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MULTI-PARAMETRIC SENSITIVITY ANALYSIS IN ACTIVE CONSTRAINTS SET OF CONVEX QUADRATIC OPTIMIZATION

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Abstract: In this paper, we study multi-parametric sensitivity analysis for the support set stability of dual problem with simultaneous perturbations in the right-hand-side of the constraints and the linear term of the objective function of the quadratic optimization. We show that the stability regions are convex polyhedral sets and we describe the set of admissible parameters by the basis vectors of the lineality space and the extreme directions of the defined cone over appropriate problems as a generalization of linear optimization.

Key words: *sensitivity analysis, optimal partition, multi-parametric programming, critical region*

Mathematics Subject Classification: *90C31, 90C20*

1 Introduction

Sensitivity analysis and parametric programming, in particular, multi-parametric programming are in focus of research [2, 4, 5]. The sensitivity analysis provides tools to study variations of the output depending on variations of the input data. This can be specified for optimization problems, in particular, for quadratic programming (QP) problems. The present paper is a step in this direction, seeking to ascertain how the problem and the optimal solutions change under variations in the input data. Usually variations occur in the right-hand-side (RHS) of the constraints and/or the coefficients of the objective function. If variation in the RHS and/or the coefficients of the objective function happens with identical parameter, the problem is called uni-parameter optimization problem [1, 6] and if there is more than one parameter in the RHS and/or the objective function, then it is called multi-parametric optimization problem. In [15], the authors proposed an approach for solving multi-parametric QP problems giving off-line piecewise affine explicit solutions to linear model predictive control (MPC) problems based on direct relations between neighboring polyhedral regions and combinations of active constraints. Variations in the data occur due to calculation errors or simply when answering the "what if?" system management questions that emerged shortly after introducing the simplex method [8, 16], which the related research area was known as basis sensitivity analysis. The primary aim of sensitivity analysis is to identify the interval where the given basic optimal solution is invariant. However, since sensitivity analysis using an optimal basis cannot be applied to an optimal non-basic solution, after Karmarkar's method [9] for solving linear optimization (LO) problems in polynomial-time which led to reconsider sensitivity analysis for LO [1, 5, 6, 14] and

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QP [5], another method has been put forward, namely support set sensitivity analysis. In this context, the aim is to find the range of parameter variations where for each parameter value in the range, an optimal solution exists with exactly the same set of positive variables as for the current optimal solution. In [10], the author studied the sensitivity analysis for the multi-parametric convex quadratic optimization problem, and then in [11] he extended this analysis for the dual and the primal-dual linear optimization, and showed that the stability regions are convex polyhedral sets which described by the basis vectors of the lineality space and the extreme directions of the defined cone. Here by using the technique in [11], we extend this analysis for QP case, also we point out the cases in which this analysis is the same as LO case.

2 Preliminaries

Let us consider the primal problem

$$\begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t: & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0, \end{array} \qquad (QP)$$

and its Wolfe dual

$$\begin{array}{ll} \max & \mathbf{b}^T\mathbf{y} - \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t: & \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c} \\ & \mathbf{s} \geq 0, \end{array} \qquad (QD)$$

where, $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix, $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are fixed data and $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ are unknown vectors. We denote the sets of feasible solutions for the primal and dual problems by

$$\mathcal{QP} = \{\mathbf{x} : \ \mathbf{A}\mathbf{x} = \mathbf{b}, \ \ \mathbf{x} \geq 0\},$$
$$\mathcal{QD} = \{(\mathbf{x}, \mathbf{y}, \mathbf{s}) : \ \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}, \ \ \mathbf{x}, \mathbf{s} \geq 0\},$$

respectively. Feasible solutions $\mathbf{x} \in \mathcal{QP}$ and $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QD}$ are optimal if and only if $\mathbf{x}^T\mathbf{s} = 0$ [3]. Also let $\mathcal{QP}^*$ and $\mathcal{QD}^*$ denote the corresponding sets of optimal solutions. Then for any $\mathbf{x} \in \mathcal{QP}^*$ and $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QD}^*$ we have

$$x_i s_i = 0, \ i = 1, 2, \ldots, n.$$

The support set of a nonnegative vector $\mathbf{x}$ is defined as

$$\sigma(\mathbf{x}) = \{i : \ x_i > 0\}.$$

The index set $\{1, 2, \ldots, n\}$ can be partitioned into three subsets

$$\begin{aligned} \mathcal{B} &= \{i : \ x_i > 0 \ \text{ for some } \mathbf{x} \in \mathcal{QP}^*\}, \\ \mathcal{N} &= \{i : \ s_i > 0 \ \text{ for some } (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QD}^*\}, \\ \mathcal{T} &= \{1, 2, \ldots, n\} \setminus (\mathcal{B} \cup \mathcal{N}) \\ &= \{i : \ x_i = s_i = 0 \ \text{ for all } \mathbf{x} \in \mathcal{QP}^* \ \text{ and } (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QD}^*\}, \end{aligned}$$

