Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 9, No. 1, pp. 61–75, January 2013

ALGORITHM FOR SOLVING THE SET-VALUED VARIATIONAL INEQUALITY PROBLEM IN EUCLIDEAN SPACE

Aviv Gibali

Abstract: In this paper we present an algorithm for solving the Variational Inequality Problem (VIP) with a point-to-set operator. In this algorithm we employ projections onto hyperplanes that separate balls from the feasible set of the VIP instead of projections onto the feasible set itself.

Key words: *paramonotone mapping, maximal monotone mapping, variational inequality, projection method, separating hyperplanes*

Mathematics Subject Classification: *47H04, 47H05, 47J20, 49J40, 90C30*

1 Introduction

In this paper we are concerned with the Variational Inequality Problem (VIP) in the Euclidean space $\mathbb{R}^n$ with a point-to-set operator. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex subset and $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ a point-to-set operator, where $\mathcal{P}(\mathbb{R}^n)$ is the power set of $\mathbb{R}^n$, i.e., the set of all the subsets of $\mathbb{R}^n$. The VIP for T and C, denoted by VIP(T, C), is to find a point $x^* \in C$ such that there exists $u^* \in T(x^*)$ satisfying

$$\langle u^*, x - x^* \rangle \geq 0 \text{ for all } x \in C. \tag{1.1}$$

We denote the solution set of (1.1) by SOL(T, C). The VIP with a single-valued operator was introduced by Hartman and G. Stampacchia [15]. It was well-studied in the last decades, see, e.g., the book of Kinderlehrer and Stampacchia [17], the treatise of Facchinei and Pang [12] and the review papers by Noor [18] and by Xiu and Zhang [19]. In particular, algorithmic approaches were investigated, using projections of different types, in order to generate a sequence of iterates that converges to a solution. See, e.g., Yang [21], Yamada and Ogura [20], Auslender and Teboulle [3] or Censor, Iusem and Zenios [10], to name but (very) few out of the existing vast literature. The importance of VIPs stems from the fact that some fundamental problems can be cast in this form, see, e.g., [12, Volume I, Subsection 1.4]. We present an iterative method for point-to-set paramonotone, maximal monotone operator $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$. Our algorithm is inspired by our δ-algorithmic scheme presented in [11]. The significance of this δ-algorithmic scheme with point-to-set paramonotone, maximal monotone operator is that it includes as special cases some earlier algorithms appearing in the literature. In addition, this algorithm employes projections onto any user chosen separating

ISSN 1348-9151

hyperplanes, therefore, beside the "freedom" in choosing this hyperplanes, the computation of these projections can be easily made.

Our paper is organized as follows. In Section 2 we present some earlier results for solving the VIP(T, C). Afterwards in Section 3 preliminaries are presented. In Section 4 the δ-algorithmic scheme is introduced and it is analyzed in Section 5. Finally, conclusions are presented in Section 6.

2 Relation with Previous Work

The literature on the VIP is vast, see, e.g., the treatise of Facchinei and Pang [12]. Some algorithms for solving the VIP with a single-valued operator $f : \mathbb{R}^n \to \mathbb{R}^n$ fit into the framework of the following general iterative scheme. Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex subset.

Algorithm 1

Initialization: Let $\{\tau_k\}_{k=0}^{\infty}$ be a user-chosen positive real sequence, select an arbitrary starting point $x^0 \in C$ and set the iteration index $k = 0$.
Iterative step: Given the current iterate x^k, calculate the next iterate

$$x^{k+1} = P_C(x^k - \tau_k f(x^k)), \tag{2.1}$$

where P_C is the orthogonal projection operator onto C.

See, Auslender [2] and consult [12, Volume 2, Subsection 12.1] for more details. Projections methods are particularly useful when the set C is simple enough to project on. However, in general, one has to solve at each iterative step the minimization problem

$$\min\{\|x - (x^k - \tau_k f(x^k))\| \mid \text{for all } x \in C\}. \tag{2.2}$$

The efficiency of such a projection method may be seriously affected by the need to solve such optimization problem at each iterative step.

An orthogonal projection of a point z onto a set C can be viewed as an orthogonal projection of z onto the hyperplane H which separates z from C, and supports C at the closest point to z in C. But, of course, at the time of performing such an orthogonal projection, neither the closest point to z in C, nor the separating and supporting hyperplane H are available. In view of the simplicity of an orthogonal projection onto a hyperplane, it is natural to ask whether one could use other separating supporting hyperplanes instead of that particular hyperplane H through the closest point to z. Aside from theoretical interest, this may lead to algorithms useful in practice, provided that the computational effort of finding such other hyperplanes favorably competes with the work involved in performing orthogonal projections directly onto the given sets.

To circumvent the difficulties associated with the orthogonal projections onto the feasible set of (1.1) Fukushima [14] developed a method that utilizes outer approximations of C. His method replaces the orthogonal projection onto the set C by a projection onto a half-space containing C, which is easier to calculate. Letting $C := \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ where $g : \mathbb{R}^n \to \mathbb{R}$ is convex, it is known that every convex set has this representation. Fukushima's iterative step is as follows.

Recently Iusem and Cruz presented a point-to-set version of Algorithms 1 and 2 with point-to-set paramonotone, maximal monotone operators. Observe that in Algorithm 2 the

Algorithm 2

Initialization: Let $\{\beta_k\}_{k=0}^{\infty}$ be a user-chosen positive real sequence, select an arbitrary starting point $x^0 \in \mathbb{R}^n$ and set the iteration index $k = 0$.
Iterative step: Given the current iterate x^k,
(1) choose a subgradient $\xi^k \in \partial g(x^k)$ of g at x^k and let

$$C_k := \{x \in \mathbb{R}^n \mid g(x^k) + \langle \xi^k, x - x^k \rangle \leq 0\}. \tag{2.3}$$

(2) Calculate the "shifted point"

$$z^k := \begin{cases} x^k - \beta_k f(x^k)/\|f(x^k)\|, & \text{if } f(x^k) \neq 0, \\ x^k, & \text{if } f(x^k) = 0. \end{cases} \tag{2.4}$$

and then the next iterate x^{k+1} is the projection of z^k onto the half-space C_k, namely,

$$x^{k+1} = P_{C_k}(z^k). \tag{2.5}$$

(3) If $x^{k+1} = x^k$ then stop, otherwise, set $k = k + 1$ and return to **(1)**.

