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AN EXTRAGRADIENT METHOD FOR GENERALIZED VARIATIONAL INEQUALITY*

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Abstract: In this paper, we propose an extragradient method for generalized variational inequality with multi-valued mapping. Our method is proven to be globally convergent to a solution of the variational inequality problem, provided the multi-valued mapping is continuous and pseudomonotone with nonempty compact convex values. We present an algorithmic framework of extragradient-type methods for multi-valued variational inequalities. Preliminary computational experience is also reported.

Key words: *generalized variational inequality, extragradient method, pseudomonotone mapping, multi-valued mapping*

Mathematics Subject Classification: *47H04, 47H10, 49J40*

1 Introduction

We consider the following generalized variational inequality: to find $x^* \in C$ and $\xi \in F(x^*)$ such that

$$\langle \xi, y - x^* \rangle \geq 0, \forall\, y \in C, \tag{1.1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$, F is a multi-valued mapping from C into $\mathbb{R}^n$ with nonempty values, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the inner product and norm in $\mathbb{R}^n$, respectively.

Extragradient-type algorithms have been extensively studied in the literature; see, for example, [5, 9, 11, 17, 18, 21, 23, 30, 31, 33] and the references therein. [9] proposed the subgradient extragradient algorithms for solving single-valued variational inequality. Further, [11] generalized the corresponding results of [9] from single-valued mapping to multi-valued one. Theory and algorithm of generalized variational inequality have been much studied in the literature [1, 3, 4, 8, 12, 13, 16, 20, 22, 26, 27, 28]. Various algorithms for computing the solution of (1.1) are proposed; see, for example, [10, 11]. Now let us compare our algorithm with algorithms in [10, 11]. First, the Armjio-type linesearsh procedures in the three algorithms are different. Secondly, the way to generate the next iterate is different. In [10], the next iterate is a projection of the initial point onto the intersection of the feasible set C and two hyperplanes, and one of the hyperplanes strictly separates

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the current iterate from the solution set of the problem. In the other algorithms, the next iterate relates to the current iterate. In [11], the next iterate is a projection onto a halfspace whose bounding hyperplane supports the feasible set C at a certain point. In our algorithm, the next iterate is a projection onto the feasible set C. Recently, [22] proposes a projection algorithm for generalized variational inequality with pseudomonotone mapping. In [22], choosing $\xi_i \in F(x_i)$ needs solving a single-valued variational inequality; see the expression (2.1) in [22]. To overcome this difficulty, [8]proposes a double projection algorithm for generalized variational inequality with pseudomonotone mapping. In [8], $\xi_i \in F(x_i)$ can be taken arbitrarily. In Algorithm 1 of [8], however, choosing the hyperplane needs computing the supremum and hence is computationally expensive; see the expression (2.2) in [8]. In this paper, we introduce an extragradient method for generalized variational inequality and prove the global convergence of the generalized iteration sequence, assuming that F is pseudomonotone on C with respect to the solution set; see the expression (1.2) below. In our method, ξ_i can be taken arbitrarily, and computing the supremum is avoided. Moreover, the Armjio-type linesearch procedure in our algorithm is also different from those in [8, 22]. At the same time, we present a algorithmic framework of extragradient-type methods for multi-valued variational inequalities and show the global convergence of the framework under standard conditions.

Let S be the solution set of (1.1), that is, those points $x^* \in C$ satisfying (1.1). Throughout this paper, we assume that the solution set S of the problem (1.1) is nonempty and F is continuous on C with nonempty compact convex values satisfying the following property:

$$\langle \zeta, y - x \rangle \geq 0, \forall\, y \in C, \zeta \in F(y), \forall\, x \in S. \tag{1.2}$$

The property (1.2) holds if F is pseudomonotone on C in the sense of Karamardian [19]. In particular, if F is monotone, then (1.2) holds.

The organization of this paper is as follows. In the next section, we recall the definition of continuous multi-valued mapping and present the details of the algorithm and prove several preliminary results for convergence analysis in Section 3. We give an algorithmic framework of extragradient-type methods for multi-valued variational inequalities in Section 4. Numerical results are reported in the last section.

2 Algorithms

Let us recall the definition of continuous multi-valued mapping. F is said to be upper semicontinuous at $x \in C$ if for every open set V containing $F(x)$, there is an open set U containing x such that $F(y) \subset V$ for all $y \in C \cap U$. F is said to be lower semicontinuous at $x \in C$ if give any sequence x_k converging to x and any $y \in F(x)$, there exists a sequence $y_k \in F(x_k)$ that converges to y. F is said to be continuous at $x \in C$ if it is both upper semicontinuous and lower semicontinuous at x. If F is single-valued, then both upper semicontinuity and lower semicontinuity reduce to the continuity of F.

