is a bounded linear operator from $H_0^1(\Omega) \times H_0^1(\Omega) \to H^{-1}(\Omega) \times H^{-1}(\Omega)$. Moreover, this operator can be shown, using standard PDE-theory, to be invertible independent of the choice of y or p. Hence, the set

$$\left\{ ||G_{\gamma_k}(y,p)^{-1}||_{\mathcal{L}(H^{-1},H_0^1)} : (y,p) \in H_0^1(\Omega) \times H_0^1(\Omega) \right\}$$

is uniformly bounded. Thus, for each γ_k we are guaranteed local superlinear convergence of the nonsmooth Newton step. If, in addition, the SUFR condition holds, then the sequence of iterates converges to a generalized Nash equilibrium.

In our future work, we will include the implementation of a more properly globalized Newton step, following [14, 16, 34], see also [21] for a function space treatment of these issues. Moreover, we note that one can easily implement a nonlinear multigrid method such as the full approximation scheme for optimization problems, see [6].

4 Numerical Experiments

Throughout this section, we let $N = 4$ and $\Omega = (0,1) \times (0,1)$. In order to discretize the problem, we considered a uniform grid with mesh size h, and we discretized the Laplacian $-\Delta$ by finite differences using a standard 5-point stencil.

In our experiments, we used a nested grid strategy using a standard 9-point prolongation to pass from the coarse to the fine mesh, cf. [18]. For the inner loop, i.e. the Newton step, we based the stopping criterion on the maximum of the observed rate of local superlinear convergence and the H^{-1}-norm of the reduced state and adjoint equations with tolerance 10^{-6} (for sufficiently fine mesh sizes). On coarse meshes, we used the less stringent stopping criterion based only on the residual of the system.

Referring to [19] Section 4. Example 1, we updated γ according to the mesh size. That is, given h, we updated γ until $C\gamma^{-1} > h^2$, for some constant $C > 0$, by setting $\gamma^{k+1} =$

	$h = 1/128$	$h = 1/256$	$h = 1/512$
q_l	0.25911	0.305	0.29818
	0.065435	0.013243	0.016935
	4.4273e-10	7.9809e-09	2.868e-08
r_l	0.035755	0.041051	0.035415
	0.0023397	0.00054362	0.00059974
	1.0358e-12	4.3386e-12	1.7201e-11
s_l	6.9274	5.1778	13.9437
	1.0617e-10	9.1398e-10	7.502e-09
	9.9346e-11	8.586e-10	7.5513e-09

Table 1: Behavior of the Newton Step in Example 4.1

$2\gamma^k$. Upon fulfillment of this inequality, we refined the mesh (reduced h) and continued. This refinement strategy for γ and h is considered optimal when no extra assumptions are made on the regularity of the solutions y. One topic of future research will be to develop a path-following strategy for the γ-updates as developed in [23, 22], in order to avoid unnecessary/costly iterations. For all our experiments, we initialized with $(y, p) = (0, 0)$.

Example 4.1. For this example, we let $a_i = -10$, $b_i = 10$ for $i = 1, \ldots, 4$, and randomly generated the α_i using: $\alpha_1 = 2.1853, \alpha_2 = 2.0942, \alpha_3 = 2.8730, \alpha_4 = 2.0866$. For $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega$ we defined the obstacle ψ by

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \cos(5\sqrt{(\mathbf{x}_1^2 - 0.5)^2 + (\mathbf{x}_2^2 - 0.5)^2}).$$

We used the partition

$$B_1 = \left]0, \frac{1}{2}\right[\times \left]0, \frac{1}{2}\right[, \; B_2 = \left]\frac{1}{2}, 1\right[\times \left]0, \frac{1}{2}\right[, \; B_3 = \left]0, \frac{1}{2}\right[\times \left]\frac{1}{2}, 1\right[, B_4 = \left]\frac{1}{2}, 1\right[\times \left]\frac{1}{2}, 1\right[,$$

and by letting $A := [0.25, 0.75] \times [0.25, 0.75] \subset \mathbb{R}^2$, we defined the fixed righthand side f by

