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CONNECTEDNESS OF THE SOLUTION SET FOR SYMMETRIC VECTOR QUASIEQUILIBRIUM PROBLEMS*

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Abstract: In this paper, by using a scalarization method, a characterrization of weak efficient solutions for a symmetric vector quasiequilibrium problem in Hausdorff topological vector spaces is obtained. Further, the existence of the weak efficient solutions and the connectedness of the set of weak efficient solutions for the symmetric vector quasiequilibrium problems are proved. The results presented in this paper generalize and improve some known results [7, 13, 23, 39].

Key words: *symmetric vector quasiequilibrium problems, connectedness, compactness, scalarization*

Mathematics Subject Classification: *49J40, 90C29*

1 Introduction

The equilibrium problem was proposed in Blum and Oettli [6] and has been intensively studied since then. It contains many important problems as special cases, such as variational inequalities, optimization problems, problems of Nash equilibrium, fixed point and coincidence point problems, saddle point problems and complementarity problems. In recent years, a great deal of papers have been devoted to the existence results of solutions for various kinds of vector equilibrium problems (see, e.g.,[1, 3, 11, 17] and the references therein).

In 1994, Noor and Oettli [35] introduced and studied a symmetric quasiequilibrium problem which is a generalization of equilibrium problem proposed by Blum and Oettli [6] and proved to be more suitable in modeling several practical situations. In 2003, Fu[20] extended the result from scalar case to the vector case in Hausdorff locally convex topological vector spaces. In 2007, Anh and Khanh [1] extended the problem considered in Fu[20] from the single-valued case to the multivalued case in Hausdorff topological vector spaces. Recently, the existence of the solution set for symmetric vector quasiequilibrium problems has been studied by many authors (see, e.g., [1, 15, 18, 20] and the references therein). However, there have been limited number of works in the literature dealing with the topological properties of the solution set for symmetric vector quasiequilibrium problems.

It is well known that the connectedness is one of the important topological properties for vector variational inequalities and vector equilibrium problems, as it provides the possibility

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of continuously moving from one solution to any other solution. Recently, Lee et al.[33] and Cheng [13] studied the connectedness of the weak efficient solution set for single-valued vector variational inequalities in finite dimensional Euclidean space. Gong [23] obtained the connectedness of the various solution sets for single-valued vector equilibrium problems in infinite dimension space. Chen et al.[7] discussed the connectedness and the compactness of the weak efficient solution set for set-valued vector equilibrium problems and the set-valued vector Hartman-Stampacchia variational inequalities in normed linear spaces. By virtue of a density result, Gong and Yao [24] studied the connectedness of the efficient solution set for single-valued vector equilibrium problems in locally convex spaces. Concerned with the connectedness and the path-connectedness of the solution set for symmetric vector equilibrium problems, we refer to the work of Zhong et al.[39]. Very recently, Chen et al. [9] studied the connectedness of approximate solution set for the vector equilibrium problems in real Hausdorff topological vector spaces. To the best of our knowledge, no paper has been devoted to the study of the connectedness of the solution set for symmetric vector quasiequilibrium problems.

Motivated and inspired by the research works mentioned above, in this paper, we consider a class of symmetric vector quasiequilibrium problems in Hausdorff topological vector spaces. By using a scalarization method, we give a characterrization of the weak efficient solutions for symmetric vector quasiequilibrium problems in Hausdorff topological vector spaces. Through the scalarization result, we obtain the existence of the weak efficient solutions and the connectedness of the weak efficient solution set for symmetric vector quasiequilibrium problems. The results presented in this paper generalize and improve some known results in [7, 13, 23, 39].

2 Preliminary Results

Throughout this paper, unless specified otherwise, we always suppose that X, Y, E and Z are real Hausdorff topological vector spaces. Let $K \subset X$ and $D \subset Y$ be nonempty convex subsets. Let $S : K \times D \to 2^K$, $T : K \times D \to 2^D$, $F : K \times D \times K \to 2^E$ and $G : D \times K \times D \to 2^Z$ be set-valued mappings. Let $C \subset E$ and $P \subset Z$ be nonempty closed convex pointed cones with $intC \neq \emptyset$ and $intP \neq \emptyset$. Let E^* and Z^* be the topological dual spaces of E and Z. Let C^* and P^* be the dual cones of C and P, respectively, that is,

$$C^* = \{f \in E^* : f(y) \geq 0, \forall y \in C\},$$

and

$$P^* = \{g \in Z^* : g(y) \geq 0, \forall y \in P\}.$$

Let

$$A = \{(x, y) \in K \times D : x \in S(x, y), y \in T(x, y)\}.$$

In this paper, we consider the following symmetric vector quasiequilibrium problem(in short, SVQEP): finding $(\bar{x}, \bar{y}) \in A$ such that

$$\begin{cases} F(\bar{x}, \bar{y}, u) \cap (-intC) = \emptyset, \forall u \in S(\bar{x}, \bar{y}), \\ G(\bar{y}, \bar{x}, v) \cap (-intP) = \emptyset, \forall v \in T(\bar{x}, \bar{y}). \end{cases}$$

We call this $(\bar{x}, \bar{y})$ a weak efficient solution to (SVQEP). Denote by $V_w(F, G)$ the set of all weak efficient solutions to (SVQEP).

Remark 2.1. (i) If for any $(x, y) \in K \times D$, $S(x, y) \equiv K$ and $T(x, y) \equiv D$, then (SVQEP) reduces to the problem considered in Zhong et al.[39].