---

bounding hyperplanes of the subgradiental half-spaces C_k, separate the current point z from the set C, the question again arises whether or not any other separating hyperplanes can be used in the algorithm while retaining the overall convergence to the solution. The answer to this question for the single-valued case is affirmative as it can be seen in our earlier work [11], and it holds under some not too restrictive conditions. Under these conditions, we showed that, as a matter of fact, the hyperplanes need to separate not just the point z from the feasible set of (1.1), but rather separate a "small" ball around z from C. This algorithm is called the δ-algorithmic scheme, our goal is to extend this algorithmic scheme for a point-to-set operators, i.e., solve (1.1). It appears that this structural algorithmic discovery for the point-to-set operator generalizes both Algorithms 1 and 2 with a point-to-set operators as it was presented in [6]. Our work is admittedly a theoretical development and no numerical advantages are claimed at this point. The large "degree of freedom" of choosing the super-sets, onto which the projections of the algorithm are performed, from a wide family of half-spaces may include specific algorithms that have not yet been explored. The construction of a δ-algorithmic scheme was originally introduced by Aharoni, Berman and Censor [1] ($\delta - \eta$ algorithm) for the Convex Feasibility Problem (CFP), see also [9, Chapte 5]. It was also applied to the Best Approximation Problem (BAP) by Bregman et al. in [7].

3 Preliminaries

Let $S \subseteq \mathbb{R}^n$ be a nonempty, closed and convex subset. For every point $x \in \mathbb{R}^n$ there exists a unique nearest point in S, denoted by $P_S(x)$, such that

$$\|x - P_S(x)\| \leq \|x - y\| \text{ for all } y \in S. \tag{3.1}$$

The operator $P_S : \mathbb{R}^n \to S$ is called the metric projection of $\mathbb{R}^n$ onto S or the orthogonal projection. It is well known that P_S is a nonexpansive operator of $\mathbb{R}^n$ onto S (see e.g. [16, Lemma 4.1]), i.e.,

$$\|P_S(x) - P_S(y)\| \le \|x - y\| \text{ for all } x, y \in \mathbb{R}^n. \tag{3.2}$$

The metric projection P_S is characterized by the following two properties:

$$P_S(x) \in S \tag{3.3}$$

and

$$\langle x - P_S(x), P_S(x) - y\rangle \ge 0 \text{ for all } x \in \mathbb{R}^n,\ y \in S, \tag{3.4}$$

and if S is a hyperplane, then (3.4) becomes an equality.

We denote by $\operatorname{dist}(x, S)$ the Euclidian distance of a point $x \in \mathbb{R}^n$ to the set S, i.e.,

$$\operatorname{dist}(x, S) := \min\{\|x - z\| \mid z \in S\}. \tag{3.5}$$

Definition 3.1. Let $A : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a point-to-set operator.

(i) A is called `monotone` if

$$\langle u - v, x - y\rangle \ge 0 \text{ for all } u \in A(x),\ v \in A(y). \tag{3.6}$$

(ii) A is called `psuedo-monotone` if

$$\langle v, x - y\rangle \ge 0 \ \Rightarrow\ \langle u, x - y\rangle \ge 0 \text{ for all } u \in A(x),\ u \in A(y). \tag{3.7}$$

(iii) A is called `paramonotone`, if it is monotone and whenever $\langle u - v, x - y\rangle = 0$, $u \in A(x)$, $v \in A(y)$ it holds that $u \in A(y)$, $v \in A(x)$.

(iv) A is called `maximal monotone` if it is monotone, and the graph $G(A)$ of A

$$G(A) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n \mid u \in A(x)\}, \tag{3.8}$$

is not properly contained in the graph of any other monotone operator.

Definition 3.2. The `domain` of a point-to-set operator $A : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is the set

$$\operatorname{Dom}(A) := \{x \in \mathbb{R}^n \mid A(x) \ne \emptyset\}. \tag{3.9}$$

The `range` of an operator A is the set

$$\operatorname{Ran}(A) = \{u \in A(x) \mid x \in \operatorname{Dom}(A)\}. \tag{3.10}$$

In next lemmas we present some properties of maximal monotone and paramonotone operators, these will have a central role in our convergence theorem.

Lemma 3.3. *Let $A : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a maximal monotone operator. Then*

(i) A is locally bounded at any point in the interior of its domain.

(ii) $G(A)$ is closed.

(iii) A is bounded on bounded subsets of the interior of its domain.

(iv) $\operatorname{SOL}(A, S)$ if nonempty, it is closed and convex.

Proof. (i) See [8, Theorem 4.6.1(ii)].

(ii) See [8, Theorem 4.2.1(ii)].

(iii) Follows easily from (i).

(iv) See [5, Lemma 2.4(ii)]. □

The following lemmas are quoted from [6, Lemma 3, 6].

Lemma 3.4. *Let $S \subseteq \mathbb{R}^n$ be a closed subset and $z \notin S$. Let $\{z^k\}_{k=0}^\infty \subseteq \mathbb{R}^n$ be such that $\lim_{k\to\infty} \|z^{k+1} - z^k\| = 0$ and both z and some point in S are cluster points of $\{z^k\}_{k=0}^\infty$. Then there exist $\varsigma > 0$ and a subsequence $\{z^{k_j}\}_{j=0}^\infty$ of $\{z^k\}_{k=0}^\infty$ such that*

$$\operatorname{dist}\left(z^{k_{j+1}}, S\right) > \operatorname{dist}\left(z^{k_j}, S\right), \tag{3.11}$$

and

$$\operatorname{dist}\left(z^{k_j}, S\right) > \varsigma. \tag{3.12}$$

Lemma 3.5. *Let $A : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be paramonotone and maximal monotone operator. Let $\{(y^k, u^k)\}_{k=0}^\infty \subset G(A)$ be a bounded sequence such that all cluster points of $\{y^k\}_{k=0}^\infty$ belong to S. Define the operator $\gamma_k : SOL(A, S) \to \mathbb{R}$, such as*

$$\gamma_k(x) := \langle u^k, y^k - x\rangle. \tag{3.13}$$

If for some $x \in SOL(A, S)$ there exists a subsequence $\{\gamma_{k_j}(x)\}_{j=0}^\infty$ of $\{\gamma_k(x)\}_{k=0}^\infty$ such that $\lim_{j\to\infty} \gamma_{k_j}(x) \le 0$, then there exists a cluster point of $\{y^{k_j}\}_{j=0}^\infty$ belong to $SOL(A, S)$.