$$f = -\chi_A \Delta\psi - 11.$$

Finally, we set $y_d^i = \chi_A \psi$. Refer to Table 2 for the performance of the outer loop, where $\gamma_{\max}$ represents the penalty parameter at which $C\gamma^{-1} > h^2$ and 'iter' is the total number of iterations on mesh size h. Here $C = 1e3$. We plotted the generalized Nash equilibrium in Figure 1 and associated state and multiplier λ^* in Figure 2. In Table 1, we provide the quantities

$$q_l := ||y_{l+1} - y_l||_{H_0^1} / ||y_l - y_{l-1}||_{H_0^1}, \quad r_l := ||y_{l+1} - y_l||_{H_0^1}$$

$$s_l := ||\Delta y_l + \sum_{i=1}^{N} \chi_{B_i} \left(\tilde{\alpha}_i p_l - (\tilde{\alpha}_i \chi_{B_i} p_l - \tilde{b}_i)_+ + (\tilde{a}_i - \tilde{\alpha}_i \chi_{B_i} p_l)_+ \right) + f||_{H^{-1}} + ||\Delta p_l + \gamma(\psi - y_l)_+ + y_l||_{H^{-1}}$$

for three loops of the nonsmooth Newton step, for $\gamma = \gamma_{max}$, to demonstrate the superlinear convergence and overall behavior of the step. Note that $q_l \to 0$ implies superlinear convergence (this is an immediate consequence of Lemma 8.2.3 in [10]).

Example 4.2. For this example, we let $a_i = -12; b_i = 12$ for $i = 1, \ldots, 4$, and randomly generated the α_i using: $\alpha_1 = 2.8859, \alpha_2 = 4.3374, \alpha_3 = 2.5921, \alpha_4 = 3.9481$. For $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega$

$\gamma_{\max}$	iter	h
8192	16	1/4
32768	5	1/8
131072	2	1/16
524288	6	1/32
2097152	5	1/64
8388608	8	1/128
33554432	8	1/256
134217728	8	1/512

Table 2: Behavior of the Outer Loop in Example 4.1

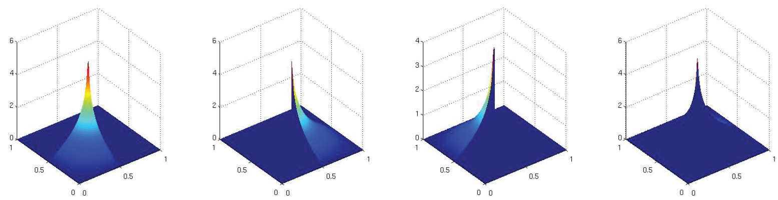

Figure 1: $(u_1^*, u_2^*, u_3^*, u_4^*)$ for Example 4.1

we defined $\psi(\mathbf{x}_1, \mathbf{x}_2) = \cos(5\sqrt{(\mathbf{x}_1^2 - z_i^1)^2 + (\mathbf{x}_2^2 - z_i^2)^2}) + .1$. We used here $B_i = \Omega$ for all i and set $f = 1$. Finally, we defined $\xi_i(\mathbf{x}_1, \mathbf{x}_2) = 1e3\max(0, 4(0.25 - \max(|\mathbf{x}_1 - z_i^1|, |\mathbf{x}_2 - z_i^2|)))$ with $z^1 := (0.25, 0.75, 0.25, 0.75, 0.5)$ and $z^2 = (0.25, 0.25, 0.75, 0.75, 0.5)$ and set $y_d^1 = \xi_1 - \xi_4$, $y_d^2 = \xi_2 - \xi_3$, $y_d^3 = \xi_3 - \xi_2$, $y_d^4 := \xi_4 - \xi_1$. As in Example 4.1, we updated γ in accordance with the mesh size h using $C\gamma_k^{-1} > h^2$, with $C = 1e3$. We refer to Table 4 for the performance of the outer loop algorithm, where $\gamma_{\max}$ and iter are defined as in Table 2. To illustrate the local superlinear convergent behavior of the algorithm, we provide Table 3. See Example 4.1 for the definition of q_l, r_l, s_l. We plotted the generalized Nash equilibrium in Figure 3 and associated state and multiplier λ^* in Figure 4 and the active sets for the control constraints in Figure 5.