(ii) If F and G are single-valued mappings, C is a closed convex cone with $intC \neq \emptyset$, $E \equiv Z$ and $C \equiv P$, then (SVQEP) reduces to the problem studied in Farajzadeh [18]; If in addition, $F(x, y, u) = f(u, y) - f(x, y)$ and $G(y, x, v) = g(x, v) - g(x, y)$, where $f : K \times D \to Z$ and $g : K \times D \to Z$ are two single-valued mappings, then (SVQEP) collapses to the problem investigated in Fu [20]; If furthermore, $Z = R$ and $C = R_+$, then (SVQEP) coincide with the scalar problem studied in Noor and Oettli [35].

(iii) If $G \equiv 0$, $Y \equiv \{y\}$ and $T(x, y) \equiv \{y\}$, then (SVQEP) reduces to the multivalued vector quasiequilibrium problem considered by many authors.

Let $(f, g) \in C^*\backslash\{0\} \times P^*\backslash\{0\}$. We also consider the following scalar symmetric vector quasiequilibrium problem(in short, SSVQEP): finding $(\bar{x}, \bar{y}) \in A$ such that

$$\begin{cases} \inf\limits_{z \in F(\bar{x}, \bar{y}, u)} f(z) \geq 0, \forall u \in S(\bar{x}, \bar{y}), \\ \inf\limits_{z \in G(\bar{y}, \bar{x}, v)} g(z) \geq 0, \forall v \in T(\bar{x}, \bar{y}). \end{cases}$$

We call this $(\bar{x}, \bar{y})$ a (f, g)-efficient solution to (SSVQEP). Denote by $V_w(f, g)$ the set of all (f, g)-efficient solutions to (SSVQEP).

For our main results, we need some definitions and lemmas as follows.

Definition 2.2. Let X and Y be two topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping.

(i) T is said to be upper semicontinuous at $x \in X$ if, for any neighborhood U of $T(x)$, there is a neighborhood V of x such that $T(t) \subset U$, for all $t \in V$. T is said to be upper semicontinuous on X if it is upper semicontinuous at each $x \in X$.

(ii) T is said to be lower semicontinuous at $x \in X$ if, for any $y \in T(x)$ and for any net $\{x_\alpha\}$ converging to x, there exists a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and y_α converges to y. T is said to be lower semicontinuous on X if it is lower semicontinuous at each $x \in X$.

(iii) T is said to be continuous on X if it is both upper semicontinuous and lower semicontinuous on X.

(iv) T is said to be closed if, $\text{Graph}(T) = \{(x, y) : x \in X, y \in T(x)\}$ is a closed subset in $X \times Y$.

Definition 2.3. Let W be a topological vector space and $D \subset W$ be a nonempty set. A set-valued mapping $G : D \to 2^E$ is said to be C-lower semicontinuous on x_0 if, for each $z \in G(x_0)$, and any neighborhood U of 0 in E, there exists a neighborhood $U(x_0)$ of x_0 such that

$$G(x) \cap (z + U - C) \neq \emptyset, \forall x \in U(x_0) \cap D.$$

Remark 2.4. Clearly, if G is lower semicontinuous on D, then G is C-lower semicontinuous on D.

Definition 2.5. Let X, Y, Z and E be topological vector spaces and C be a closed convex cone in E. Let $H : K \times D \times M \to 2^E$ be a set-valued mapping, where $K \subset X$ and $D \subset Y$ are nonempty sets, $M \subset Z$ is a nonempty convex set. For any fixed $(x, y) \in K \times D$, $H(x, y, \cdot)$ is said to be

(i) C-convex if, for every $z_1, z_2 \in M$ and $t \in [0,1]$, one has

$$tH(x,y,z_1) + (1-t)H(x,y,z_2) \subset H(x,y,tz_1 + (1-t)z_2) + C;$$

(ii) C-quasiconvex if, for every $z_1, z_2 \in M$ and $t \in [0,1]$, one has either

$$H(x,y,z_1) \subset H(x,y,tz_1 + (1-t)z_2) + C;$$

or

$$H(x,y,z_2) \subset H(x,y,tz_1 + (1-t)z_2) + C.$$

Definition 2.6 ([5])**.** A subset $D \subset X$ is said to be arcwise connected iff, for every pair of points $x, z \in D$, there exists a continuous mapping $\varphi_{x,z}$, called an arc, defined on the unit interval $[0,1] \subseteq R$ and with values in D such that

$$\varphi_{x,z}(0) = x, \varphi_{x,z}(1) = z.$$

In the sequel, $\varphi_{x,z}$ will denote a continuous arc connecting x and z. It is easy to see that every convex set is arcwise connected.

Definition 2.7 ([32])**.** Let $D \subset X$ be a nonempty arcwise connected set, a set-valued mapping $F : D \to 2^Y$ is said to be C-arcwise connected iff

$$tF(z) + (1-t)F(x) \subset F(\varphi_{x,z}(t)) + C, \forall t \in [0,1], \forall x, z \in D.$$

Remark 2.8. If $F : D \to 2^Y$ is a C-arcwise connected set-valued mapping, then $F(D) + C$ is a convex set.

Remark 2.9 ([32])**.** The C-convex set-valued mapping is C-arcwise connected. However, there exist C-arcwise connected set-valued mappings which are not C-convex set-valued mappings.