where is known as the optimal partition of the index set $\{1, 2, \ldots, n\}$ for problems (QP) and (QD), and is denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. The uniqueness of the optimal partition follows from the convexity of the optimal solution sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$. A *maximally complementary solution* [7] $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ is a pair of primal-dual optimal solutions of QP and QD for which

$$x_i > 0 \text{ if and only if } i \in \mathcal{B},$$

$$s_i > 0 \text{ if and only if } i \in \mathcal{N}.$$

The existence of maximally complementary solution is a consequence of the convexity of the optimal solution sets $\mathcal{QP}^*$ and $\mathcal{QD}^*$ [12]. Knowing a maximally complementary solution, one can easily determine the optimal partition as well. If $\mathcal{T} = \emptyset$ for some optimal partition, then any maximally complementary solution is strictly complementary. It is worth to mention that we have $\sigma(\mathbf{x}^*) \subseteq \mathcal{B}$ and $\sigma(\mathbf{s}^*) \subseteq \mathcal{N}$ for any pair of primal-dual optimal solutions $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$.

Let $\lambda \in \mathbb{R}^k$ and $\epsilon \in \mathbb{R}^{k'}$ be two vectors of parameters. We consider the parametric primal problem in the general form

$$\begin{array}{ll} \min & \mathbf{c}^T(\epsilon)\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t: & \mathbf{A}\mathbf{x} = \mathbf{b}(\lambda) \\ & \mathbf{x} \geq 0, \end{array} \qquad (QPP)$$

with its dual

$$\begin{array}{ll} \max & \mathbf{b}^T(\lambda)\mathbf{y} - \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ s.t: & \mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}(\epsilon) \\ & \mathbf{s} \geq 0, \end{array} \qquad (QDP)$$

where, $\mathbf{b}(\lambda) = \mathbf{b} + \sum_{j=1}^{k} \alpha_j \lambda_j$ and $\mathbf{c}(\epsilon) = \mathbf{c} + \sum_{j=1}^{k'} \beta_j \epsilon_j$ in which α_j and β_j are given. Also let $\mathcal{QPP}^*$ and $\mathcal{QPD}^*$ denote the sets of optimal solutions of the problems (QPP) and (QPD), respectively. Let $\mathbf{x}^*$ and $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ be optimal solutions of (QP) and (QD), respectively. The corresponding optimal partition is denoted by $\pi = (\mathcal{B}, \mathcal{N}, \mathcal{T})$. Let us define the support set sensitivity analysis for the dual problem and the primal-dual problem.

Support set sensitivity analysis for the dual and the primal-dual problem: Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ be a primal-dual optimal solution of (QP) and (QD) with $P = \sigma(\mathbf{x}^*)$ and $\hat{P} = \sigma(\mathbf{s}^*)$. Thus, the index set {1, 2, . . . , n} can be partitioned as $(\hat{P}, \hat{Z})$, where $\hat{P} = \{i : \ s_i^* > 0\}$ and $\hat{Z} = \{1, 2, \ldots, n\} \setminus \hat{P}$. Further, another partition of the index set can be defined as $(P, \tilde{Z}, \hat{P})$, where $P = \{i : x_i^* > 0\}$ and $\tilde{Z} = \{1, 2, \ldots, n\} \setminus (P \cup \hat{P})$. Support set sensitivity analysis for the dual problem (QD) deals with finding a region of variation of parameters (λ, ϵ) , such that there is a primal-dual optimal solution $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ with the property $\sigma(\mathbf{s}) = \hat{P}$. In other words, we want the active set of constraints in the dual problem (QD) remains active in the perturbed problem (QDP) for any (λ, ϵ) in this region. On the other hand, in support set sensitivity analysis for the primal-dual problems, we want the positive variables in the given primal optimal solution of the problem (QP) remain positive in the perturbed problem (QPP), as well as keeping the active set of constraints in the dual problem (QD), active in the perturbed problem (QDP) for any (λ, ϵ) in this region.

The corresponding sets of sensitivity analysis are denoted by $\Upsilon_{\hat{P}}(\mathbf{s}^*)$ and $\Upsilon(\mathbf{x}^*, \mathbf{s}^*)$, respectively, which are referred to critical regions.

The purpose of this paper is to extend support set sensitivity analysis of the dual and the primal-dual problems when two independent sets of parameters applied for the objective function and the right-hand-side of the constraints, simultaneously.

The following definitions and Theorem are quoted from [13].

Definition 2.1. A set $\mathbf{C} \subseteq \mathbb{R}^n$ is a polyhedron if and only if there exist an $m \times n$ matrix $\mathbf{H}$ and a vector $\mathbf{h} \in \mathbb{R}^m$ such that $\mathbf{C} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{Hx} \leq \mathbf{h}\}$.

Definition 2.2. The lineality space of $\mathbf{C}$ is defined as $L_C = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{Hx} = 0\}$.