Let Π_C denote the projector onto C and let $\mu > 0$ be a parameter.

Proposition 2.1. *$x \in C$ and $\xi \in F(x)$ solves the problem* (1.1) *if and only if*

$$r_\mu(x, \xi) := x - \Pi_C(x - \mu\xi) = 0. \tag{2.1}$$

Algorithm 2.2. Choose $x_0 \in C$ and three parameters $\sigma > 0, 0 < \mu < \min\{1, 1/\sigma\}$ and $\gamma \in (0, 1)$. Set $i = 0$.

Step 1. If $r_\mu(x_i, \xi) = 0$ for some $\xi \in F(x_i)$, stop; else take arbitrarily $\xi_i \in F(x_i)$.

Step 2. Let k_i be the smallest nonnegative integer k satisfying

$$\langle \xi_i - y_k, r_\mu(x_i, \xi_i)\rangle \leq \sigma \|r_\mu(x_i, \xi_i)\|^2. \tag{2.2}$$

where $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i,\xi_i))}(\xi_i)$. Set $\eta_i = \gamma^{k_i}$.

Step 3. Compute $x_{i+1} := \Pi_C(x_i - \alpha_i d_i)$, where

$$d_i = \eta_i r_\mu(x_i, \xi_i) - \mu\eta_i\xi_i + y_{k_i}, \tag{2.3}$$

$$\alpha_i = \frac{\eta_i\langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i + y_{k_i}\rangle}{\|d_i\|^2}. \tag{2.4}$$

Let $i := i + 1$ and go to step 1.

Remark 2.3. Since F has compact convex values, F has closed convex values. Therefore, y_i in Step 2 is uniquely determined by k_i.

Remark 2.4. If F is a single-valued mapping, the Armijo-type linesearch procedure (2.2) becomes that of Algorithm 2.1 in [15].

We show that Algorithm 2.2 is well-defined and implementable.

Proposition 2.5. *If x_i is not a solution of the problem* (1.1), *then there exists a nonnegative integer k_i satisfying* (2.2).

Proof. Suppose that for all k, we have

$$\langle \xi_i - y_k, r_\mu(x_i, \xi_i)\rangle > \sigma \|r_\mu(x_i, \xi_i)\|^2, \tag{2.5}$$

where $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i,\xi_i))}(\xi_i)$. Since F is lower semicontinuous, $\xi_i \in F(x_i)$, and $x_i - \gamma^k r_\mu(x_i, \xi_i) \to x_i$ as $k \to \infty$, for each k, there is $u_k \in F(x_i - \gamma^k r_\mu(x_i, \xi_i))$ such that $\lim_{k\to\infty} u_k = \xi_i$. Since $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i,\xi_i))}(\xi_i)$,

$$\|y_k - \xi_i\| \leq \|u_k - \xi_i\| \to 0, \ \textit{as } k \to \infty.$$

So $\lim_{k\to\infty} y_k = \xi_i$. Let $k \to \infty$ in (2.5), we have $0 = \|\xi_i - \xi_i\| \geq \sigma\|r_\mu(x_i, \xi_i)\| > 0$. This contradiction completes the proof. □

Lemma 2.6. *For every $x \in C$ and $\xi \in F(x)$,*

$$\langle \xi, r_\mu(x, \xi)\rangle \geq \mu^{-1}\|r_\mu(x, \xi)\|^2.$$

Proof. See [Lemma 2.3, 22]. □

Lemma 2.7. *Let C be a closed convex subset of $\mathbb{R}^n$. For any $x, y \in \mathbb{R}^n$ and $z \in C$, the following statements hold:*

(i) $\langle \Pi_C(x) - x, z - \Pi_C(x)\rangle \geq 0$.

(ii) $\|\Pi_C(x) - \Pi_C(y)\|^2 \leq \|x - y\|^2 - \|\Pi_C(x) - x + y - \Pi_C(y)\|^2$.

Proof. See[32]

For $x^* \in S$, define

$$h(x) = \frac{1}{2}\|x - x^*\|^2, \ x \in \mathbb{R}^n.$$

The following lemma shows that $-d_i$ in Step 3 is a descent direction of $h(x)$ at x_i. □

Lemma 2.8. *If the condition* (1.2) *holds and* $x_i \notin S$, *then for any* $x^* \in S$,

$$\langle d_i, x_i - x^* \rangle \geq (\mu^{-1} - \sigma)\eta_i \|r_\mu(x_i, \xi_i)\|^2 > 0. \tag{2.6}$$

Proof. Let $x^* \in S$. By (1.2) and $\mu > 0$, we have

$$\langle y_{k_i}, x_i - \eta_i r_\mu(x_i, \xi_i) - x^* \rangle \geq 0. \tag{2.7}$$