The next lemma is quoted from [14, Lemma 2].

Lemma 3.6. *Let $\{\xi_k\}_{k=0}^\infty$ and $\{\nu_k\}_{k=0}^\infty$ be sequences of nonnegative numbers, and let $\mu \in [0, 1)$ be a constant. If the inequalities*

$$\xi_{k+1} \le \mu\xi_k + \nu_k \text{ for all } k \ge 0 \tag{3.14}$$

hold and if $\lim\limits_{k\to\infty} \nu_k = 0$, then $\lim\limits_{k\to\infty} \xi_k = 0$.

4 The δ-Algorithmic Scheme

Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be paramonotone and maximal monotone operator and $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex subset. For the convergence of our δ-algorithmic scheme we assume the following conditions.

Condition 4.1. $\mathrm{SOL}(T, C) \ne \emptyset$.

Condition 4.2. There exist $y \in C$ and a bounded set $D \subset \mathbb{R}^n$ such that

$$\langle u, x - y\rangle \ge 0, \text{ for all } x \notin D, \text{ and for all } u \in T(x). \tag{4.1}$$

In order to present the δ-algorithmic scheme a few definitions are needed.

Definition 4.3. Given $\delta \in [0, 1]$, and a point $x \in \mathbb{R}^n$,

(i) the ball centered at x with radius $\delta \operatorname{dist}(x, C)$ is

$$B(x, C, \delta) := B(x, \delta \operatorname{dist}(x, C)) = \{z \in \mathbb{R}^n \mid \|x - z\| \le \delta \operatorname{dist}(x, C)\}, \tag{4.2}$$

(ii) for any $x \notin \operatorname{int} C$, denote by $\mathcal{H}(x, C, \delta)$ the set of all hyperplanes which separate $B(x, C, \delta)$ from C,

(iii) for $x, y \in \mathbb{R}^n$, define the mapping

$$\mathcal{A}_{C,\delta}(x, y) := \begin{cases} \{x\}, & \text{if } x \in \operatorname{int} C, \\ \{P_{H^-}(y) \mid H \in \mathcal{H}(x, C, \delta)\}, & \text{if } x \notin \operatorname{int} C, \end{cases} \tag{4.3}$$

where P_{H^-} is the projection operator onto the half-space whose bounding hyperplane is H and such that $C \subseteq H^-$.

The mapping $\mathcal{A}$ defined above maps a quadruple (x, y, C, δ) onto a set. A *selection* from $\mathcal{A}_{C,\delta}(x, y)$ means that if $x \notin \operatorname{int} C$ a specific hyperplane $H \in \mathcal{H}(x, C, \delta)$ is chosen and $P_{H^-}(y)$ is selected. If $x \in \operatorname{int} C$ then x is selected.

Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k = \infty, \text{ and } \sum_{k=0}^{\infty} \beta_k^2 < \infty. \tag{4.4}$$

Our δ-algorithmic scheme for solving (1.1) is as follows.

Algorithm 3

Initialization: Let $\{\beta_k\}_{k=0}^{\infty}$ be a user-chosen positive real sequence that fulfills (4.4). Choose a constant $\delta \in (0, 1]$, select a starting point $x^0 \in C$ and set $k = 0$.
Iterative step: Given the current iterate x^k, if $0 \in T\left(x^k\right)$, then stop. Otherwise,
(1) take $u^k \in T\left(x^k\right)$, $u^k \neq 0$, and choose $\eta_k = \max\left\{1, \left\|u^k\right\|\right\}$
(2) calculate x^{k+1} as a selection from

$$\mathcal{A}_{C,\delta}\left(x^k, x^k - \frac{\beta_k}{\eta_k} u^k\right). \tag{4.5}$$

(3) If $x^{k+1} = x^k$, stop, otherwise, set $k = k + 1$ and return to **(1)**.

In what follows, we shall denote by P_k the projection operator onto H_k^- where H_k is the selected hyperplane $H_k \in \mathcal{H}(x^k, C, \delta)$. Thus, in (4.5) if we denote by $z^k = x^k - \beta_k/\eta_k u^k$, we get

$$x^{k+1} = \begin{cases} x^k, & \text{if } x^k \in \operatorname{int} C, \\ P_k(z^k), & \text{if } x^k \notin \operatorname{int} C. \end{cases} \tag{4.6}$$

The iterative step of this algorithmic scheme is illustrated in Figure 1.

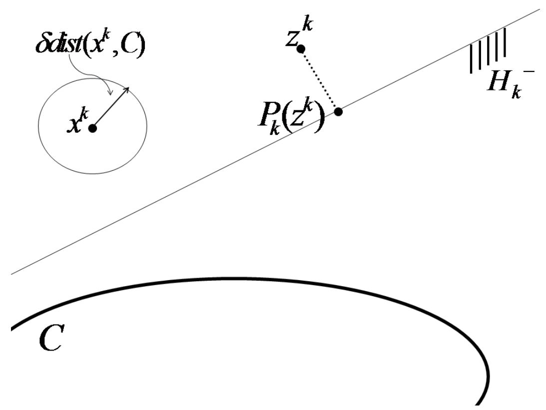

Figure 1: Illustration of the iterative step of Algorithm 3.

Remark 4.4. Observe that there is no need to calculate in practice the radius $\delta \operatorname{dist}(x^k, C)$ of the ball $B(x^k, C, \delta)$. If there would have been a need to calculate this then it would, obviously, amount to preforming a projection of x^k onto C, which is the very thing that we are trying to circumvent. All that is needed, when deriving from the algorithmic scheme a specific algorithm, is to show that the specific algorithm indeed “chooses” the hyperplanes in concert with the requirement of separating such $B(x^k, C, \delta)$ balls from the feasible set of (1.1). We demonstrate this later on.