	$h = 1/128$	$h = 1/256$	$h = 1/512$
q_l	0.86665	2.7241	1.2221
	0.30953	0.40752	0.25405
	0.77642	0.31822	0.3131
	1.3829e-11	0.26581	0.40521
		1.1597e-10	0.0026788
			3.1146e-07
r_l	0.51824	1.6909	1.058
	0.16041	0.68906	0.26878
	0.12455	0.21927	0.084154
	1.7224e-12	0.058284	0.0341
		6.7594e-12	9.1348e-05
			2.8451e-11
s_l	21.8022	332.9043	222.4632
	4.9505e-10	8.6492	3.2809e-08
	5.0033e-10	4.0237e-09	0.21287
	4.7248e-10	3.9826e-09	0.25647
		4.0435e-09	3.2643e-08
			3.2628e-08

Table 3: Behavior of the Newton Step in Example 4.2

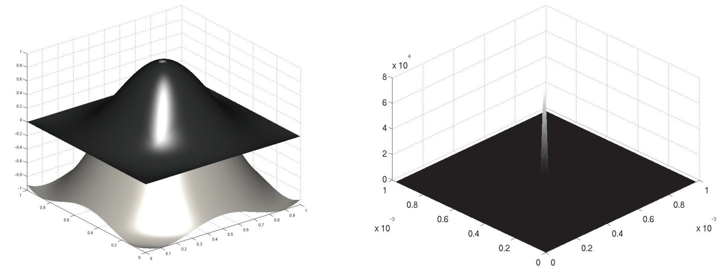

Figure 2: l: y^* (dark) and ψ (light); r: λ^* for Example 4.1

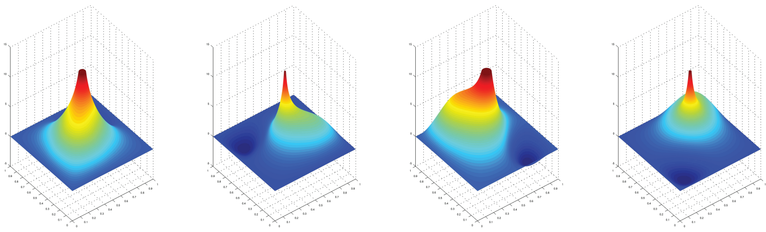

Figure 3: $(u_1^*, u_2^*, u_3^*, u_4^*)$ for Example 4.2

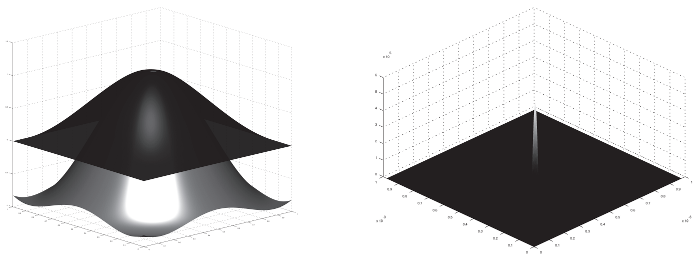

Figure 4: l: y^* (dark) and ψ (light); r: λ^* for Example 4.2

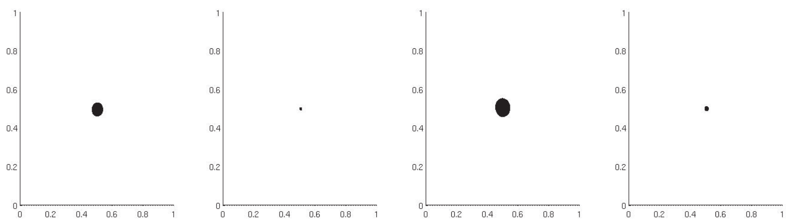

Figure 5: Active Sets for Example 4.2 $(u_i = b_i)$ black

$\gamma_{\max}$	iter	h
8192	17	1/4
32768	3	1/8
131072	4	1/16
524288	6	1/32
2097152	7	1/64
8388608	11	1/128
33554432	12	1/256
134217728	13	1/512

Table 4: Behavior of the Outer Loop in Example 4.2

References

[1] R.A. Adams and J.J.-F. Fournier, *Sobolev Spaces*, Second ed. Elsevier, Amsterdam, 2008.