Since $x^* \in C$, from (2.1) and Lemma 2.7(i) we have

$$\langle (x_i - r_\mu(x_i, \xi_i)) - (x_i - \mu\xi_i), x^* - (x_i - r_\mu(x_i, \xi_i)) \rangle \geq 0,$$

which implies that

$$\langle r_\mu(x_i, \xi_i) - \mu\xi_i, x_i - x^* \rangle \geq \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i \rangle. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned} \langle d_i, x_i - x^* \rangle &= \langle \eta_i r_\mu(x_i, \xi_i) - \mu\eta_i\xi_i + y_{k_i}, x_i - x^* \rangle \\ &= \eta_i \langle r_\mu(x_i, \xi_i) - \mu\xi_i, x_i - x^* \rangle + \eta_i \langle r_\mu(x_i, \xi_i), y_{k_i} \rangle \\ &\quad + \langle x_i - \eta_i r_\mu(x_i, \xi_i) - x^*, y_{k_i} \rangle \\ &\geq \eta_i \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i + y_{k_i} \rangle. \end{aligned} \tag{2.9}$$

Thus, we have

$$\begin{aligned} \langle d_i, x_i - x^* \rangle &\geq \eta_i \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i + y_{k_i} \rangle \\ &= \eta_i \|r_\mu(x_i, \xi_i)\|^2 - \mu\eta_i \langle r_\mu(x_i, \xi_i), \xi_i \rangle + \eta_i \langle r_\mu(x_i, \xi_i), y_{k_i} \rangle \\ &\geq (1 - \sigma)\eta_i \|r_\mu(x_i, \xi_i)\|^2 + (1 - \mu)\eta_i \langle r_\mu(x_i, \xi_i), \xi_i \rangle \\ &\geq (\mu^{-1} - \sigma)\eta_i \|r_\mu(x_i, \xi_i)\|^2, \end{aligned} \tag{2.10}$$

where the second inequality follows from (2.2) and the last one follows from Lemma 2.6 and $\mu < 1$. This completes the proof. □

Next we present a fundamental existence result for variational inequality problem (1.1) that will be used for proving the conclusion of Theorem 3.2.

Lemma 2.9. *Let* $C \subset \mathbb{R}^n$ *be a nonempty bounded closed convex set and the mapping* $F : C \to 2^{\mathbb{R}^n}$ *be lower semicontinuous with nonempty closed convex values. Then, the solution set* S *of* $GVI(F, C)$ *is nonempty.*

Proof. Since the multifunction F is lower semicontinuous and has nonempty closed convex values, by Michael's selection theorem (see for instance Theorem 24.1 in [6]), it admits a continuous selection; that is, there exists a continuous mapping $G : C \to \mathbb{R}^n$ such that $G(x) \in F(x)$ for every $x \in C$. Since C is a nonempty bounded closed convex set, the variational inequality problem VI(C, G), which consists of finding an $x \in C$ such that

$$\langle G(x), y - x \rangle \geq 0, \forall\, y \in C$$

has a solution(see Lemma 3.1 in [14]), i.e. the solution set S' of VI(C, G) is nonempty. It follows from $S' \subset S$ that S is nonempty. □

3 Main Results

By using Lemma 2.8, we conclude the global convergence of Algorithm 2.2.

Theorem 3.1. *If $F : C \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C and the condition* (1.2) *holds, then either Algorithm 2.2 terminates in a finite number of iterations or generates an infinite sequence $\{x_i\}$ converging to a solution of* (1.1).

Proof. Let $x^* \in S$. It follows from Lemma 2.7(ii), (2.3),(2.4) and (2.10) that

$$\begin{aligned} \|x_{i+1} - x^*\|^2 &\leq \|x_i - x^* - \alpha_i d_i\|^2 \\ &= \|x_i - x^*\|^2 - 2\alpha_i \langle d_i, x_i - x^*\rangle + \alpha_i^2 \|d_i\|^2 \\ &\leq \|x_i - x^*\|^2 - \frac{(\eta_i \langle r_\mu(x_i,\xi_i), r_\mu(x_i,\xi_i) - \mu\xi_i + y_{k_i}\rangle)^2}{\|d_i\|^2} \\ &\leq \|x_i - x^*\|^2 - (\mu^{-1} - \sigma)^2 \frac{\eta_i^2 \|r_\mu(x_i,\xi_i)\|^4}{\|\eta_i r_\mu(x_i,\xi_i) - \mu\eta_i\xi_i + y_{k_i}\|^2}. \end{aligned} \tag{3.1}$$