5 Convergence

In this section we establish the convergence theorem for Algorithm 3. We divide our proof into several lemmas, similar as in [6]. The next Lemma is quoted from [11, Lemma 14].

Lemma 5.1. *Let $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex subset, and let $\delta \in (0, 1]$. Let $W \subseteq \mathbb{R}^n$ be a nonempty, convex and compact subset, let $W \backslash C := \{x \in W \mid x \notin C\}$. Denote by $\theta(x)$ a selection from $\mathcal{A}_{C,\delta}(x, x)$, then there exists a constant $\mu \in [0, 1)$ such that*

$$\operatorname{dist}(\theta(x), C) \leq \mu \operatorname{dist}(x, C), \text{ for all } x \in W \backslash C. \tag{5.1}$$

Now we prove that if Algorithm 3 stops then it has reached a solution of the VIP (1.1).

Theorem 5.2. *If $x^{k+1} = x^k$ occurs for some $k \geq 0$ in Algorithm 3, then $x^k \in SOL(T, C)$.*

Proof. Suppose that $x^{k+1} = x^k$, then the radius of $B(x^k, C, \delta)$ is zero which implies that $x^k \in C$ since $\delta > 0$. By the characterization of the metric projection with respect to H_k^- ((3.4)), we get

$$\left\langle \left(x^k - \frac{\beta_k}{\eta_k} u^k \right) - x^{k+1}, w - x^{k+1} \right\rangle \leq 0 \text{ for all } w \in H_k^-. \tag{5.2}$$

By taking $x^{k+1} = x^k$ in (5.2), we obtain

$$\left\langle -\frac{\beta_k}{\eta_k} u^k, w - x^k \right\rangle \leq 0 \text{ for all } w \in H_k^-. \tag{5.3}$$

Since $\beta_k > 0$, $\eta_k > 0$ and $C \subseteq H_k^-$ we get that

$$\langle u^k, w - x^k \rangle \geq 0 \text{ for all } w \in C, \tag{5.4}$$

meaning that $x^k \in \mathrm{SOL}(T, C)$. □

In the remainder of this section we suppose that Algorithm 3 generates an infinite sequence $\{x^k\}_{k=0}^\infty$. The next lemmas are central for the convergence theorem of Algorithm 3, the proof follows similar lines as in [6].

Lemma 5.3. *Let y and D be as in Condition 4.2, choose $\lambda > 0$ such that $\left\|x^0 - y\right\| \leq \lambda$, and $D \subseteq B(y, \lambda)$. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm 3 have the following properties.*

(i) *if $x^k \in D$ then $\|x^{k+1} - y\|^2 \leq \lambda^2 + \beta_k^2 + 2\beta_k\lambda$,*

(ii) *if $x^k \notin D$ then $\|x^{k+1} - y\|^2 \leq \|x^k - y\|^2 + \beta_k^2$.*

Proof. Since $y \in C$, and $C \subseteq H_k^-$ it follows that $y \in H_k^-$ for all $k \geq 0$, i.e., $y = P_k(y)$. Due to the nonexpansivness of the operator P_k with respect to H_k^-, we get

$$\begin{aligned}
\|x^{k+1} - y\|^2 &= \left\| P_k \left(x^k - \frac{\beta_k}{\eta_k} u^k \right) - P_k(y) \right\|^2 \leq \left\| x^k - \frac{\beta_k}{\eta_k} u^k - y \right\|^2 \\
&= \left\| x^k - y \right\|^2 + \left(\frac{\beta_k}{\eta_k} \right)^2 \left\| u^k \right\|^2 - 2 \frac{\beta_k}{\eta_k} \langle u^k, x^k - y \rangle \\
&\leq \left\| x^k - y \right\|^2 + \beta_k^2 - 2 \frac{\beta_k}{\eta_k} \langle u^k, x^k - y \rangle. \qquad (5.5)
\end{aligned}$$

Consider the following two cases:

(i) if $x^k \in D$, apply the Cauchy-Schwartz inequality, the definition of η_k and the assumption that $D \subseteq B(y, \lambda)$ to (5.5) and obtain that

$$\begin{aligned}
\|x^{k+1} - y\|^2 &\leq \left\| x^k - y \right\|^2 + \beta_k^2 + 2 \frac{\beta_k}{\eta_k} \left\| u^k \right\| \left\| x^k - y \right\| \\
&\leq \lambda^2 + \beta_k^2 + 2 \beta_k \lambda. \qquad (5.6)
\end{aligned}$$

(ii) if $x^k \notin D$, by Condition 4.2, we get that $\langle u^k, x^k - y \rangle \geq 0$. In addition, since $\beta_k / \eta_k > 0$ we obtain from (5.5) that

$$\left\| x^{k+1} - y \right\|^2 \leq \left\| x^k - y \right\|^2 + \beta_k^2, \qquad (5.7)$$

as asserted. ☐

Lemma 5.4. *Assume that Condition 4.2 hold. Let* $\{x^k\}_{k=0}^\infty$ *and* $\{u^k\}_{k=0}^\infty$ *be any two sequences generated by Algorithm 3. Then,*

(i) *the sequences* $\{x^k\}_{k=0}^\infty$ *and* $\{u^k\}_{k=0}^\infty$ *are bounded,*

(ii) $\lim_{k \to \infty} \operatorname{dist}(x^k, C) = 0$,

(iii) $\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0$,

(iv) *all cluster points of* $\{x^k\}_{k=0}^\infty$ *belong to* C.

Proof. (i) Let the point y and the set D be as in Condition 4.2, choose $\lambda > 0$ and $\overline{\beta} > 0$ such that $\|x^0 - y\| \leq \lambda$, $D \subseteq B(y, \lambda)$ and $\beta_k \leq \overline{\beta}$, for all $k \geq 0$. Observe that the existence of $\overline{\beta}$ is guaranteed by (4.4). Denote by $\sigma := \sum_{k=0}^\infty \beta_k^2$ and $\overline{\lambda} := \sqrt{\lambda^2 + \sigma + 2\overline{\beta}\lambda}$. We will prove the boundedness of $\{x^k\}_{k=0}^\infty$ by showing that

$$\{x^k\}_{k=0}^\infty \subseteq B(y, \overline{\lambda}). \qquad (5.8)$$

Consider the two cases:

(i) If $x^k \in B(y, \lambda)$ then $x^k \in B(y, \overline{\lambda})$ since $\overline{\lambda} > \lambda$.