[2] K.J. Arrow and G. Debreu, Existence of an equilibrium for a competitive economy, *Econometrica* 22 (1954) 265–290.

[3] H. Attouch and G. Buttazzo and G. Michaille, *Variational Analysis in Sobolev and BV Spaces*, MPS-SIAM Series on Optimization, SIAM and MPS, Philadelphia, 2006.

[4] G. Beer, *Topologies on Closed and Closed Convex Sets*, Mathematics and its Applications Vol. 268, Kluwer Academic Publishers Group, Dordrecht, 1993.

[5] A. Borzì and C. Kanzow, Formulation and numerical solution of Nash equilibrium multiobjective elliptic control problems, Preprint University of Würzburg, Institute of Mathematics, Feb 3, 2012, (Revised) July 16, 2012.

[6] A. Borzì and V. Schulz, Multigrid methods for PDE optimization, *SIAM Rev.* 51 (2009) 361–395.

[7] F. Bozorgnia, Numerical algorithm for spatial segregation of competitive systems, *SIAM J. Sci. Comput.* 31 (2009) 3946–3958.

[8] X. Chen, Z. Nashed and L. Qi, Smoothing methods and semismooth methods for nondifferentiable operator equations, *SIAM J. Numer. Anal.* 38 (2000) 1200–1216 (electronic).

[9] G. Debreu, A social equilibrium existence theorem, *Proc. Natl. Acad. Sci.* 38 (1952) 886–893.

[10] J. E. Dennis, Jr. and R.B. Schnabel, Numerical methods for unconstrained optimization and nonlinear equations, Classics in Applied Mathematics Vol. 16, Corrected reprint of the 1983 original, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1996.

[11] N. Dunford and J.T. Schwartz, *Linear operators. Part I*, Wiley Classics Library, John Wiley & Sons Inc., New York.

[12] F. Facchinei and C. Kanzow, Generalized Nash equilibrium problems, *4OR* 5 (2007) 173–210.

[13] F. Facchinei and C. Kanzow, Penalty methods for the solution of generalized Nash equilibrium problems, *SIAM J. Optim.* 20 (2010) 2228–2253.

[14] F. Facchinei and J.-S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. II*, Springer Series in Operations Research, Springer-Verlag, New York, 2003.

[15] K. Fan, Fixed points and minimax theorems in locally convex topological linear spaces, *Proceedings of the National Academy of Science* 28 (1952) 121-126.

[16] M.C. Ferris and T.S. Munson, Interfaces to PATH 3.0: design, implementation and usage, *Comput. Optim. Appl.* 12 (1999) 207–227.

[17] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, Vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[18] W. Hackbusch, *Multigrid Methods and Applications*, Springer Series in Computational Mathematics, Vol. 4, Springer-Verlag, Berlin, 1985.

[19] M. Hintermüller and M. Hinze, Moreau-Yosida regularization in state constrained elliptic control problems: error estimates and parameter adjustment, *SIAM J. Numer. Anal.* 47 (2009) 1666–1683.

[20] M. Hintermüller, K. Ito and K. Kunisch, The primal-dual active set strategy as a semismooth Newton method, *SIAM J. Optim.* 13 (2002) 865–888.

[21] M. Hintermüller and I. Kopacka, A smooth penalty approach and a nonlinear multigrid algorithm for elliptic MPECs, *Computational Optimization and Applications* 2009, DOI 10.1007/s10589-009-9307-9.

[22] M. Hintermüller and K. Kunisch, Feasible and noninterior path-following in constrained minimization with low multiplier regularity, *SIAM J. Control Optim.* 45 (2006) 1198–1221.

[23] M. Hintermüller and K. Kunisch, Path-following methods for a class of constrained minimization problems in function space, *SIAM J. Optim.* 17 (2006) 159–187 (electronic).