It follows that the squence $\{\|x_{i+1} - x^*\|^2\}$ is nonincreasing, and hence is a convergent sequence. Therefore, $\{x_i\}$ is bounded. Since F is continuous with compact values, Proposition 3.11 in [2] implies that $\{F(x_i) : i \in N\}$ is a bounded set, and so are $\{\xi_i\}, \{r_\mu(x_i,\xi_i)\}$ and $\{y_{k_i}\}$. Thus, $\{\eta_i r_\mu(x_i,\xi_i) - \mu\eta_i\xi_i + y_{k_i}\}$ is bounded. Then, there exists a positive number M_1 such that

$$\|\eta_i r_\mu(x_i,\xi_i) - \mu\eta_i\xi_i + y_{k_i}\| \leq M_1.$$

It follows from (3.1) that

$$\|x_{i+1} - x^*\|^2 \leq \|x_i - x^*\|^2 - (\mu^{-1} - \sigma)^2 M_1^{-2} \eta_i^2 \|r_\mu(x_i,\xi_i)\|^4. \tag{3.2}$$

Therefore,

$$\lim_{i\to\infty} \eta_i \|r_\mu(x_i,\xi_i)\| = 0. \tag{3.3}$$

By the boundedness of $\{x_i\}$, there exists a convergent subsequence $\{x_{i_j}\}$ converging to $\overline{x}$.

If $\overline{x}$ is a solution of the problem (1.1), we show next that the whole sequence $\{x_i\}$ converges to $\overline{x}$. Replacing x^* by $\overline{x}$ in the preceding argument, we obtain that the sequence $\{\|x_i - \overline{x}\|\}$ is nonincreasing and hence converges. Since $\overline{x}$ is an accumulation point of $\{x_i\}$, some subsequence of $\{\|x_i - \overline{x}\|\}$ converges to zero. This shows that the whole sequence $\{\|x_i - \overline{x}\|\}$ converges to zero, hence $\lim_{i\to\infty} x_i = \overline{x}$.

Suppose now that $\overline{x}$ is not a solution of the problem (1). We show first that k_i in Algorithm 2.2 cannot tend to ∞. Since F is continuous with compact values, Proposition 3.11 in [2] implies that $\{F(x_i) : i \in N\}$ is a bounded set, and so the sequence $\{\xi_i\}$ is bounded. Therefore, there exists a subsequence $\{\xi_{i_j}\}$ converging to $\overline{\xi}$. Since F is upper semicontinuous with compact values, Proposition 3.7 in [2] implies that F is closed, and so $\overline{\xi} \in F(\overline{x})$. By the definition of k_i, we have

$$\langle \xi_i - u_i, r_\mu(x_i,\xi_i)\rangle > \sigma \|r_\mu(x_i,\xi_i)\|^2, \forall\ u_i = \Pi_{F(x_i - \gamma^{k_i - 1} r_\mu(x_i,\xi_i))}(\xi_i).$$

If $k_{i_j} \to \infty$, then $x_{i_j} - \gamma^{k_{i_j}-1} r_\mu(x_{i_j},\xi_{i_j}) \to \overline{x}$. The lower continuity of F, in turn, implies the existence of $\overline{\xi}_{i_j} \in F(x_{i_j} - \gamma^{k_{i_j}-1} r_\mu(x_{i_j},\xi_{i_j}))$ such that $\overline{\xi}_{i_j}$ converges to $\overline{\xi}$. Since $u_{i_j} = \Pi_{F(x_{i_j} - \gamma^{k_{i_j}-1} r_\mu(x_{i_j},\xi_{i_j}))}(\xi_{i_j})$, $u_{i_j} \in F(x_{i_j} - \gamma^{k_{i_j}-1} r_\mu(x_{i_j},\xi_{i_j}))$ and $\|u_{i_j} - \xi_{i_j}\| \leq \|\overline{\xi}_{i_j} - \xi_{i_j}\|$. Therefore $\lim_{j\to\infty} u_{i_j} = \overline{\xi}$ and

$$\langle \xi_{i_j} - u_{i_j}, r_\mu(x_{i_j},\xi_{i_j})\rangle > \sigma \|r_\mu(x_{i_j},\xi_{i_j})\|^2.$$

Letting $j \to \infty$, we obtain the contradiction

$$0 \geq \sigma \|r_\mu(\overline{x}, \overline{\xi})\|^2 > 0,$$

being $r_\mu(\cdot,\cdot)$ continuous. Therefore, $\{k_i\}$ is bounded and so is $\{\eta_i\}$.

By the boundedness of $\{\eta_i\}$, it follows from (3.3) that $\lim_{i\to\infty} \|r_\mu(x_i, \xi_i)\| = 0$. Since $r_\mu(\cdot,\cdot)$ is continuous and the sequences $\{x_i\}$ and $\{\xi_i\}$ are bounded, there exists an accumulation point $(\overline{x}, \overline{\xi})$ of $\{(x_i, \xi_i)\}$ such that $r_\mu(\overline{x}, \overline{\xi}) = 0$. This implies that $\overline{x}$ solves the variational inequality (1). Similar to the preceding proof, we obtain that $\lim_{i\to\infty} x_i = \overline{x}$.