(ii) If $x^k \notin B(y, \lambda)$, denote by $\ell(k) = \max \left\{ \ell < k \mid x^\ell \in B(y, \lambda) \right\}$, which is well defined since $\|x^0 - y\| \leq \lambda$, i.e., $x^0 \in B(y, \lambda)$. Using Lemma 5.3(i) to obtain

$$\left\| x^{\ell(k)+1} - y \right\|^2 \leq \lambda^2 + \beta_{\ell(k)}^2 + 2\beta_{\ell(k)} \lambda \leq \lambda^2 + \beta_{\ell(k)}^2 + 2\overline{\beta}\lambda. \qquad (5.9)$$

Now, for $\ell(k) + 1 < j \leq k - 1$, $x^j \notin D$, we get

$$\left\| x^{j+1} - y \right\|^2 \leq \left\| x^j - y \right\|^2 + \beta_j^2. \qquad (5.10)$$

Summing up (5.9) with $\ell(k)+1<j\leq k-1$,

$$\left\|x^k-y\right\|^2 \leq \left\|x^{\ell(k)+1}-y\right\|^2 + \sum_{j=\ell(k)+1}^{k-1} \beta_j^2. \tag{5.11}$$

Combining inequalities (5.9) and (5.11) yield

$$\begin{aligned}\left\|x^k-y\right\|^2 &\leq \lambda^2 + \sum_{j=\ell(k)}^{k-1} \beta_j^2 + 2\overline{\beta}\lambda \leq \lambda^2 + \sum_{j=0}^{\infty} \beta_j^2 + 2\overline{\beta}\lambda \\ &= \lambda^2+\sigma+2\overline{\beta}\lambda = \overline{\lambda}^2. \end{aligned}\tag{5.12}$$

Therefore, $x^k \in B(y,\overline{\lambda})$ which implies that the sequence $\{x^k\}_{k=0}^{\infty}$ is bounded. Since $\{x^k\}_{k=0}^{\infty}$ is bounded, it follows by Lemma 3.3(iii) that so is $\{u^k\}_{k=0}^{\infty}$.

(ii) Observe that

$$\begin{aligned}\|x^{k+1}-P_k(x^k)\| &= \left\|P_k\left(x^k-\frac{\beta_k}{\eta_k}u^k\right)-P_k(x^k)\right\| \\ &\leq \frac{\beta_k}{\eta_k}\|u^k\| \leq \beta_k \end{aligned}\tag{5.13}$$

for all $k\geq 0$. By Lemma 5.1 with $W=B(y,\overline{\lambda})$ there exists $\widetilde{\mu}\in[0,1)$ such that

$$\operatorname{dist}(P_k(x),C) \leq \widetilde{\mu}\operatorname{dist}(x,C), \text{ for all } x\in B(y,\overline{\lambda})\backslash C. \tag{5.14}$$

So, for all $k\geq 0$ such that $x^k\notin C$, (5.8) implies that

$$\operatorname{dist}(P_k(x^k),C)\leq \widetilde{\mu}\operatorname{dist}(x^k,C). \tag{5.15}$$

If $x^k\in C$ then $\widetilde{\mu}=0$ since $C\subseteq H_k^-$. Denote by $c_k=P_C\left(P_k\left(x^k\right)\right)$, namely,

$$\|P_k(x^k)-c_k\| = \operatorname{dist}(P_k(x^k),C). \tag{5.16}$$

Then, by the triangle inequality, we get

$$\begin{aligned}\|x^{k+1}-c_k\| &= \|x^{k+1}-P_k(x^k)+P_k(x^k)-c_k\| \\ &\leq \|x^{k+1}-P_k(x^k)\|+\|P_k(x^k)-c_k\|. \end{aligned}\tag{5.17}$$

Since $c_k\in C$, we have

$$\operatorname{dist}(x^{k+1},C)\leq \|x^{k+1}-c_k\|. \tag{5.18}$$

It follows from (5.13), (5.15)–(5.18) that

$$\begin{aligned}\operatorname{dist}(x^{k+1},C) &\leq \|x^{k+1}-P_k(x^k)\|+\operatorname{dist}(P_k(x^k),C) \\ &\leq \beta_k+\widetilde{\mu}\operatorname{dist}(x^k,C). \end{aligned}\tag{5.19}$$

Therefore, by applying Lemma 3.6 with $\xi_k=\operatorname{dist}(x^k,C)$, $\nu_k=\beta_k$ and $\mu=\widetilde{\mu}$, we get that

$$\lim_{k\to\infty}\operatorname{dist}(x^k,C)=0. \tag{5.20}$$

(iii) By (5.13) and the triangle inequality, we have

$$\begin{aligned}\|x^{k+1}-x^k\| &\leq \left\|x^{k+1}-P_k(x^k)\right\|+\left\|P_k(x^k)-x^k\right\| \\ &\leq \beta_k+\operatorname{dist}(x^k,C). \end{aligned} \tag{5.21}$$

Since $\lim_{k\to\infty}\beta_k=0$ by (ii) and (5.21) we get that $\lim_{k\to\infty}\|x^{k+1}-x^k\|=0$.

(iv) Follows immediately from (ii). □

Theorem 5.5. *Let $T:\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ be a paramonotone operator. Assume that Condition 4.1 holds and let $\{x^k\}_{k=0}^\infty$ be any sequence generated by Algorithm 3. Then any cluster point of $\{x^k\}_{k=0}^\infty$ belongs to $SOL(T,C)$.*

Proof. Let $\{x^k\}_{k=0}^\infty$ and $\{u^k\}_{k=0}^\infty$ be any two sequences generated by Algorithm 3 and let the operator γ_k be as (3.13). Using the facts that $C\subseteq H_k^-$, P_k is nonexpansivness and the definition of γ_k we have that for all $x^*\in\mathrm{SOL}(T,C)$