Clearly, $0 \in L_C$ and we have $L_C = \{0\}$ if and only if $r(\mathbf{H}) = n$, where $r(\mathbf{H})$ denotes the "rank" of matrix $\mathbf{H}$. Let

$$L_C^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{y}^T\mathbf{x} = 0 \ \text{ for all } \mathbf{y} \in \mathrm{L_C}\},$$

be the orthogonal complement of L_C in $\mathbb{R}^n$. In this way, we have $dim L_C = n - r(\mathbf{H})$ and $dim L_C^{\perp} = r(\mathbf{H})$ and $L_C = \{0\}$ if and only if $L_C^{\perp} = \mathbb{R}^n$. Let the rows of the matrix $\mathbf{G}$ correspond to the vectors of a basis of the space L_C. Then $\mathbf{G}$ has $n - r(\mathbf{H})$ rows and n columns $r(\mathbf{G}) = n - r(\mathbf{H})$ and $L_C^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{Gx} = 0\}$.

Definition 2.3. Let $\mathbf{S} \subseteq \mathbb{R}^n$ be any set. Then the set

$$\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \sum_{i=1}^{t} \mu_i \mathbf{x}^i, \ \sum_{i=1}^{t} \mu_i = 1, \ \mu_i \geq 0, \ \mathbf{x}^i \in \mathbf{S}, \ 0 \leq t < \infty\},$$

is the convex hull of $\mathbf{S}$, denoted by conv($\mathbf{S}$), and the set

$$\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \sum_{i=1}^{t} \mu_i \mathbf{x}^i, \ \mu_i \geq 0, \ \mathbf{x}^i \in \mathbf{S}, \ 0 \leq t < \infty\},$$

is the conical hull of $\mathbf{S}$, denoted by cone($\mathbf{S}$).

Theorem 2.4. *Let $\mathbf{C}$ be a polyhedron, L_C its lineality space and $\mathbf{C}^0 = \mathbf{C} \cap L_C^{\perp}$. Denote by $\mathbf{S} = \{\mathbf{x}^1, \ldots, \mathbf{x}^q\}$ the extreme points and by $\mathbf{T} = \{\mathbf{y}^1, \ldots, \mathbf{y}^r\}$ the extreme directions of $\mathbf{C}^0$. Then $\mathbf{C}^0 = conv(\mathbf{S}) + cone(\mathbf{T})$ and $\mathbf{C} = L_C + conv(\mathbf{S}) + cone(\mathbf{T})$.*

3 Stability Regions

To identify the sets $\Upsilon_{\hat{P}}(\mathbf{s}^*)$ and $\Upsilon(\mathbf{x}^*, \mathbf{s}^*)$, a computational method is introduced in this section.

3.1 Support Set Sensitivity Analysis for Dual Problem

Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ be a primal-dual optimal solution of (QP) and (QD) with $\hat{P} = \sigma(\mathbf{s}^*)$ and $\hat{Z} = \{1, 2, \ldots, n\} \backslash \hat{P}$. Consider the partition $(\hat{P}, \hat{Z})$ of the index set $\{1, 2, \ldots, n\}$ for matrices $\mathbf{Q}$, $\mathbf{A}$ and the vectors $\mathbf{x}, \mathbf{c}$ and $\mathbf{s}$ as follows:

$$\begin{aligned} &\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{\hat{P}\hat{P}} & \mathbf{Q}_{\hat{P}\hat{Z}} \\ \mathbf{Q}_{\hat{P}\hat{Z}}^T & \mathbf{Q}_{\hat{Z}\hat{Z}} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A}_{\hat{P}} & \mathbf{A}_{\hat{Z}} \end{pmatrix}, \\ &\mathbf{c} = \begin{pmatrix} \mathbf{c}_{\hat{P}} \\ \mathbf{c}_{\hat{Z}} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{x}_{\hat{P}} \\ \mathbf{x}_{\hat{Z}} \end{pmatrix} \text{ and } \quad \mathbf{s} = \begin{pmatrix} \mathbf{s}_{\hat{P}} \\ \mathbf{s}_{\hat{Z}} \end{pmatrix}. \end{aligned} \tag{3.1}$$

We want to identify the set $\Upsilon_{\hat{P}}(\mathbf{s}^*)$. Let I and J be the index set of a basis of the lineality space and the extreme directions of the convex polyhedron cone corresponded to a polyhedron, respectively.

Theorem 3.1. *Let* $(\mathbf{h}_i^{\hat{Z}}, \mathbf{h}_i)$, $i \in I$, *be a basis of the lineality space*

$$L = \{(\mathbf{u}, \mathbf{v}) : \mathbf{A}_{\hat{Z}}^T \mathbf{u} - \mathbf{Q}_{\hat{P}\hat{Z}}^T \mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}\hat{Z}}^T \mathbf{v}_{\hat{Z}} = 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} = 0\},$$

and let $(\mathbf{g}_j^{\hat{Z}}, \mathbf{g}_j)$, $j \in J$, *be all extreme directions of the convex polyhedron cone*

$$S = \{(\mathbf{u}, \mathbf{v}) : \mathbf{A}_{\hat{Z}}^T \mathbf{u} - \mathbf{Q}_{\hat{P}\hat{Z}}^T \mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}\hat{Z}}^T \mathbf{v}_{\hat{Z}} \geq 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} \geq 0\} \cap L^{\perp},$$

where I *and* J *are indices of the basis and directions, respectively. Then*

$$\Upsilon_{\hat{P}}(\mathbf{s}^*) = \{(\lambda, \epsilon) : \mathbf{b}(\lambda)^T \mathbf{h}_i^{\hat{Z}} + \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0,\ i \in I,\ \mathbf{b}(\lambda)^T \mathbf{g}_j^{\hat{Z}} + \mathbf{c}(\epsilon)^T \mathbf{g}_j > 0,\ j \in J\}.$$