The following theorem shows that, if the solution set S is empty, the sequence $\{x_i\}$ generated by Algorithm 2.2 is unbounded.

Theorem 3.2. *If $F : C \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C and suppose $S = \emptyset$. Then, the sequence $\{x_i\}$ generated by Algorithm 2.2 must be unbounded.*

Proof. By Step 1 of Algorithm 2.2, we know that Algorithm 2.2 generates an infinite sequence if $S = \emptyset$. Suppose, on the contrary, the sequence $\{x_i\}$ is bounded. Then, there exists a positive number M_2 such that

$$\{x_i\} \subseteq B(0, M_2),$$

where

$$B(0, M_2) := \{x \in \mathbb{R}^n : \|x\| \leq M_2\}.$$

Since $F(x)$ is continuous with compact values, Proposition 3.11 in [2] implies that $\{F(x_i)\}$ is a bounded set, and so $\{x_i - \mu\xi_i : \xi_i \in F(x_i)\}$ is bounded. Without loss of generality, we assume

$$\{x_i - \mu\xi_i : \xi_i \in F(x_i)\} \subseteq B(0, M_2).$$

Consider the variational inequality GVI(F, C') where

$$C' = C \cap B(0, 2M_2).$$

From Lemma 2.9, we know that the solution set of GVI(F, C'), denoted by S', is nonempty. We apply Algorithm 2.2 to GVI(F, C') with the starting point x_0, then an infinite sequence, denoted by $\{\widetilde{x}_i\}$, is generated. It follows from Theorem 3.1 that $\{\widetilde{x}_i\}$ converges to a solution of GVI(F, C'). By the definitions of C' and the projection operator, along with the procedure of Algorithm 2.2, we have

$$\widetilde{x}_i = x_i, \ \forall\ i \geq 0.$$

Thus, the limit of $\{\widetilde{x}_i\}$ is also a solution of GVI(F, C) which contradicts the supposition that $S = \emptyset$. □

Now we provide a result on the convergence rate of the iterative sequence generated by Algorithm 2.2. To establish this result, we need a certain error bound to hold locally(see (3.4) below). The research on error bound is a large topic in mathematical programming. One can refer to the survey [24] for the roles played by error bounds in the convergence analysis of iterative algorithms; more recent developments on this topic are included in Chapter 6 in [7]. A condition similar to (16) has also been used in [29] (see expression (2.3) therein) to analyze the convergence rate in very general framework.

For any $\delta > 0$, define

$$P(\delta) := \{(x, \xi) \in C \times \mathbb{R}^n : \xi \in F(x),\ \|r_\mu(x, \xi)\| \leq \delta\}.$$

We say that F is Lipschitz continuous on C if there exists a constant $L > 0$ such that, for all $x, y \in C$, $H(F(x), F(y)) \leq L\|x - y\|$, where H denotes the Hausdorff metric.

Theorem 3.3. *In addition to the assumptions in the above theorem, if F is Lipschitz continuous with modulus $L > 0$ and if there exist positive constants c and δ such that*

$$dist(x, S) \leq c\|r_\mu(x, \xi)\|, \ \forall \ (x, \xi) \in P(\delta), \tag{3.4}$$

then there is a constant $\alpha > 0$ such that for sufficiently large i,

$$dist(x_i, S) \leq \frac{1}{\sqrt{\alpha i + dist^{-2}(x_0, S)}}.$$

Proof. Put $\eta := \min\{1/2, L^{-1}\gamma\sigma\}$. We first prove that $\eta_i > \eta$ for all i. By the construction of η_i, we have $\eta_i \in (0, 1]$. If $\eta_i = 1$,then clearly $\eta_i > 1/2 \geq \eta$. Now we assume that $\eta_i < 1$. Since $\eta_i = \gamma^{k_i}$, it follows that the nonnegative integer $k_i \geq 1$. Thus the construction of k_i implies that

$$\langle \xi_i - u_i, r_\mu(x_i, \xi_i)\rangle > \sigma\|r_\mu(x_i, \xi_i)\|^2, \forall \ u_i = P_{F(x_i - \gamma^{k_i - 1} r_\mu(x_i, \xi_i))}(\xi_i),$$

and hence, as $k_i \geq 1$,

$$\|u_i - \xi_i\| > \sigma\|r_\mu(x_i, \xi_i)\|, \forall \ u_i = P_{F(x_i - \gamma^{k_i - 1} r_\mu(x_i, \xi_i))}(\xi_i). \tag{3.5}$$