[24] M. Hintermüller, and K. Kunisch, PDE-constrained optimization subject to pointwise constraints on the control, the state, and its derivative, *SIAM J. Optim.* 20 (2009) 1133–1156.

[25] M. Hintermüller and M. Ulbrich, A mesh-independence result for semismooth Newton methods, *Math. Program., Ser. B* 101 (2004) 151–184.

[26] M. Hinze, R. Pinnau, M. Ulbrich and S. Ulbrich, *Optimization with PDE Constraints*, Mathematical Modelling: Theory and Applications, Vol. 23, Springer, New York, 2009.

[27] J. Kadlec,The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain, *Czechoslovak Math. J.* 14 (1964) 386–393.

[28] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York and London, 1980.

[29] M. Kunze and J.F. Rodrigues, An elliptic quasi-variational inequality with gradient constraints and some of its applications, *Math. Methods Appl. Sci.* 23 (2000) 897–908.

[30] J.-L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.* 20 (1967) 493–519.

[31] J. Nečas, Sur la coercivité des formes sesquilinéaires, elliptiques, *Rev. Roumaine Math. Pures Appl.* 9 (1964) 47–69.

[32] P. Neittaanmäki, J. Sprekels and D. Tiba, *Optimization of Elliptic Systems*, Springer Monographs in Mathematics, Springer, New York, 2006.

[33] J.-S. Pang and M. Fukushima, Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games, *Comput. Manag. Sci.* 2 (2005) 21–56.

[34] D. Ralph, Global convergence of damped Newton's method for nonsmooth equations via the path search, *Math. Oper. Res.* 19 (1994) 352–389.

[35] A.M. Ramos, R. Glowinski and J. Periaux, Nash equilibria for the multiobjective control of linear partial differential equations, *J. Optim. Theory Appl.* 112 (2002) 457–498.

[36] A.M. Ramos, R. Glowinski and J. Periaux, Pointwise control of the Burgers equation and related Nash equilibrium problems: computational approach, *J. Optim. Theory Appl.* 112 (2002) 499–516.

[37] A.M. Ramos and T. Roubíček, Nash equilibria in noncooperative predator-prey games, *Appl. Math. Optim.* 56 (2007) 211–241.

[38] T. Roubíček, Noncooperative games with elliptic systems, in *Optimal Control of Partial Differential Equations (Chemnitz, 1998)*, Internat. Ser. Numer. Math., Vol. 133, Birkhäuser, Basel, 1999, pp. 245–255.

[39] T. Roubíček, On noncooperative nonlinear differential games, *Kybernetika (Prague)* 35 (1999) 487–498.

[40] Z. Tang, J.-A. Désidéri and J. Périaux, Multicriterion aerodynamic shape design optimization and inverse problems using control theory and Nash games, *J. Optim. Theory Appl.* 135 (2007) 599–622.

[41] D. Tiba and C. Zălinescu, On the necessity of some constraint qualification conditions in convex programming, *J. Convex Anal.* 11 (2004) 95–110.

[42] F. Tröltzsch, Fredi, *Optimal control of partial differential equations*, Graduate Studies in Mathematics, Vol. 112, American Mathematical Society, Providence, RI, 2010.

[43] M. Ulbrich, Semismooth Newton methods for operator equations in function spaces, *SIAM J. Optim.* 13 (2002) 805–842.

[44] M. Vidyasagar, A new approach to N-person, nonzero-sum, linear differential games, *JOTA* 18 (1976), 477–487.

[45] J. Wloka, *Partielle Differentialgleichungen*, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982.

[46] K. Yosida, *Functional Analysis*, Spinger-Verlag, Berlin, 1980.

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M. Hintermüller
Department of Mathematics, Humboldt University of Berlin, Germany
and Department of Mathematics and Scientific Computing
Karl-Franzens-University of Graz, Austria
E-mail address: hint@math.hu-berlin.de

T. Surowiec
Department of Mathematics, Humboldt University of Berlin, Germany
E-mail address: surowiec@math.hu-berlin.de