Proof. To prove, we identify the set of (λ, ϵ) such that support set of the given solution remains invariant, that is,

$$\begin{aligned}
\Upsilon_{\hat{P}}(\mathbf{s}^*) &= \{(\lambda, \epsilon) : \exists(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \text{ with } \sigma(\mathbf{s}) = \hat{P}\} \\
&= \{(\lambda, \epsilon) : \exists(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \text{ s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}(\lambda), \\
&\qquad \mathbf{A}^T \mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}(\epsilon),\ \mathbf{x}, \mathbf{s} \geq 0,\ \mathbf{x}^T \mathbf{s} = 0,\ \sigma(\mathbf{s}) = \hat{P}\} \\
&= \{(\lambda, \epsilon) : \exists(\mathbf{x}, \mathbf{y}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \text{ s.t. } \mathbf{A}_{\hat{Z}} \mathbf{x}_{\hat{Z}} = \mathbf{b}(\lambda), \mathbf{x}_{\hat{Z}} \geq 0, \\
&\qquad \mathbf{A}_{\hat{P}}^T \mathbf{y} - \mathbf{Q}_{\hat{P}\hat{Z}} \mathbf{x}_{\hat{Z}} < \mathbf{c}_{\hat{P}}(\epsilon), \mathbf{A}_{\hat{Z}}^T \mathbf{y} - \mathbf{Q}_{\hat{Z}\hat{Z}} \mathbf{x}_{\hat{Z}} = \mathbf{c}_{\hat{Z}}(\epsilon)\}.
\end{aligned}$$

Therefore, it is sufficient to determine the set of (λ, ϵ) for which the system

$$\begin{aligned}
\mathbf{A}_{\hat{Z}} \mathbf{x}_{\hat{Z}} &= \mathbf{b}(\lambda) \\
\mathbf{A}_{\hat{P}}^T \mathbf{y} - \mathbf{Q}_{\hat{P}\hat{Z}} \mathbf{x}_{\hat{Z}} &< \mathbf{c}_{\hat{P}}(\epsilon) \\
\mathbf{A}_{\hat{Z}}^T \mathbf{y} - \mathbf{Q}_{\hat{Z}\hat{Z}} \mathbf{x}_{\hat{Z}} &= \mathbf{c}_{\hat{Z}}(\epsilon) \\
\mathbf{x}_{\hat{Z}} &\geq 0,
\end{aligned} \tag{3.2}$$

is solvable. But the system (3.2) is solvable if and only if the corresponding problem

$$\begin{aligned}
\max \quad & 0^T \mathbf{y} + 0^T \mathbf{x}_{\hat{Z}} \\
s.t: \quad & \mathbf{A}_{\hat{Z}} \mathbf{x}_{\hat{Z}} = \mathbf{b}(\lambda) \\
& \mathbf{A}_{\hat{P}}^T \mathbf{y} - \mathbf{Q}_{\hat{P}\hat{Z}} \mathbf{x}_{\hat{Z}} \leq \mathbf{c}_{\hat{P}}(\epsilon) - \eta_{\hat{P}} \\
& \mathbf{A}_{\hat{Z}}^T \mathbf{y} - \mathbf{Q}_{\hat{Z}\hat{Z}} \mathbf{x}_{\hat{Z}} = \mathbf{c}_{\hat{Z}}(\epsilon) \\
& \mathbf{x}_{\hat{Z}} \geq 0,
\end{aligned} \tag{3.3}$$

has an optimal solution for sufficiently small $\eta_{\hat{P}} > 0$. By duality theory in linear programming, it is equivalent to the optimality of the following dual problem

$$\begin{aligned}
\min \quad & \mathbf{b}(\lambda)^T \mathbf{u} + \mathbf{c}(\epsilon)^T \mathbf{v} - \eta_{\hat{P}}(\epsilon)^T \mathbf{v}_{\hat{P}} \\
s.t: \quad & \mathbf{A}_{\hat{Z}}^T \mathbf{u} - \mathbf{Q}_{\hat{P}\hat{Z}}^T \mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}\hat{Z}}^T \mathbf{v}_{\hat{Z}} \geq 0 \\
& \mathbf{A}\mathbf{v} = 0 \\
& \mathbf{v}_{\hat{P}} \geq 0.
\end{aligned} \tag{3.4}$$

Now, let L denote the lineality space of the problem (3.4), that is,

$$L = \{(\mathbf{u}, \mathbf{v}) : \mathbf{A}_{\hat{Z}}^T \mathbf{u} - \mathbf{Q}_{\hat{P}\hat{Z}}^T \mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}\hat{Z}}^T \mathbf{v}_{\hat{Z}} = 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} = 0\}.$$