Since $u_i = P_{F(x_i - \gamma^{k_i - 1} r_\mu(x_i, \xi_i))}(\xi_i)$,

$$\|u_i - \xi_i\| \leq \|y - \xi_i\|, \ \forall \ y \in F(x_i - \gamma^{k_i - 1} r_\mu(x_i, \xi_i)). \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\|y - \xi_i\| > \sigma\|r_\mu(x_i, \xi_i)\|, \ \forall \ y \in F(x_i - \gamma^{k_i - 1} r_\mu(x_i, \xi_i)). \tag{3.7}$$

Since $\xi_i \in F(x_i)$ and F is compact-valued, the definition of the Hausdorff metric implies the existence of $\zeta_i \in F(x_i - \gamma^{-1}\eta_i r_\mu(x_i, \xi_i))$ such that

$$\sigma\|r_\mu(x_i, \xi_i)\| < \|\zeta_i - \xi_i\| \leq H(F(x_i - \gamma^{-1}\eta_i r_\mu(x_i, \xi_i)), F(x_i)) \leq L\gamma^{-1}\eta_i\|r_\mu(x_i, \xi_i)\|$$

Therefore $\eta_i > L^{-1}\gamma\sigma \geq \eta$.

Let $x^* \in \Pi_S(x_i)$. By the proof of Theorem 3.1 and (3.4), we obtain that for sufficiently large i,

$$\begin{aligned}
\text{dist}^2(x_{i+1}, S) &\leq \|x_{i+1} - x^*\|^2 \leq \|x_i - x^*\|^2 - M_1^{-2}(\mu^{-1} - \sigma)^2{\eta_i}^2\|r_\mu(x_i, \xi_i)\|^4 \\
&\leq \|x_i - x^*\|^2 - M_1^{-2}(\mu^{-1} - \sigma)^2\eta^2\|r_\mu(x_i, \xi_i)\|^4 \\
&\leq \text{dist}^2(x_i, S) - M_1^{-2}(\mu^{-1} - \sigma)^2\eta^2 c^{-4}\text{dist}^4(x_i, S).
\end{aligned}$$

Write α for $M_1^{-2}(\mu^{-1} - \sigma)^2\eta^2 c^{-4}$. Applying Lemma 6 in Chapter 2 of [25], we have

$$\text{dist}(x_i, S) \leq \text{dist}(x_0, S)/\sqrt{\alpha i \, \text{dist}^2(x_0, S) + 1} = 1/\sqrt{\alpha i + \text{dist}^{-2}(x_0, S)}.$$

This completes the proof. ∎

4 Algorithmic Framework

Thus, we present our algorithmic framework for solving (1.1).

Algorithm 4.1. Choose $x_0 \in C$ and three parameters $\sigma > 0, \mu \in (0, 1/\sigma)$ and $\gamma \in (0,1)$. Set $i = 0$.

Step 1. If $r_\mu(x_i, \xi) = 0$ for some $\xi \in F(x_i)$, stop; else take arbitrarily $\xi_i \in F(x_i)$.

Step 2. Let k_i be the smallest nonnegative integer satisfying

$$\langle \xi_i - y_k, r_\mu(x_i, \xi_i)\rangle \le \sigma \|r_\mu(x_i, \xi_i)\|^2. \tag{4.1}$$

where $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i,\xi_i))}(\xi_i)$. Set $\eta_i = \gamma^{k_i}$.

Step 3. Compute $x_{i+1} := \Pi_C(x_i - \alpha(x_i, \eta_i, \mu)d(x_i, \eta_i, \mu))$, where $d(x_i, \eta_i, \mu)$ is a direction vector and $\alpha(x_i, \eta_i, \mu)$ is a step-size defined by

$$\alpha(x_i, \eta_i, \mu) := \frac{g(x_i, \eta_i, \mu)}{\|d(x_i, \eta_i, \mu)\|^2},\ g(x_i, \eta_i, \mu) > 0. \tag{4.2}$$

Let $i := i + 1$ and go to step 1.

Remark 4.2. Since the inequality

$$\langle y_k, r_\mu(x_i, \xi_i)\rangle \ge (\mu^{-1} - \sigma)\|r_\mu(x_i, \xi_i)\|^2$$

implies (4.1), the stepsize rule of η_i in Algorithm 4.1 can be replaced also by the aforementioned inequality.

Remark 4.3. The positive parameters σ and μ can vary with the different choice of $\alpha(x_i, \eta_i, \mu)$ and $d(x_i, \eta_i, \mu)$(see μ in Algorithm 4.5, for example).

Next we conclude the global convergence of Algorithm 4.1.