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &= \left\|P_k\left(x^k-\frac{\beta_k}{\eta_k}u^k\right)-P_k(x^*)\right\|^2 \leq \left\|\left(x^k-\frac{\beta_k}{\eta_k}u^k\right)-x^*\right\|^2 \\ &= \|x^k-x^*\|^2+\left(\frac{\beta_k}{\eta_k}\right)^2\left\|u^k\right\|^2-2\frac{\beta_k}{\eta_k}\langle u^k,x^k-x^*\rangle \\ &\leq \|x^k-x^*\|^2-\beta_k\left(2\frac{\gamma_k(x^*)}{\eta_k}-\beta_k\right). \end{aligned} \tag{5.22}$$

Since $\{x^k\}_{k=0}^\infty$ and $\{u^k\}_{k=0}^\infty$ are bounded, so is $\{(x^k,u^k)\}_{k=0}^\infty$. Therefore by Lemma 3.5 it is suffices to prove that $\{\gamma_k(x^*)\}_{k=0}^\infty$ has a non-positive cluster point for some $x^*\in\mathrm{SOL}(T,C)$. Assume, by negation, that this is not true, and take any $\overline{x}\in\mathrm{SOL}(T,C)$. Then there exists $\overline{k}\geq 0$ and $\rho>0$ such that

$$\gamma_k(\overline{x})\geq\rho \text{ for all } k\geq\overline{k}. \tag{5.23}$$

Since $\{u^k\}_{k=0}^\infty$ is bounded, there exists $M>1$ such that $\|u^k\|\leq M$ for all $k\geq 0$. Therefore

$$\eta_k=\max\left\{1,\left\|u^k\right\|\right\}\leq\max\{1,M\}=M \text{ for all } k\geq 0. \tag{5.24}$$

Thus, there exists $\overline{\rho}>0$ such that

$$\frac{\gamma_k(\overline{x})}{\eta_k}\geq\frac{\gamma_k(\overline{x})}{M}\geq\overline{\rho}, \tag{5.25}$$

applying this with $\overline{x}\in\mathrm{SOL}(T,C)$ to (5.22)

$$\|x^{k+1}-\overline{x}\|^2\leq\|x^k-\overline{x}\|^2-\beta_k\left(2\overline{\rho}-\beta_k\right) \text{ for all } k\geq\overline{k}. \tag{5.26}$$

By (4.4) $\lim_{k\to\infty}\beta_k=0$, then there exists $k^{'}\geq\overline{k}$ such that

$$\beta_k\leq\overline{\rho} \text{ for all } k^{'}\geq\overline{k}. \tag{5.27}$$

So, we get for all $k\geq k^{'}$

$$\overline{\rho}\beta_k\leq\|x^k-\overline{x}\|^2-\|x^{k+1}-\overline{x}\|^2. \tag{5.28}$$

Summing up (5.28) with $m\geq k\geq k^{'}$ and deduce that

$$\begin{aligned}\overline{\rho}\sum_{k=k'}^m\beta_k &\leq\sum_{k=k'}^m\left(\|x^k-\overline{x}\|^2-\|x^{k+1}-\overline{x}\|^2\right) \\ &\leq\|x^{k^{'}}-\overline{x}\|^2-\|x^{m+1}-\overline{x}\|^2\leq\|x^{k^{'}}-\overline{x}\|^2. \end{aligned} \tag{5.29}$$

By taking the limit as $m \to \infty$ in (5.29) we contradict (4.4). Therefore, there exists a cluster point of $\{x^k\}_{k=0}^{\infty}$ belonging to $\mathrm{SOL}(T, C)$.

Now in order to show that all cluster points of $\{x^k\}_{k=0}^{\infty}$ belong to $\mathrm{SOL}(T, C)$, suppose that this is not true, i.e., there exists a cluster point z of $\{x^k\}_{k=0}^{\infty}$ such that $z \notin \mathrm{SOL}(T, C)$. By Lemmas 3.3(iv) and 5.4(iii) we get that $\mathrm{SOL}(T, C)$ is closed and $\lim_{k\to\infty} \|x^{k+1}-x^k\| = 0$, so by using Lemma 3.4, we can obtain a subsequence $\{x^{k_j}\}_{j=0}^{\infty}$ of $\{x^k\}_{k=0}^{\infty}$ and a real number $\varsigma > 0$ such that

$$\operatorname{dist}\left(x^{k_{j+1}}, SOL(T, C)\right) > \operatorname{dist}\left(x^{k_j}, SOL(T, C)\right), \tag{5.30}$$

and

$$\operatorname{dist}\left(x^{k_j}, SOL(T, C)\right) > \varsigma. \tag{5.31}$$

Let the operator $\gamma_k(x)$ be as in (3.13) and define the operator $\gamma : \mathrm{SOL}(T, C) \to \mathbb{R}$ as

$$\gamma(x^*) := \liminf_{j\to\infty} \gamma_{k_j}(x^*). \tag{5.32}$$

By Lemma 5.4(ii), $\{\gamma_k(x^*)\}_{k=0}^{\infty}$ is bounded. Next, we prove that γ is continuous and actually $\gamma : \mathrm{SOL}(T, C) \to (0, \infty)$. Take $x^*, x^{'} \in \mathrm{SOL}(T, C)$. Note that

$$\begin{aligned}\gamma_{k_j}(x^*) &= \langle u^{k_j}, x^{k_j} - x^*\rangle = \langle u^{k_j}, x^{k_j} - x^{'}\rangle + \langle u^{k_j}, x^{'} - x\rangle \\ &\le \gamma_{k_j}(x^{'}) + M\|x^* - x^{'}\|. \end{aligned}\tag{5.33}$$

Thus, $\gamma(x^*) \le \gamma(x^{'}) + M\|x^* - x^{'}\|$, where M is a upper bound of $\{\|u^k\|\}_{k=0}^{\infty}$. Now by reversing the role of $x^*, x^{'}$, we obtain

$$\mid \gamma(x^*) - \gamma(x^{'}) \mid \le M\|x^* - x^{'}\|, \tag{5.34}$$

meaning that γ is continuous. Now, in order to show that $\gamma(x^*) > 0$ for all $x^* \in \mathrm{SOL}(T, C)$, assume that this is not true, then by Lemma 3.5 $\{x^{k_j}\}_{j=0}^{\infty}$ has a cluster point in $\mathrm{SOL}(T, C)$, in contradiction with (5.31). Now, denote by U the set of cluster points of $\{x^k\}_{k=0}^{\infty}$. We prove that $U \subseteq \mathrm{SOL}(T, C)$. By the above arguments, $U\cap \mathrm{SOL}(T, C) \neq \emptyset$ and since $\{x^k\}_{k=0}^{\infty}$ is bounded, the sets U and $U\cap \mathrm{SOL}(T, C)$ are compact. By the continuity of γ, it follows that there exists $x^* \in U\cap \mathrm{SOL}(T, C)$ such that

$$\gamma(x^{'}) \ge \gamma(x^{^*}) > 0 \text{ for all } x^{'} \in U \cap \mathrm{SOL}(T, C). \tag{5.35}$$