Definition 2.10. Let X, Y, Z and E be topological vector spaces and C be a closed convex cone in E. $K \subset X$, $D \subset Y$ and $M \subset Z$ are nonempty convex sets. A set-valued mapping $H : K \times D \times M \to 2^E$ is said to be C-concave on $K \times D \times M$ if, for every $(x_1, y_1, z_1), (x_2, y_2, z_2) \in K \times D \times M$ and $t \in [0,1]$, one has

$$H(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2, tz_1 + (1-t)z_2) \subset tH(x_1, y_1, z_1) + (1-t)H(x_2, y_2, z_2) + C.$$

Lemma 2.11. *([4])Let X and Y be two Hausdorff topological vector spaces and $T : X \to 2^Y$ be a set-valued mapping.*

(i) *If T is upper semicontinuous with closed values, then T is closed;*

(ii) *If T is closed and Y is compact, then T is upper semicontinuous.*

Lemma 2.12 ([14])**.** *Let $\{K_i\}_{i \in I}$ be a family of nonempty convex subsets where each K_i is contained in a Hausdorff topological vector space X_i. For each $i \in I$, let $Q_i : K = \prod_{i \in I} K_i \to 2^{K_i}$ be a set-valued mapping such that*

(i) *for each $i \in I$, $Q_i(x)$ is convex;*

(ii) *for each $x \in K$, $x_i \notin Q_i(x)$;*

(iii) *for each* $y_i \in K_i$, $Q_i^{-1}(y_i)$ *is open in* K*;*

(iv) *if* K *is not compact, then there exists a nonempty compact subset* $\overline{K}$ *of* K *and a nonempty compact convex subset* B_i *of* K_i *such that for each* $x \in K \backslash \overline{K}$*, there exists a* $i \in I$ *such that* $Q_i(x) \cap B_i \neq \emptyset$*.*

Then there exists $x \in K$ *such that* $Q_i(x) = \emptyset$ *for all* $i \in I$*.*

Lemma 2.13 ([37])**.** *Let* X *and* Y *be two topological vector spaces,* S *be a connected subset of* X*,* $F : S \to 2^Y$ *be a set-valued mapping. If* F *is upper semicontinuous on* S *and* $F(x)$ *is connected subset of* Y *for each* $x \in S$*, then* $F(S) = \cup_{x\in S} F(x)$ *is a connected subset of* Y*.*

Lemma 2.14. *Let* $S : K \times D \to 2^K$ *and* $T : K \times D \to 2^D$ *be set-valued mappings with nonempty convex values. Suppose that* $F(x, y, \cdot)$ *is a* C*-convex mapping on* K *and* $F(y, x, \cdot)$ *is a* P*-convex mapping on* D*. Then,* $F(x, y, S(x, y)) + C$ *and* $G(y, x, T(x, y)) + P$ *are convex.*

Proof. Let $z_1, z_2 \in F(x, y, S(x, y)) + C$ and $t \in [0, 1]$. It follows that there exist $u_1, u_2 \in S(x, y)$ such that $z_1 \in F(x, y, u_1) + C$ and $z_2 \in F(x, y, u_2) + C$. Since $S(x, y)$ is convex and $F(x, y, \cdot)$ is C-convex, we have

$$\begin{aligned} tz_1 + (1-t)z_2 &\in tF(x, y, u_1) + tC + (1-t)F(x, y, u_2) + (1-t)C \\ &\subset F(x, y, tu_1 + (1-t)u_2) + C \subset F(x, y, S(x, y)) + C. \end{aligned}$$

Hence, $F(x, y, S(x, y)) + C$ is convex. Similarly, we can prove that $G(y, x, T(x, y)) + P$ is convex. □

Lemma 2.15. *Suppose that for any* $(\bar{x}, \bar{y}) \in A$*,* $F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y})) + C$ *and* $G(\bar{y}, \bar{x}, T(\bar{x}, \bar{y})) + P$ *are convex sets. Then*

$$V_w(F, G) = \bigcup_{(f,g)\in C^*\backslash\{0\}\times P^*\backslash\{0\}} V_w(f, g).$$

Proof. Suppose that

$$(\bar{x}, \bar{y}) \in \bigcup_{(f,g)\in C^*\backslash\{0\}\times P^*\backslash\{0\}} V_w(f, g).$$

Then there exists $(f, g) \in C^*\backslash\{0\} \times P^*\backslash\{0\}$ such that

$$\begin{cases} \inf\limits_{z\in F(\bar{x},\bar{y},u)} f(z) \geq 0, \forall u \in S(\bar{x}, \bar{y}), \\ \inf\limits_{z\in G(\bar{y},\bar{x},v)} g(z) \geq 0, \forall v \in T(\bar{x}, \bar{y}). \end{cases} \tag{2.1}$$

Now, we claim that

$$F(\bar{x}, \bar{y}, u) \cap (-intC) = \emptyset, \forall u \in S(\bar{x}, \bar{y}).$$

In fact, if there exists some $\bar{u} \in S(\bar{x}, \bar{y})$ and $\bar{z} \in F(\bar{x}, \bar{y}, \bar{u})$ such that $\bar{z} \in -intC$, then for $f \in C^*\backslash\{0\}$, we have

$$\inf_{z\in F(\bar{x},\bar{y},\bar{u})} f(z) \leq f(\bar{z}) < 0,$$

which contradicts (2.1). Similarly, we can prove that

$$G(\bar{y}, \bar{x}, v) \cap (-intP) = \emptyset, \forall v \in T(\bar{x}, \bar{y}).$$

Hence, $(\bar{x}, \bar{y}) \in V_w(F, G)$.