Let $(\mathbf{h}_i^{\hat{Z}}, \mathbf{h}_i),\ i \in I$, denote the vectors of basis of L, and $(\mathbf{g}_j^{\hat{Z}}, \mathbf{g}_j),\ j \in J$, denote the extreme directions of $S = \{(\mathbf{u}, \mathbf{v}) : \mathbf{A}_{\hat{Z}}^T\mathbf{u} - \mathbf{Q}_{\hat{P}\hat{Z}}^T\mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}\hat{Z}}^T\mathbf{v}_{\hat{Z}} \geq 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} \geq 0\} \cap L^{\perp}$. Any solution of problem (3.4) can be written as

$$(\mathbf{u}, \mathbf{v}) = \sum_{i\in I}\mu_i(\mathbf{h}_i^{\hat{Z}}, \mathbf{h}_i) + \sum_{j\in J}\mu_j^{'}(\mathbf{g}_j^{\hat{Z}}, \mathbf{g}_j)$$

$$= \Big(\sum_{i\in I}\mu_i\mathbf{h}_i^{\hat{Z}} + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j^{\hat{Z}},\ \sum_{i\in I}\mu_i\mathbf{h}_i + \sum_{j\in J}\mu_j^{'}\mathbf{g}\Big), \qquad \mu_j^{'} \geq 0,\ \ \forall j \in J, \tag{3.5}$$

by Theorem 2.4. It follows from weak duality theorem that

$$\mathbf{b}(\lambda)^T(\sum_{i\in I}\mu_i\mathbf{h}_i^{\hat{Z}} + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j^{\hat{Z}}) + \mathbf{c}(\epsilon)^T\big(\sum_{i\in I}\mu_i\mathbf{h}_i + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j\big) - \eta_{\hat{P}}(\epsilon)^T\big(\sum_{i\in I}\mu_i\mathbf{h}_i + \sum_{j\in J}\mu_j^{'}\mathbf{g}_j\big)_{\hat{P}} \geq 0,$$

which holds if and only if

$$\begin{cases} \mathbf{b}(\lambda)^T\mathbf{h}_i^{\hat{Z}} + \mathbf{c}(\epsilon)^T\mathbf{h}_i - \eta_{\hat{P}}(\epsilon)^T(\mathbf{h}_i)_{\hat{P}} = 0, & i \in I, \\ \mathbf{b}(\lambda)^T\mathbf{g}_j^{\hat{Z}} + \mathbf{c}(\epsilon)^T\mathbf{g}_j - \eta_{\hat{P}}(\epsilon)^T(\mathbf{g}_j)_{\hat{P}} \geq 0, & j \in J. \end{cases} \tag{3.6}$$

Since $\eta_{\hat{P}}(\epsilon)^T(\mathbf{h}_i)_{\hat{P}} = 0$ for each $i \in I$ and $\eta_{\hat{P}}(\epsilon)^T(\mathbf{g}_j)_{\hat{P}} > 0$ for every $j \in J$, it follows from (3.6) that

$$\begin{cases} \mathbf{b}(\lambda)^T\mathbf{h}_i^{\hat{Z}} + \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0, & i \in I, \\ \mathbf{b}(\lambda)^T\mathbf{g}_j^{\hat{Z}} + \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0, & j \in J. \end{cases} \tag{3.7}$$

Therefore, (3.7) describes the set $\Upsilon_{\hat{P}}(\mathbf{s}^*)$. □

Remark 3.2. If $Q = 0$, then the problems (QPP) and (QDP) reduce to linear optimization problems. In this case, we have

$$\Upsilon_{\hat{P}}(\mathbf{s}^*) = \{\lambda : \mathbf{A}_{\hat{Z}}\mathbf{x}_{\hat{Z}} = \mathbf{b}(\lambda), \mathbf{x}_{\hat{Z}} \geq 0\} \times\ \{\epsilon : \mathbf{A}_{\hat{P}}^T\mathbf{y} < \mathbf{c}_{\hat{P}}(\epsilon),\ \mathbf{A}_{\hat{Z}}^T\mathbf{y} = \mathbf{c}_{\hat{Z}}(\epsilon)\}.$$

Therefore,

$$\Upsilon_{\hat{P}}(\mathbf{s}^*) = \{\lambda : \mathbf{b}(\lambda)^T\mathbf{h}_i^{\hat{Z}} = 0,\ i \in I,\ \mathbf{b}(\lambda)^T\mathbf{g}_j^{\hat{Z}} \geq 0,\ j \in J\}\times \{\epsilon : \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0,\ i \in I,\ \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0,\ j \in J\},$$

where, $\mathbf{h}_i^{\hat{Z}}$ $(i \in I)$ are the basis vectors of

$$L_1 = \{\mathbf{u} : \mathbf{A}_{\hat{Z}}^T\mathbf{u} = 0\},$$

and $\mathbf{g}_j^{\hat{Z}}$ $(j \in J)$ are the extreme directions of

$$S_1 = \{\mathbf{u} : \mathbf{A}_{\hat{Z}}^T\mathbf{u} \geq 0\} \cap L_1^{\perp},$$

respectively. Also $\mathbf{h}_i$ $(i \in I)$ are the basis vectors of

$$L_2 = \{\mathbf{v} : \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} = 0\},$$

and $\mathbf{g}_j$ $(j \in J)$ are the extreme directions of

$$S_2 = \{\mathbf{v} : \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} \geq 0\} \cap L_2^{\perp}.$$

This is matched with Theorem 2.1 in [11].