By (5.32) and (4.4) there exists $\widehat{j}$ such that for all indexes $j \ge \widehat{j}$, we have

$$\gamma_{k_j}(x^{'}) \ge \frac{\gamma(x^{^*})}{2}, \tag{5.36}$$

and

$$\beta_{k_j} < \frac{\gamma(x^{^*})}{M}. \tag{5.37}$$

In view of (5.22), using (5.36) and (5.37), we get, for all $x \in U\cap\mathrm{SOL}(T, C)$ and all $j \ge \widehat{j}$,

$$\begin{aligned}\|x^{k_{j+1}} - x^*\|^2 &\le \|x^{k_j} - x^*\|^2 - \beta_{k_j}\left(2\frac{\gamma_{k_j}(x^*)}{\eta_{k_j}} - \beta_{k_j}\right) \\ &\le \|x^{k_j} - x^*\|^2 - \beta_{k_j}\left(\frac{\gamma(x^*)}{M} - \beta_{k_j}\right) < \|x^{k_j} - x^*\|^2. \end{aligned}\tag{5.38}$$

So, it follows that

$$\operatorname{dist}\left(x^{k_{j+1}}, U \cap SOL(T,C)\right) \leq \operatorname{dist}\left(x^{k_j}, U \cap SOL(T,C)\right)$$

for all $j \geq \widehat{j}$, in contradiction with (5.30). Therefore all clusters points of $\{x^k\}_{k=0}^{\infty}$ solve the VIP(T,C). $\square$

In the next theorem we summarize the convergence sequence properties of Algorithm 3.

Theorem 5.6. *Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be paramonotone and maximal monotone operator. Assume that Condition 4.1 holds and let $\{x^k\}_{k=0}^{\infty}$ be any sequence generated by Algorithm 3. Then $\{x^k\}_{k=0}^{\infty}$ is bounded, $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$ and all cluster points of $\{x^k\}_{k=0}^{\infty}$ belong to $SOL(T,C)$. If the VIP(T,C) has a unique solution then the whole sequence $\{x^k\}_{k=0}^{\infty}$ converge to it.*

Remark 5.7. In [14] for the single-valued case, convergence is proved under continuity, strongly monotonicity and the following condition on $T : \mathbb{R}^n \to \mathbb{R}^n$.

There exist $y \in C$, $\beta > 0$ and a bounded set $D \subset \mathbb{R}^n$ such that

$$\langle T(x), x - y\rangle \geq \beta \|T(x)\| \text{ for all } x \notin D. \tag{5.39}$$

So, it can be easily verified that our Condition 4.2 is weaker than (5.39) (see [6] for more details). In addition continuity and strong monotonicity imply uniqueness of the solution to VIP(T,C), and also strict monotonicity of T, therefore in this case, according to Theorem 5.6 any sequence $\{x^k\}_{k=0}^{\infty}$ generated by Algorithm 3 converge to it.

5.1 Special cases of the δ-algorithmic scheme

We now recall the example given in [11] as an illustration that additional algorithms can be derived from Algorithm 3. This particular realization requires that (the interior) $\operatorname{int} C$ is nonempty. The idea of using an interior point as an anchor to generate a separating hyperplane appeared previously in [1] for the Convex Feasibility Problem and in [13] for an outer approximation method. Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a point-to-set operator and $C \subseteq \mathbb{R}^n$ be a nonempty, closed and convex subset.

Algorithm 4

Initialization: Let $y \in \operatorname{int} C$ be fixed and given. Select an arbitrary starting point $x^0 \in \mathbb{R}^n$ and set $k = 0$.
Iterative step: Given the current iterate x^k, if $0 \in T\left(x^k\right)$, then stop. If $x^k \in C$ set $x^{k+1} = x^k$ and again stop. Otherwise,
(1) take $u^k \in T\left(x^k\right)$, $u^k \neq 0$, and choose $\eta_k = \max\left\{1, \left\|u^k\right\|\right\}$
(2) calculate the "shifted point"

$$z^k = x^k - \frac{\beta_k}{\eta_k} u^k \tag{5.40}$$

and construct the line L_k through the points x^k and y.
(3) Denote by w^k the point closet to x^k in the set $L_k \cap C$.
(4) Construct a hyperplane H_k separating x^k from C and supporting C at w^k.
(5) Compute $x^{k+1} = P_{H_k^-}(z^k)$, where H_k^- is the half-space whose bounding hyperplane is H_k and $C \subseteq H_k^-$, set $k = k + 1$ and return to **(1)**.

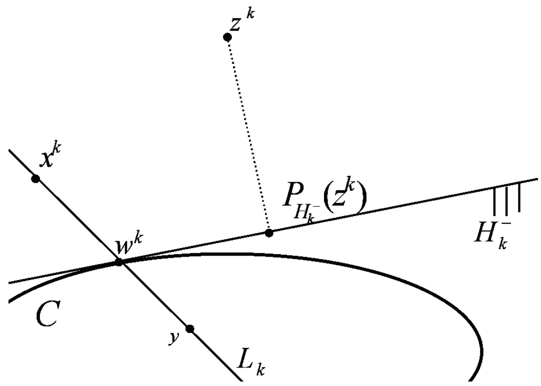

Figure 2: Illustration of the iterative step of Algorithm 4: Interior anchor point.

The iterative step of this algorithm is illustrated in Figure 2. We show that Algorithm 4 generates sequences that converge to a solution of problem (1.1) by showing that it is a special case of Algorithm 3.