Conversely, let $(\bar{x}, \bar{y}) \in V_w(F, G)$. Then

$$\begin{cases} F(\bar{x}, \bar{y}, u) \cap (-intC) = \emptyset, \forall u \in S(\bar{x}, \bar{y}), \\ G(\bar{y}, \bar{x}, v) \cap (-intP) = \emptyset, \forall v \in T(\bar{x}, \bar{y}). \end{cases}$$

It follows that

$$\begin{cases} F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y})) \cap (-intC) = \emptyset, \\ G(\bar{y}, \bar{x}, T(\bar{x}, \bar{y})) \cap (-intP) = \emptyset \end{cases}$$

and so

$$\begin{cases} (F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y})) + C) \cap (-intC) = \emptyset, \\ (G(\bar{y}, \bar{x}, T(\bar{x}, \bar{y})) + P) \cap (-intP) = \emptyset. \end{cases}$$

Since $F(\bar{x}, \bar{y}, S(\bar{x}, \bar{y})) + C$ and $G(\bar{y}, \bar{x}, T(\bar{x}, \bar{y})) + P$ are convex sets, by the separation theorem of convex sets, there exists some $(f, g) \in E^*\backslash\{0\} \times Z^*\backslash\{0\}$ such that

$$\inf\{f(z + c) : c \in C, \bar{u} \in S(\bar{x}, \bar{y}), z \in F(\bar{x}, \bar{y}, \bar{u})\} \geq \sup\{f(c) : c \in -intC\} \tag{2.2}$$

and

$$\inf\{g(z + p) : p \in P, \bar{v} \in T(\bar{x}, \bar{y}), z \in G(\bar{y}, \bar{x}, \bar{v})\} \geq \sup\{g(p) : p \in -intP\}. \tag{2.3}$$

Since C is a cone, $f(c) \geq 0$ for all $c \in C$, we know that $f \in C^*$. This together with (2.2) yields $f \in C^*\backslash\{0\}$ and

$$\inf_{z \in F(\bar{x}, \bar{y}, u)} f(z) \geq 0, \forall u \in S(\bar{x}, \bar{y}).$$

Also, by (2.3), we see that $g \in P^*\backslash\{0\}$ and

$$\inf_{z \in G(\bar{y}, \bar{x}, v)} g(z) \geq 0, \forall v \in T(\bar{x}, \bar{y}).$$

This means that $(\bar{x}, \bar{y}) \in V_w(f, g)$. This completes the proof. □

Remark 2.16. (i) Lemma 2.15 generalizes Theorem 3.1 of [7] from set-valued vector equilibrium problem to symmetric vector quasiequilibrium problem.

(ii) Lemma 2.15 generalizes Lemma 2.1 of [39] from symmetric vector equilibrium problem to symmetric vector quasiequilibrium problem.

3 Existence of the Solutions

In this section, we give an existence theorem of the solutions for scalar symmetric vector quasiequilibrium problem.

Theorem 3.1. *Assume that*

(i) *For each $(x, y) \in K \times D$, $F(x, y, x) \subset C$ and $G(y, x, y) \subset P$;*

(ii) *For each $u \in K$, $F(\cdot, \cdot, u)$ is C-lower semicontinuous on $K \times D$ and for each $v \in D$, $F(\cdot, \cdot, v)$ is P-lower semicontinuous on $D \times K$;*

(iii) *For each $(x, y) \in K \times D$, $F(x, y, \cdot)$ is C-quasiconvex on K or C-convex on K and for each $(x, y) \in K \times D$, $G(y, x, \cdot)$ is P-quasiconvex on D or P-convex on D;*

(iv) S and T are upper semicontinuous with nonempty closed convex values on $K \times D$ and for each $(x, y) \in K \times D$, $S^{-1}(x)$ and $T^{-1}(y)$ are open in $K \times D$;

(v) *If $K \times D$ is not compact, there exists a nonempty compact subset $K_0 \times D_0$ of $K \times D$ and a nonempty compact convex subset $K_1 \times D_1$ of $K \times D$ such that for each $(x, y) \in (K \times D)\backslash(K_0 \times D_0)$, there exists a $\bar{x} \in K_1 \cap S(x, y)$ such that $F(x, y, \bar{x}) \cap (-intC) \neq \emptyset$ or a $\bar{y} \in D_1 \cap T(x, y)$ such that $G(y, x, \bar{y}) \cap (-intP) \neq \emptyset$.*

Then $V_w(f, g) \neq \emptyset$ for all $(f, g) \in C^*\backslash\{0\} \times P^*\backslash\{0\}$.

Proof. Let $(f, g) \in C^*\backslash\{0\} \times P^*\backslash\{0\}$. For any $(x, y) \in K \times D$, set

$$P_1 = \{(x, y) \in K \times D : x \in S(x, y)\},$$

$$P_2 = \{(x, y) \in K \times D : y \in T(x, y)\},$$

$$Q_1(x, y) = \{\bar{x} \in K : \inf_{z \in F(x,y,\bar{x})} f(z) < 0\},$$

$$Q_2(x, y) = \{\bar{y} \in D : \inf_{z \in G(y,x,\bar{y})} g(z) < 0\},$$

$$R_1(x, y) = \begin{cases} S(x, y) \cap Q_1(x, y), & \text{if } (x, y) \in P_1; \\ S(x, y), & \text{otherwise} \end{cases}$$

and

$$R_2(x, y) = \begin{cases} T(x, y) \cap Q_2(x, y), & \text{if } (x, y) \in P_2; \\ T(x, y), & \text{otherwise.} \end{cases}$$

Next we will show that R_1 and R_2 satisfy all the conditions of Lemma 2.12. For any $x \in K$, we have

$$\begin{aligned} R_1^{-1}(x) &= \{(\bar{x}, \bar{y}) \in P_1 : x \in S(\bar{x}, \bar{y}) \cap Q_1(\bar{x}, \bar{y})\} \cup \{(\bar{x}, \bar{y}) \in K \times D\backslash P_1 : x \in S(\bar{x}, \bar{y})\} \\ &= \{(\bar{x}, \bar{y}) \in P_1 : (\bar{x}, \bar{y}) \in S^{-1}(x) \cap Q_1^{-1}(x)\} \cup \{(\bar{x}, \bar{y}) \in K \times D\backslash P_1 : (\bar{x}, \bar{y}) \in S^{-1}(x)\} \\ &= \{P_1 \cap S^{-1}(x) \cap Q_1^{-1}(x)\} \cup \{[(K \times D)\backslash P_1] \cap S^{-1}(x)\} \\ &= \{[(K \times D)\backslash P_1] \cup Q_1^{-1}(x)\} \cap S^{-1}(x). \end{aligned} \tag{3.1}$$