Theorem 4.4. *If $F : C \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C, if the sequence $\{x_i\}$ generated by Algorithm 4.1 satisfies*

$$\theta(\sigma, \mu)\eta_i \|r_\mu(x_i, \xi_i)\|^2 \le g(x_i, \eta_i, \mu) \le \langle d(x_i, \eta_i, \mu), x_i - x^*\rangle, \forall\, x^* \in S \tag{4.3}$$

and if there exists a parameter $M > 0$ such that

$$\|d(x_i, \eta_i, \mu)\| \le M, \tag{4.4}$$

where $\theta(\sigma, \mu)$ is a positive parameter depending on σ and μ, then either Algorithm 4.1 terminates in a finite number of iterations or generates an infinite sequence $\{x_i\}$ converging to a solution $\overline{x}$ of (1.1).

Proof. Let $x^* \in S$. It follows from Lemma 2.7(ii), (4.2), (4.3) and (4.4) that

$$\begin{aligned}
\|x_{i+1} - x^*\|^2 &\le \|x_i - x^* - \alpha(x_i, \eta_i, \mu)d(x_i, \eta_i, \mu)\|^2 \\
&= \|x_i - x^*\|^2 - 2\alpha(x_i, \eta_i, \mu)\langle d(x_i, \eta_i, \mu), x_i - x^*\rangle \\
&+ \alpha^2(x_i, \eta_i, \mu)\|d(x_i, \eta_i, \mu)\|^2 \\
&\le \|x_i - x^*\|^2 - \frac{g^2(x_i, \eta_i, \mu)}{\|d(x_i, \eta_i, \mu)\|^2} \\
&\le \|x_i - x^*\|^2 - \theta^2(\sigma, \mu)\frac{\eta_i^2 \|r_\mu(x_i, \xi_i)\|^4}{\|d(x_i, \eta_i, \mu)\|^2} \\
&\le \|x_i - x^*\|^2 - \theta^2(\sigma, \mu)M^{-2}\eta_i^2\|r_\mu(x_i, \xi_i)\|^4.
\end{aligned} \tag{24}$$

The remainder is similar to the proof of Theorem 3.1 and we omit it. □

Setting $d(x_i, \eta_i, \mu) = y_{k_i} + \eta_i r_\mu(x_i, \xi_i)$, $g(x_i, \eta_i, \mu) = \eta_i \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i + y_{k_i}\rangle$, we obtain the following algorithm for solving (1.1).

Algorithm 4.5. Choose $x_0 \in C$ and three parameters $\sigma > 0, 0 < \mu < \min\{1, 1/\sigma\}$ and $\gamma \in (0,1)$. Set $i = 0$.

Step 1. If $r_\mu(x_i, \xi) = 0$ for some $\xi \in F(x_i)$, stop; else take arbitrarily $\xi_i \in F(x_i)$.

Step 2. Let k_i be the smallest nonnegative integer k satisfying

$$\langle \xi_i - y_k, r_\mu(x_i, \xi_i)\rangle \le \sigma \|r_\mu(x_i, \xi_i)\|^2. \tag{4.5}$$

where $y_k = \Pi_{F(x_i - \gamma^k r_\mu(x_i, \xi_i))}(\xi_i)$. Set $\eta_i = \gamma^{k_i}$.

Step 3. Compute $x_{i+1} := \Pi_C(x_i - \alpha(x_i, \eta_i, \mu) d(x_i, \eta_i, \mu))$, where

$$d(x_i, \eta_i, \mu) = y_{k_i} + \eta_i r_\mu(x_i, \xi_i), \tag{4.6}$$

$$\alpha(x_i, \eta_i, \mu) = \frac{\eta_i \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i + y_{k_i}\rangle}{\|d(x_i, \eta_i, \mu)\|^2}. \tag{4.7}$$

Let $i := i + 1$ and go to step 1.

Theorem 4.6. *If $F : C \to 2^{\mathbb{R}^n}$ is continuous with nonempty compact convex values on C and the condition* (1.2) *holds, then either Algorithm 4.5 terminates in a finite number of iterations or generates an infinite sequence $\{x_i\}$ converging to a solution of* (1.1).

Proof. In view of Theorem 4.4, we only need to show that the sequence $\{x_i\}$ generated by Algorithm 4.5 satisfies (4.3) and (4.4). For any $x^* \in S$, by (1.2)and $\mu > 0$, we have

$$\langle \mu\xi_i, x_i - x^*\rangle \ge 0, \tag{4.8}$$

and

$$\langle y_{k_i}, x_i - \eta_i r_\mu(x_i, \xi_i) - x^*\rangle \ge 0. \tag{4.9}$$

Therefore,

$$\begin{aligned} \langle y_{k_i}, x_i - x^*\rangle &= \langle y_{k_i}, x_i - \eta_i r_\mu(x_i, \xi_i) - x^* + \eta_i r_\mu(x_i, \xi_i)\rangle \\ &= \langle y_{k_i}, x_i - \eta_i r_\mu(x_i, \xi_i) - x^*\rangle + \eta_i \langle y_{k_i}, r_\mu(x_i, \xi_i)\rangle \\ &\ge \eta_i \langle y_{k_i}, r_\mu(x_i, \xi_i)\rangle, \end{aligned} \tag{4.10}$$

where the last inequality follows from (4.9).