Theorem 5.8. *Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be paramonotone and maximal monotone operator, and assume that Conditions 4.1 and 4.2 hold and* $\operatorname{int} C \neq \emptyset$. *Then any sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 4, converges to $x^* \in SOL(T, C)$.*

Proof. Algorithm 4 is obviously a special case of Algorithm 3 where we choose at each step a separating hyperplane which also supports C at the point w^k. The stopping criterion is valid by Theorem 5.2. In order to invoke Theorem 5.6 we have to show that for such an algorithm $\delta \in (0, 1]$ always holds. By Lemma 5.4, $\{x^k\}_{k=0}^{\infty}$ is bounded and, since $x^k \notin C$, we have

$$\|P_{H_k^-}(x^k) - x^k\| = \frac{\|x^k - w^k\| \|y - P_{H_k}(y)\|}{\|y - w^k\|}, \tag{5.41}$$

and we also have

$$\|x^k - w^k\| \geq \operatorname{dist}(x^k, C). \tag{5.42}$$

Defining $d := \operatorname{dist}(y, \operatorname{bd} C)$, where $\operatorname{bd} C$ is the boundary of C. Since $y \in \operatorname{int} C$,

$$\|y - P_{H_k}(y)\| \geq d > 0. \tag{5.43}$$

From the boundedness of $\{x^k\}_{k=0}^{\infty}$ we know that there exists a positive N such that $\|y - w^k\| \leq N$, for all $k \geq 0$. Combining these inequalities with (5.41) implies that

$$\|P_{H_k^-}(x^k) - x^k\| \geq \frac{d}{N} \operatorname{dist}(x^k, C),$$

which shows that the algorithm is of the same type of Algorithm 3 with $\delta := d/N > 0$. We can choose N big enough so that $N > d$ and then $\delta \in (0, 1]$ as required. □

6 Conclusions

In this paper we proposed an algorithmic scheme for solving the variational inequality problem in the Euclidean space $\mathbb{R}^n$ with a point-to-set operator T. This scheme, which we call the δ-algorithmic scheme entails a large “degree of freedom” of choosing the half-spaces, onto which the projections of the algorithm are performed, from a wide family of half-spaces; this “degree of freedom”, besides including some existing results may include specific algorithms that have not yet been explored. The convergence of our algorithmic scheme is guarantee under paramonotonicity and maximal monotonicity of T. It is known that there exist projection algorithms for set-valued variational inequality that require only pseudo-monotonicity of of T in order of the whole sequence to converges to a solution, (see e.g., Bao and Khanh [4]) but these algorithms require projections on the C which in general is hard to obtain.

Acknowledgments

We are grateful to an anonymous referee for many helpful comments on an earlier version of the paper.

References

[1] R. Aharoni, A. Berman and Y. Censor, An interior points algorithm for the convex feasibility problem, *Advances in Applied Mathematics* 4 (1983) 479–489.

[2] A. Auslender, *Optimisation: Méthodes Numériques*, Masson, Paris, 1976.

[3] A. Auslender and M. Teboulle, Interior projection-like methods for monotone variational inequalities, *Mathematical Programming*, Series A, 104 (2005) 39–68.

[4] T. Q. Bao and P. Q. Khanh, A projection-type algorithm for pseudo-monotone non-Lipschitzian multivalued variational inequalities, *Nonconvex Optimization and Its Applications* 77 (2005) 113–129.

[5] J. Y. Bello Cruz and A. N. Iusem, A Strongly Convergent Direct Method for Monotone Variational Inequalities in Hilbert Spaces, *Numerical Functional Analysis and Optimization* 30 (2009) 23–36.

[6] J. Y. Bello Cruz and A. N. Iusem, Convergence of direct methods for paramonotone variational inequalities, *Computational Optimization and Applications* 46 (2010) 247–263.

[7] L. M. Bregman, Y. Censor, S. Reich and Y. Zepkowitz-Malachi, Finding the projection of a point onto the intersection of convex sets via projections onto halfspaces, *Journal of Approximation Theory* 124 (2003) 194–218.

[8] R. S. Burachik and A. N Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, Springer, Berlin, 2008.

[9] Y. Censor and S. A. Zenios, *Parallel Optimization: Theory, Algorithms, and Applications*, Oxford University Press, New York, NY, USA, 1997.

[10] Y. Censor, A. N. Iusem and S. A. Zenios, An interior point method with Bregman functions for the variational inequality problem with paramonotone operators, *Mathematical Programming* 81 (1998) 373–400.

[11] Y. Censor and A. Gibali, Projections onto super-half-spaces for monotone variational inequality problems in finite-dimensional spaces, *Journal of Nonlinear and Convex Analysis* 9 (2008) 461–475.

[12] F. Facchinei and J. S. Pang, *Finite-Dimensional Variational Inequalities and Complementarity Problems, Volume I and Volume II,* Springer-Verlag, New York, NY, USA, 2003.

[13] M. Fukushima, On the convergence of a class of outer approximation algorithms for convex programs, *Journal of Computational and Applied Mathematics* 10 (1984) 147–156.

[14] M. Fukushima, A relaxed projection method for variational inequalities, *Mathematical Programming* 35 (1986) 58–70.

[15] P. Hartman and G. Stampacchia, On some non-linear elliptic differential-functional equations, *Acta Mathematica* 115 (1966) 271–310.

[16] A. N. Iusem and L. R. Lucambio Pérez, An extragradient-type method for non-smooth variational inequalities, *Optimization* 48 (2000) 309–332.

[17] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, NY, USA, 1980.

[18] M. A. Noor, Some developments in general variational inequalities, *Applied Mathematics and Computation* 152 (2004) 197–277.

[19] N. Xiu and J. Zhang, Some recent advances in projection-type methods for variational inequalities, *Journal of Computational and Applied Mathematics* 152 (2003) 559–585.

[20] I. Yamada and N. Ogura, Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings, *Numerical Functional Analysis and Optimization* 25 (2004) 619–655.

[21] Q. Yang, On variable-step relaxed projection algorithm for variational inequalities, *Journal of Mathematical Analysis and Applications* 302 (2005) 166–179.

Manuscript received 12 May 2011
revised 22 November 2011
accepted for publication 9 April 2012

Aviv Gibali
Department of Mathematics,
The Technion - Israel Institute of Technology
Technion City, Haifa 32000, Israel
E-mail address: avivg@techunix.technion.ac.il