It follows from (3.1) that

$$\begin{aligned} (K \times D)\backslash R_1^{-1}(x) &= (K \times D)\backslash\{[((K \times D)\backslash P_1) \cup Q_1^{-1}(x)] \cap S^{-1}(x)\} \\ &= \{(K \times D)\backslash[((K \times D)\backslash P_1) \cup Q_1^{-1}(x)]\} \cup [(K \times D)\backslash S^{-1}(x)] \\ &= \{[(K \times D)\backslash Q_1^{-1}(x)] \cap P_1\} \cup [(K \times D)\backslash S^{-1}(x)]. \end{aligned} \tag{3.2}$$

Since S is upper semicontinuous with closed values, we know that P_1 is closed. Next we show that

$$(K \times D)\backslash Q_1^{-1}(x) = \{(\bar{x}, \bar{y}) \in K \times D : \inf_{z \in F(\bar{x},\bar{y},x)} f(z) \geq 0\}$$

is closed. Let $\{(x_\alpha, y_\alpha)\} \subset (K \times D)\backslash Q_1^{-1}(x)$ with $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y})$. Since $\{(x_\alpha, y_\alpha)\} \subset (K \times D)\backslash Q_1^{-1}(x)$, we have

$$\inf_{z \in F(x_\alpha,y_\alpha,x)} f(z) \geq 0. \tag{3.3}$$

We claim that

$$\inf_{z \in F(\bar{x},\bar{y},x)} f(z) \geq 0.$$

If not, then there exists $\bar{z} \in F(\bar{x}, \bar{y}, x)$ such that $f(\bar{z}) < 0$. Set $U = \{z \in E : |f(z)| < \frac{1}{2}|f(\bar{z})|\}$. Clearly, U is a neighborhood of 0 in E. By the C-lower semicontinuity of $F(\cdot, \cdot, x)$, there exists a neighborhood V of $(\bar{x}, \bar{y})$ such that

$$F(x', y', x) \cap (\bar{z} + U - C) \neq \emptyset, \quad \forall (x', y') \in V. \tag{3.4}$$

It follows from $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y})$ and (3.4) that there exists α_0 such that

$$F(x_\alpha, y_\alpha, x) \cap (\bar{z} + U - C) \neq \emptyset, \quad \forall \alpha \geq \alpha_0.$$

Hence, there exists $z_\alpha \in F(x_\alpha, y_\alpha, x)$ such that $z_\alpha \in \bar{z} + U - C$. Therefore, there exists $u_\alpha \in U$ and $c_\alpha \in C$ such that $z_\alpha = \bar{z} + u_\alpha - c_\alpha$. So

$$f(z_\alpha) = f(\bar{z}) + f(u_\alpha) - f(c_\alpha) < f(\bar{z}) + \frac{1}{2}|f(\bar{z})| - f(c_\alpha) = \frac{1}{2}f(\bar{z}) - f(c_\alpha) < 0. \tag{3.5}$$

It follows from (3.3) and $z_\alpha \in F(x_\alpha, y_\alpha, x)$ that

$$f(z_\alpha) \geq \inf_{z \in F(x_\alpha, y_\alpha, x)} f(z) \geq 0,$$

which contradicts (3.5). Hence, $(\bar{x}, \bar{y}) \in (K \times D) \backslash Q_1^{-1}(x)$. Therefore, it follows from (3.2) that $(K \times D) \backslash R_1^{-1}(x)$ is closed. Thus, $R_1^{-1}(x)$ is open in $K \times D$ for each $x \in K$. Similarly, we can prove that $R_2^{-1}(y)$ is open in $K \times D$ for each $y \in D$.

Next we show that $Q_1(x, y)$ and $Q_2(x, y)$ are convex for any $(x, y) \in K \times D$. In fact, for any $x_1, x_2 \in Q_1(x, y)$, we have

$$\inf_{z \in F(x, y, x_1)} f(z) < 0, \qquad \inf_{z \in F(x, y, x_2)} f(z) < 0.$$

Hence, there exist $z_1 \in F(x, y, x_1)$ and $z_2 \in F(x, y, x_2)$ such that $f(z_1) < 0$ and $f(z_2) < 0$. When $F(x, y, \cdot)$ is C-convex set-valued mapping on K, we have

$$(1-t)F(x, y, x_1) + tF(x, y, x_2) \subset F(x, y, (1-t)x_1 + tx_2) + C,$$

and so there exist $\bar{z} \in F(x, y, (1-t)x_1 + tx_2)$ and $c \in C$ such that $(1-t)z_1 + tz_2 = \bar{z} + c$. This shows that

$$\inf_{z \in F(x, y, (1-t)x_1 + tx_2)} f(z) \leq f(\bar{z}) = (1-t)f(z_1) + tf(z_2) - f(c) < 0.$$

When $F(x, y, \cdot)$ is C-quasiconvex on K, we have

either $F(x, y, x_1) \subset F(x, y, tx_1 + (1-t)x_2) + C$ or $F(x, y, x_2) \subset F(x, y, tx_1 + (1-t)x_2) + C$.