Remark 3.3. Let $Q = 0$ and $\epsilon = 0$. In this case, the set of the dual optimal solutions are invariant [14]. Therefore,

$$\Upsilon_{\hat{P}}(\mathbf{s}^*) = \{\lambda : \mathbf{b}(\lambda)^T \mathbf{h}_i^{\hat{Z}} = 0,\ i \in I,\ \mathbf{b}(\lambda)^T \mathbf{g}_j^{\hat{Z}} \geq 0,\ j \in J\},$$

where, $\mathbf{h}_i^{\hat{Z}}$ $(i \in I)$ and $\mathbf{g}_j^{\hat{Z}}$ $(j \in J)$ are the same as Remark 3.2.

Remark 3.4. Let $Q = 0$ and $\lambda = 0$. In this case, the optimal solution set of the primal problem is invariant [14]. Therefore,

$$\Upsilon_{\hat{P}}(\mathbf{s}^*) = \{\epsilon : \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0,\ i \in I,\ \mathbf{c}(\epsilon)^T \mathbf{g}_j > 0,\ j \in J\},$$

where, $\mathbf{h}_i$ $(i \in I)$ and $\mathbf{g}_j$ $(j \in J)$ are the same as Remark 3.2.

3.2 Support Set Sensitivity Analysis for the Primal-Dual Problems

Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{s}^*)$ be a primal-dual optimal solution of (QP) and (QD) with $P = \sigma(\mathbf{x}^*)$, $\hat{P} = \sigma(\mathbf{s}^*)$, $Z = \{1, \ldots, n\} \setminus P$ and $\hat{Z} = \{1, 2, \ldots, n\} \setminus \hat{P}$. Consider the partitions $(\hat{P}, \hat{Z})$ and (P, Z) of the index set $\{1, 2, \ldots, n\}$ for matrices $\mathbf{Q}$, $\mathbf{A}$ and vectors $\mathbf{x}, \mathbf{c}$ and $\mathbf{s}$ as follows:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{\hat{P}P} & \mathbf{Q}_{\hat{P}Z} \\ \mathbf{Q}_{\hat{Z}P} & \mathbf{Q}_{\hat{Z}Z} \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} \mathbf{A}_P & \mathbf{A}_Z \end{pmatrix}, \quad \mathbf{A}^T = \begin{pmatrix} \mathbf{A}_{\hat{P}}^T \\ \mathbf{A}_{\hat{Z}}^T \end{pmatrix},$$
$$\mathbf{c} = \begin{pmatrix} \mathbf{c}_{\hat{P}} \\ \mathbf{c}_{\hat{Z}} \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \mathbf{x}_P \\ \mathbf{x}_Z \end{pmatrix} \text{ and } \quad \mathbf{s} = \begin{pmatrix} \mathbf{s}_{\hat{P}} \\ \mathbf{s}_{\hat{Z}} \end{pmatrix}. \tag{3.8}$$

We want to identify the set $\Upsilon(\mathbf{x}^*, \mathbf{s}^*)$.

Theorem 3.5. *Let* $(\mathbf{h}_i^P, \mathbf{h}_i)$, $i \in I$, *be a basis of the lineality space*

$$L = \{(\mathbf{u}, \mathbf{v}) : \mathbf{A}_P^T \mathbf{u} - \mathbf{Q}_{\hat{P}P}^T \mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}P}^T \mathbf{v}_{\hat{Z}} = 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} = 0\},$$

and let $(\mathbf{g}_j^P, \mathbf{g}_j)$, $j \in J$, *be all extreme directions of the convex polyhedron cone*

$$S = \{(\mathbf{u}, \mathbf{v}) : \mathbf{A}_P^T \mathbf{u} - \mathbf{Q}_{\hat{P}P}^T \mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}P}^T \mathbf{v}_{\hat{Z}} \geq 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} \geq 0\} \cap L^{\perp}.$$

Then

$$\Upsilon(\mathbf{x}^*, \mathbf{s}^*) = \{(\lambda, \epsilon) : \mathbf{b}(\lambda)^T \mathbf{h}_i^P + \mathbf{c}(\epsilon)^T \mathbf{h}_i = 0,\ i \in I,$$
$$\mathbf{b}(\lambda)^T \mathbf{g}_j^P + \mathbf{c}(\epsilon)^T \mathbf{g}_j > 0,\ j \in J\}.$$