Since $x^* \in C$, from (2.1) and Lemma 2.7(i) we have

$$\langle (x_i - r_\mu(x_i, \xi_i)) - (x_i - \mu\xi_i), x^* - (x_i - r_\mu(x_i, \xi_i))\rangle \ge 0,$$

which implies that

$$\langle r_\mu(x_i, \xi_i), x_i - x^*\rangle \ge \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i\rangle + \mu\langle \xi_i, x_i - x^*\rangle. \tag{4.11}$$

Combining (4.8) and (4.11) yields that

$$\langle r_\mu(x_i, \xi_i), x_i - x^*\rangle \ge \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i\rangle. \tag{4.12}$$

It follows from (4.10), (4.12) and (2.10) that

$$\begin{aligned} \langle d(x_i, \eta_i, \mu), x_i - x^*\rangle &= \langle y_{k_i} + \eta_i r_\mu(x_i, \xi_i), x_i - x^*\rangle \\ &\ge \eta_i \langle r_\mu(x_i, \xi_i), r_\mu(x_i, \xi_i) - \mu\xi_i + y_{k_i}\rangle \\ &\ge (\mu^{-1} - \sigma)\eta_i \|r_\mu(x_i, \xi_i)\|^2. \end{aligned} \tag{4.13}$$

Similar to the proof of Theorem 3.1, we know that there exists a positive number M such that

$$\|d(x_i, \eta_i, \mu)\| \le M.$$

□

5 Numerical Experiments

In this section, we present some numerical experiments for the proposed algorithm. The MATLAB codes are run on a PC(with CPU Intel P-T2390)under MATLAB Version 7.0.1.24704(R14)Service Pack 1. We compare the performance of our Algorithm 2.2, [22, Algorithm 1]and [8, Algorithm 1]. In the following tables, 'It.' denotes number of iteration, and 'CPU' denotes the CPU time in seconds. The tolerance ε means when $\|r_\mu(x,\xi)\| \leq \varepsilon$, the procedure stops.

Example 1. Let $n = 3$,

$$C := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1\}$$

and $F : C \to 2^{\mathbb{R}^n}$ be defined by

$$F(x) := \{(t, t - x_1, t - x_2) : t \in [0, 1]\}$$

Then the set C and the mapping F satisfy the assumptions of Theorem 3.1 and (0,0,1) is a solution of the generalized variational inequality. Example 1 is tested in [22]. We choose $\sigma = 0.4, \gamma = 0.9$ and $\mu = 1$ for our algorithm; $\sigma = 0.5, \gamma = 0.8$ and $\mu = 1$ for Algorithm 1 in [22].

Table 1
Example 1

		Algorithm 2.2		[22, Algorithm 1]	
Initial point	ε	It.	CPU	It.	CPU
(1,0,0)	10^{-5}	19	0.3125	23	0.40625
(0,1,0)	10^{-5}	17	0.296875	18	0.34375
(0.5,0.5,0)	10^{-5}	18	0.3125	20	0.390625
(1,0,0)	10^{-7}	25	0.34375	31	0.46875
(0,1,0)	10^{-7}	23	0.328125	26	0.40625
(0.5,0.5,0)	10^{-7}	24	0.359375	29	0.453125

Example 2. Let $n = 4$,

$$C := \{x \in \mathbb{R}^n_+ : \sum_{i=1}^{n} x_i = 1\}$$

and $F : C \to 2^{\mathbb{R}^n}$ be defined by

$$F(x) = \{(t, t + 2x_2, t + 3x_3, t + 4x_4) : t \in [0, 1]\}$$

Then the set C and the mapping F satisfy the assumptions of Theorem 3.1 and (1,0,0,0) is a solution of the generalized variational inequality. Example 1 is tested in [8]. We choose $\sigma = 0.5, \gamma = 0.8$ and $\mu = 1$ for our algorithm; $\sigma = 2, \gamma = 0.9$ and $\mu = 0.1$ for Algorithm 1 in [8].

Table 2
Example 2

		Algorithm 2.2		[8, Algorithm 1]	
Initial point	ε	It.	CPU	It.	CPU
(0,0,0,1)	10^{-5}	41	0.6875	129	0.6875
(0,0,1,0)	10^{-5}	29	0.453125	128	0.6875
(0.5,0,0.5,0)	10^{-5}	24	0.421875	118	0.625
(0,0,0,1)	10^{-7}	49	0.734375	195	0.984375
(0,0,1,0)	10^{-7}	37	0.53125	194	0.984375
(0.5,0,0.5,0)	10^{-7}	32	0.484375	184	0.921875

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