If $F(x, y, x_1) \subset F(x, y, tx_1 + (1-t)x_2) + C$, then there exist $\bar{z} \in F(x, y, tx_1 + (1-t)x_2)$ and $c \in C$ such that $z_1 = \bar{z} + c$ and so

$$\inf_{z \in F(x, y, tx_1 + (1-t)x_2)} f(z) \leq f(\bar{z}) = f(z_1) - f(c) < 0.$$

If $F(x, y, x_2) \subset F(x, y, tx_1 + (1-t)x_2) + C$, then

$$\inf_{z \in F(x, y, tx_1 + (1-t)x_2)} f(z) < 0$$

and so $tx_1 + (1-t)x_2) \in Q_1(x,y)$. Thus, $Q_1(x,y)$ is convex. By the definition of $R_1(x,y)$, we know that $R_1(x,y)$ is convex. Similarly, we can prove that $Q_2(x,y)$ and $R_2(x,y)$ are convex.

Due to the fact that $F(x,y,x) \subset C$ for each $(x,y) \in K \times D$, we have

$$\inf_{z \in F(x,y,x)} f(z) \geq 0$$

and hence $x \notin Q_1(x,y)$. If $(x,y) \in P_1$, then $x \notin R_1(x,y)$. If $(x,y) \in (K \times D) \backslash P_1$, then $x \notin S(x,y)$ and hence $x \notin R_1(x,y)$. Similarly, we have $y \notin R_2(x,y)$ for any $(x,y) \in K \times D$.

By condition (v), for each $(x,y) \in (K \times D) \backslash (K_0 \times D_0)$, if there exists $\bar{x} \in K_1 \cap S(x,y)$ such that $F(x,y,\bar{x}) \cap (-intC) \neq \emptyset$, then there exists $\bar{z} \in F(x,y,\bar{x})$ such that $\bar{z} \in -intC$. It follows from

$$\inf_{z \in F(x,y,\bar{x})} f(z) \leq f(\bar{z}) < 0$$

that $\bar{x} \in Q_1(x,y)$ and so $K_1 \cap R_1(x,y) \neq \emptyset$. If there exists $\bar{y} \in D_1 \cap T(x,y)$ such that $G(y,x,\bar{y}) \cap (-intP) \neq \emptyset$, then there exists $\bar{z} \in G(y,x,\bar{y})$ such that $\bar{z} \in -intP$. It follows from

$$\inf_{z \in G(y,x,\bar{y})} g(z) \leq g(\bar{z}) < 0$$

that $\bar{y} \in Q_2(x,y)$ and so $D_1 \cap R_2(x,y) \neq \emptyset$.

It follows that all the assumptions of Lemma 2.12 are satisfied and so there exists $(\bar{x},\bar{y}) \in K \times D$ such that $R_1(\bar{x},\bar{y}) = R_2(\bar{x},\bar{y}) = \emptyset$. Since $S(\bar{x},\bar{y})$ and $T(\bar{x},\bar{y})$ are nonempty sets, $(\bar{x},\bar{y})$ must be in $P_1 \cap P_2$. Hence, $R_1(\bar{x},\bar{y}) = S(\bar{x},\bar{y}) \cap Q_1(\bar{x},\bar{y}) = \emptyset$ and $R_2(\bar{x},\bar{y}) = T(\bar{x},\bar{y}) \cap Q_2(\bar{x},\bar{y}) = \emptyset$. Thus, for all $x \in S(\bar{x},\bar{y})$ and $y \in T(\bar{x},\bar{y})$, we have $x \notin Q_1(\bar{x},\bar{y})$ and $y \notin Q_2(\bar{x},\bar{y})$, i.e.,

$$\begin{cases} \inf\limits_{z \in F(\bar{x},\bar{y},x)} f(z) \geq 0, & \forall x \in S(\bar{x},\bar{y}), \\ \inf\limits_{z \in G(\bar{y},\bar{x},y)} g(z) \geq 0, & \forall y \in T(\bar{x},\bar{y}). \end{cases}$$

Hence, $V_w(f,g) \neq \emptyset$ for all $(f,g) \in C^* \backslash \{0\} \times P^* \backslash \{0\}$. This completes the proof. □

4 Connectedness of the Solution Set

In this section, we discuss the connectedness and the compactness of the weak efficient solution set for symmetric vector quasiequilibrium problem.

Theorem 4.1. *Assume that*

(i) *For each $(x,y) \in K \times D$, $F(x,y,x) \subset C$ and $G(y,x,y) \subset P$;*

(ii) *F is C-lower semicontinuous on $K \times D \times K$ and G is P-lower semicontinuous on $D \times K \times D$;*

(iii) *For each $(x,y) \in K \times D$, $F(x,y,\cdot)$ is C-quasiconvex, C-arcwise connected on K and $G(y,x,\cdot)$ is P-quasiconvex, P-arcwise connected on D; F is C-concave on $K \times D \times K$ and G is P-concave on $D \times K \times D$;*

(iv) *S and T are continuous with nonempty closed convex values on $K \times D$ and for each $(x,y) \in K \times D$, $S^{-1}(x)$ and $T^{-1}(y)$ are open in $K \times D$;*

(v) *If $K \times D$ is not compact, then there exists a nonempty compact subset $K_0 \times D_0$ of $K \times D$ and a nonempty compact convex subset $K_1 \times D_1$ of $K \times D$ such that, for each $(x, y) \in (K \times D) \backslash (K_0 \times D_0)$, there exists a $\bar{x} \in K_1 \cap S(x, y)$ such that $F(x, y, \bar{x}) \cap (-intC) \neq \emptyset$ or a $\bar{y} \in D_1 \cap T(x, y)$ such that $G(y, x, \bar{y}) \cap (-intP) \neq \emptyset$.*

(vi) *$\{F(x, y, u) : x \in K, y \in D, u \in K\}$ is a bounded subset in E and $\{G(x, y, u) : x \in D, y \in K, u \in D\}$ is a bounded subset in Z;*

(vii) *For any $t \in [0, 1]$, $(x_1, y_1), (x_2, y_2) \in K \times D$, $tS(x_1, y_1) + (1-t)S(x_2, y_2) = S(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$ and $tT(x_1, y_1) + (1-t)T(x_2, y_2) = T(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$.*

Then $V_w(F, G)$ is nonempty connected compact.