Proof. To prove, we identify the set of (λ, ϵ) such that support set of the primal-dual of the given solution remains invariant, that is,

$$\Upsilon(\mathbf{x}^*, \mathbf{s}^*) = \{(\lambda, \epsilon) : \exists (\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \ \text{ with } P = \sigma(\mathbf{x}),\ \sigma(\mathbf{s}) = \hat{P}\}$$

$$= \{(\lambda,\epsilon) : \exists(\mathbf{x},\mathbf{y},\mathbf{s}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \ \text{ s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}(\lambda),$$
$$\mathbf{A}^T\mathbf{y} + \mathbf{s} - \mathbf{Q}\mathbf{x} = \mathbf{c}(\epsilon), \mathbf{x},\mathbf{s} \geq 0,\ \mathbf{x}^T\mathbf{s} = 0,\ \sigma(\mathbf{x}) = P, \sigma(\mathbf{s}) = \hat{P}\}$$
$$= \{(\lambda,\epsilon) : \exists(\mathbf{x},\mathbf{y}) \in \mathcal{QPP}^* \times \mathcal{QDP}^* \ \text{ s.t. } \mathbf{A}_P\mathbf{x}_P = \mathbf{b}(\lambda),$$
$$\mathbf{A}_{\hat{Z}}^T\mathbf{y} - \mathbf{Q}_{\hat{Z}P}\mathbf{x}_P = \mathbf{c}_{\hat{Z}}(\epsilon),\ \mathbf{A}_{\hat{P}}^T\mathbf{y} - \mathbf{Q}_{\hat{P}P}\mathbf{x}_P < \mathbf{c}_{\hat{P}}(\epsilon),\ \mathbf{x}_P > 0\}.$$

Therefore, it is sufficient to determine the set of (λ,ϵ) for which the system

$$\begin{aligned} \mathbf{A}_P\mathbf{x}_P &= \mathbf{b}(\lambda) \\ \mathbf{A}_{\hat{Z}}^T\mathbf{y} - \mathbf{Q}_{\hat{Z}P}\mathbf{x}_P &= \mathbf{c}_{\hat{Z}}(\epsilon) \\ \mathbf{A}_{\hat{P}}^T\mathbf{y} - \mathbf{Q}_{\hat{P}P}\mathbf{x}_P &< \mathbf{c}_{\hat{P}}(\epsilon) \\ \mathbf{x}_P &> 0, \end{aligned} \tag{3.9}$$

is solvable. The system (3.9) is solvable if and only if the corresponding problem

$$\begin{aligned} \max \quad & 0^T\mathbf{y} + 0^T\mathbf{x}_P \\ s.t: \quad & \mathbf{A}_P\mathbf{x}_P = \mathbf{b}(\lambda) \\ & \mathbf{A}_{\hat{Z}}^T\mathbf{y} - \mathbf{Q}_{\hat{Z}P}\mathbf{x}_P = \mathbf{c}_{\hat{Z}}(\epsilon) \\ & \mathbf{A}_{\hat{P}}^T\mathbf{y} - \mathbf{Q}_{\hat{P}P}\mathbf{x}_P \leq \mathbf{c}_{\hat{P}}(\epsilon) - \eta_{\hat{P}} \\ & -\mathbf{x}_P \leq -\zeta_P, \end{aligned} \tag{3.10}$$

has an optimal solution for sufficiently small vectors $\eta_{\hat{P}} > 0$ and $\zeta_P > 0$. By duality theory in linear programming, it is equivalent to the optimality of the following dual problem

$$\begin{aligned} \min \quad & \big(\mathbf{b}(\lambda) - \mathbf{A}_P\zeta_P\big)^T\mathbf{u} + \big(\mathbf{c}(\epsilon)^T + \zeta_P^T[\mathbf{Q}_{\hat{P}P}^T, \mathbf{Q}_{\hat{Z}P}^T]\big)\mathbf{v} - \eta_{\hat{P}}^T\mathbf{v}_{\hat{P}} \\ s.t: \quad & \mathbf{A}_P^T\mathbf{u} - \mathbf{Q}_{\hat{P}P}^T\mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}P}^T\mathbf{v}_{\hat{Z}} \geq 0 \\ & \mathbf{A}\mathbf{v} = 0 \\ & \mathbf{v}_{\hat{P}} \geq 0. \end{aligned} \tag{3.11}$$

Now, let L denote the lineality space of the problem (3.11), that is,

$$L = \{(\mathbf{u},\mathbf{v}) : \mathbf{A}_P^T\mathbf{u} - \mathbf{Q}_{\hat{P}P}^T\mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}P}^T\mathbf{v}_{\hat{Z}} = 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} = 0\}\ .$$

Let $(\mathbf{h}_i^P, \mathbf{h}_i),\ i \in I$, denote the vectors of basis of L, and $(\mathbf{g}_j^P, \mathbf{g}_j),\ j \in J$, denote the extreme directions

$$S = \{(\mathbf{u},\mathbf{v}) : \mathbf{A}_P^T\mathbf{u} - \mathbf{Q}_{\hat{P}P}^T\mathbf{v}_{\hat{P}} - \mathbf{Q}_{\hat{Z}P}^T\mathbf{v}_{\hat{Z}} \geq 0,\ \mathbf{A}\mathbf{v} = 0,\ \mathbf{v}_{\hat{P}} \geq 0\} \cap L^\perp.$$