Proof. We define the set-valued mapping $H : C^* \backslash \{0\} \times P^* \backslash \{0\} \to 2^{K \times D}$ by

$$H(f, g) = V_w(f, g), \forall (f, g) \in C^* \backslash \{0\} \times P^* \backslash \{0\}.$$

By Theorem 3.1, we have $H(f, g) \neq \emptyset$ for each $(f, g) \in C^* \backslash \{0\} \times P^* \backslash \{0\}$. Hence, $V_w(F, G) \neq \emptyset$. It is clear that $C^* \backslash \{0\} \times P^* \backslash \{0\}$ is convex and so it is connected. Now we prove that, for each $(f, g) \in C^* \backslash \{0\} \times P^* \backslash \{0\}$, $H(f, g)$ is a connected set. Suppose that $(x_1, y_1), (x_2, y_2) \in H(f, g)$. Then $(x_1, y_1) \in S(x_1, y_1) \times T(x_1, y_1)$, $(x_2, y_2) \in S(x_2, y_2) \times T(x_2, y_2)$,

$$\begin{cases} \inf\limits_{z \in F(x_1, y_1, u)} f(z) \geq 0, \quad \forall u \in S(x_1, y_1), \\ \inf\limits_{z \in G(y_1, x_1, v)} g(z) \geq 0, \quad \forall v \in T(x_1, y_1). \end{cases} \tag{4.1}$$

and

$$\begin{cases} \inf\limits_{z \in F(x_2, y_2, u)} f(z) \geq 0, \quad \forall u \in S(x_2, y_2), \\ \inf\limits_{z \in G(y_2, x_2, v)} g(z) \geq 0, \quad \forall v \in T(x_2, y_2). \end{cases} \tag{4.2}$$

By condition (vii), we have

$$tx_1 + (1-t)x_2 \in tS(x_1, y_1) + (1-t)S(x_2, y_2) = S(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$$

and

$$ty_1 + (1-t)y_2 \in tT(x_1, y_1) + (1-t)T(x_2, y_2) = T(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

For any $z \in S(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$, there exist $z_1 \in S(x_1, y_1)$ and $z_2 \in S(x_2, y_2)$ such that $z = tz_1 + (1-t)z_2$ through the condition (vii). It follows from condition (iii) that

$$\begin{aligned} F(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2, z) &= F(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2, tz_1 + (1-t)z_2) \\ &\subset tF(x_1, y_1, z_1) + (1-t)F(x_2, y_2, z_2) + C. \end{aligned} \tag{4.3}$$

Hence, for any $r \in F(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2, z)$, there exist $r_1 \in F(x_1, y_1, z_1)$, $r_2 \in F(x_2, y_2, z_2)$ and $c \in C$ such that $r = tr_1 + (1-t)r_2 + c$. It follows from (4.1) and (4.2) that

$$f(r) = tf(r_1) + (1-t)f(r_2) + f(c) \geq t \inf_{z \in F(x_1, y_1, z_1)} f(z) + (1-t) \inf_{z \in F(x_2, y_2, z_2)} f(z) + f(c) \geq 0.$$

This means that

$$\inf_{r \in F(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2, z)} f(r) \geq 0, \forall z \in S(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2).$$

Similarly, we can prove that

$$\inf_{r\in G(ty_1+(1-t)y_2,tx_1+(1-t)x_2,z)} g(r)\geq 0, \forall z\in T(tx_1+(1-t)x_2, ty_1+(1-t)y_2).$$

Hence, $t(x_1,y_1)+(1-t)(x_2,y_2)\in H(f,g)$. Thus $H(f,g)$ is convex and therefore it is also a connected set.

Now we show that H is upper semicontinuous on $C^*\backslash\{0\}\times P^*\backslash\{0\}$. Since $V_w(f,g)\subset V_w(F,G)\subset K_0\times D_0$ and $K_0\times D_0$ is compact, we only need to show that H is closed. Let $\{(f_\alpha,g_\alpha),(x_\alpha,y_\alpha):\alpha\in I\}\subset Graph(H)$ with $(f_\alpha,g_\alpha)\to(f_0,g_0)$ and $(x_\alpha,y_\alpha)\to(x_0,y_0)$, where $f_\alpha\to f_0$ means that $\{f_\alpha\}$ converges to f_0 with respect to the strong topological $\beta(E^*,E)$ in E^*, $g_\alpha\to g_0$ means that $\{g_\alpha\}$ converges to g_0 with respect to the strong topological $\beta(Z^*,Z)$ in Z^*. Since $(x_\alpha,y_\alpha)\in H(f_\alpha,g_\alpha)$, we have

$$\begin{cases}\inf\limits_{z\in F(x_\alpha,y_\alpha,u)} f_\alpha(z)\geq 0, & \forall u\in S(x_\alpha,y_\alpha),\\ \inf\limits_{z\in G(y_\alpha,x_\alpha,v)} g_\alpha(z)\geq 0, & \forall v\in T(x_\alpha,y_\alpha).\end{cases} \tag{4.4}$$