It follows from Theorems 2.4 and weak duality that

$$\begin{cases} \mathbf{b}(\lambda)^T\mathbf{h}_i^P + \mathbf{c}(\epsilon)^T\mathbf{h}_i - \zeta_P^T\big(\mathbf{A}_P^T\mathbf{h}_i^P - [\mathbf{Q}_{\hat{P}P}^T, \mathbf{Q}_{\hat{Z}P}^T]\mathbf{h}_i\big) - \eta_{\hat{P}}^T(\mathbf{h}_i)_{\hat{P}} = 0, & i \in I, \\ \mathbf{b}(\lambda)^T\mathbf{g}_j^P + \mathbf{c}(\epsilon)^T\mathbf{g}_j - \zeta_P^T\big(\mathbf{A}_P^T\mathbf{g}_j^P - [\mathbf{Q}_{\hat{P}P}^T, \mathbf{Q}_{\hat{Z}P}^T]\mathbf{g}_j\big) - \eta_{\hat{P}}^T(\mathbf{g}_j)_{\hat{P}} > 0, & j \in J. \end{cases} \tag{3.12}$$

Since $\mathbf{A}_P^T\mathbf{h}_i^P - [\mathbf{Q}_{\hat{P}P}^T, \mathbf{Q}_{\hat{Z}P}^T]\mathbf{h}_i = 0$, $\eta_{\hat{P}}^T(\mathbf{h}_i)_{\hat{P}} = 0$, $0 \neq \mathbf{A}_P^T\mathbf{g}_j^P - [\mathbf{Q}_{\hat{P}P}^T, \mathbf{Q}_{\hat{Z}P}^T]\mathbf{g}_j \geq 0$ and $\eta_{\hat{P}}^T(\mathbf{g}_j)_{\hat{P}} > 0$, it follows from (3.12) that

$$\begin{cases} \mathbf{b}(\lambda)^T\mathbf{h}_i^P + \mathbf{c}(\epsilon)^T\mathbf{h}_i = 0, & i \in I, \\ \mathbf{b}(\lambda)^T\mathbf{g}_j^P + \mathbf{c}(\epsilon)^T\mathbf{g}_j > 0, & j \in J. \end{cases} \tag{3.13}$$

Therefore, (3.13) describes the set $\Upsilon(\mathbf{x}^*, \mathbf{s}^*)$. □

Note: Similar remarks such as Remarks 3.2, 3.3 and 3.4 hold in this case.

Example 3.6. Consider the problem

$$\begin{array}{ll} \max & 2x_1 + 2x_2 + \frac{1}{2}(x_1 + x_2)^2 \\ s.t: & x_1 + x_2 + x_3 = 10 \\ & x_1 + 4x_2 + x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

It is easy to verify that $\mathbf{x}^* = (10, 0, 0, 0)^T, \mathbf{y}^* = (0, 12)^T, \mathbf{s}^* = (0, 36, 0, 12)^T$ is a primal-dual optimal solution with $\sigma(\mathbf{s}^*) = \{2, 4\}$. Let $\mathbf{b}(\lambda) = (10 - 2\lambda_1 + 3\lambda_2, 10 + \lambda_1 - 2\lambda_2)^T$ and $\mathbf{c}(\epsilon) = (2 - 2\epsilon_1 + \epsilon_2, 2 + \epsilon_1 - \epsilon_2, 0, 0)^T$. The lineality space and convex polyhedron cone are as follows:

$$L = \{(\mathbf{u}, \mathbf{v}) : u_1 + u_2 - v_1 - v_2 = 0, u_1 = 0,\ v_1 + v_2 + v_3 = 0, v_1 + 4v_2 + v_4 = 0, v_2 = v_4 = 0\},$$

$$S = \{(\mathbf{u}, \mathbf{v}) : u_1 + u_2 - v_1 - v_2 \geq 0, u_1 \geq 0,\ v_1 + v_2 + v_3 = 0,\ v_1 + 4v_2 + v_4 = 0,\ v_2, v_4 \geq 0\}.$$

Since $L = \{0\}$, there is no basis for the lineality space and the extreme directions of the set S are

$$g_1^{\hat{Z}} = (0, -1)^T, \quad g_1 = (-1, 0, 1, 1), \quad g_2^{\hat{Z}} = (0, -3)^T, \quad g_2 = (-4, 1, 3, 0)^T.$$

Therefore,we get

$$\Upsilon_{\hat{P}}(\mathbf{s}^*) = \{(\lambda, \epsilon) : -\lambda_1 + 2\lambda_2 + 2\epsilon_1 - \epsilon_2 < 12, \\ -3\lambda_1 + 6\lambda_2 + 9\epsilon_1 - 5\epsilon_2 < 36\}.$$

4 Conclusion

We studied multi-parametric sensitivity analysis for quadratic optimization under support set stability for the dual and the primal-dual problems as a generalization of linear optimization [11]. The resulting critical regions are determined by linear equations and linear inequalities or strict inequalities which represent polyhedral set. We stated them for linear optimization with simultaneously perturbations in the right-hand-side of the constraints and the objective coefficients, and compared them with independently perturbations.

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