Note that $W=\{F(x,y,u):x\in K,y\in D,u\in K\}$ is a bounded subset of E. For each $y^*\in E^*$, we define

$$P_W(y^*):=\sup\{|y^*(s)|:s\in W\}.$$

It is easy to see that P_W is a seminorm of E^*. For any $\epsilon>0$,

$$U=\{y^*\in E^*:P_W(y^*)<\epsilon\}$$

is a neighborhood of zero with respect to $\beta(E^*,E)$ in E^*. Since $f_\alpha\to f_0$, there exists $\alpha_0\in I$ such that $f_\alpha-f_0\in U$ for all $\alpha\geq\alpha_0$. It follows that

$$P_W(f_\alpha-f_0)=\sup\{|(f_\alpha-f)(s)|:s\in W\}<\varepsilon,\forall\alpha\geq\alpha_0. \tag{4.5}$$

We claim that

$$\inf_{z\in F(x_0,y_0,u)} f_0(z)\geq 0,\forall u\in S(x_0,y_0).$$

If not, there exist $u_0\in S(x_0,y_0)$ and $z_0\in F(x_0,y_0,u_0)$ such that $f_0(z_0)<0$. By the lower semicontinuity of S, there exists $u_\alpha\in S(x_\alpha,y_\alpha)$ such that $u_\alpha\to u_0$. It follows from (4.4) that

$$\inf_{z\in F(x_\alpha,y_\alpha,u_\alpha)} f_\alpha(z)\geq 0. \tag{4.6}$$

Set $U'=\{z\in E:|f_0(z)|<\frac{1}{2}|f_0(z_0)|\}$. Clearly, U' is a neighborhood of 0 in E. By the C-lower semicontinuity of F, there exists a neighborhood V of (x_0,y_0,u_0) such that

$$F(x',y',u')\cap(z_0+U'-C)\neq\emptyset,\forall(x',y',u')\in V. \tag{4.7}$$

It follows from $(x_\alpha,y_\alpha,u_\alpha)\to(x_0,y_0,u_0)$ and (4.7) that there exists $\alpha_1\geq\alpha_0$ such that

$$F(x_\alpha,y_\alpha,u_\alpha)\cap(z_0+U'-C)\neq\emptyset,\forall\alpha\geq\alpha_1.$$

Hence, there exists $z_\alpha\in F(x_\alpha,y_\alpha,u_\alpha)$ such that $z_\alpha\in z_0+U'-C$ and so there exist $v_\alpha\in U'$ and $c_\alpha\in C$ such that $z_\alpha=z_0+v_\alpha-c_\alpha$. Thus,

$$f_0(z_\alpha)=f_0(z_0)+f_0(v_\alpha)-f_0(c_\alpha)<f_0(z_0)+\frac{1}{2}|f_0(z_0)|-f_0(c_\alpha)=\frac{1}{2}f_0(z_0)-f_0(c_\alpha)\leq\frac{1}{2}f_0(z_0) \tag{4.8}$$

It follows from (4.6) and $z_\alpha \in F(x_\alpha, y_\alpha, u_\alpha)$ that

$$f_\alpha(z_\alpha) \geq \inf_{z \in F(x_\alpha, y_\alpha, u_\alpha)} f_\alpha(z) \geq 0. \tag{4.9}$$

By (4.5), we have

$$|(f_\alpha - f_0)(z_\alpha)| < \epsilon, \forall \alpha \geq \alpha_1,$$

which implies that

$$\lim\, [f_\alpha(z_\alpha) - f_0(z_\alpha)] = 0. \tag{4.10}$$

Now, (4.8) and (4.10) show that

$$\begin{aligned} \text{limsup } f_\alpha(z_\alpha) &= \text{limsup } (f_\alpha(z_\alpha) - f_0(z_\alpha) + f_0(z_\alpha)) \\ &= \lim\, (f_\alpha(z_\alpha) - f_0(z_\alpha)) + \text{limsup } f_0(z_\alpha) \\ &\leq \frac{1}{2} f_0(z_0) < 0, \end{aligned} \tag{4.11}$$

which contradicts (4.9). In a similar way, we can prove that

$$\inf_{z \in G(y_0, x_0, v)} g_0(z) \geq 0, \quad \forall v \in T(x_0, y_0).$$

Thus, $\{(f_0, g_0), (x_0, y_0)\} \subset Graph(H)$ and so H is a closed mapping. By Lemma 2.14, for each $(x, y) \in A$, $F(x, y, S(x, y)) + C$ and $G(y, x, T(x, y)) + P$ are convex. It follows from Lemma 2.15 that

$$V_w\,(F, G) = \bigcup_{(f,g) \in C^* \setminus \{0\} \times P^* \setminus \{0\}} V_w\,(f, g).$$

By Lemma 2.13, $V_w(F, G)$ is a connected set.

Next, we show that $V_w(F, G)$ is compact. Let $(x_\alpha, y_\alpha) \in V_w(F, G)$ with $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Then $(x_\alpha, y_\alpha) \in S(x_\alpha, y_\alpha) \times T(x_\alpha, y_\alpha)$ and

$$\begin{cases} F(x_\alpha, y_\alpha, u) \cap (-intC) = \emptyset, & \forall u \in S(x_\alpha, y_\alpha), \\ G\,(y_\alpha, x_\alpha, v) \cap (-intP) = \emptyset, & \forall v \in T(x_\alpha, y_\alpha). \end{cases} \tag{4.12}$$

Due to the fact that S and T are upper semicontinuous with closed values, we have $(x_0, y_0) \in S(x_0, y_0) \times T(x_0, y_0)$. We claim that $(x_0, y_0) \in V_w(F, G)$. If not, then there exists some $u_0 \in S(x_0, y_0)$ such that

$$F(x_0, y_0, u_0) \cap (-intC) \neq \emptyset, \tag{4.13}$$

or there exists some $v_0 \in T(x_0, y_0)$ such that

$$G(y_0, x_0, v_0) \cap (-intP) \neq \emptyset. \tag{4.14}$$

If (4.13) holds, then there exists some $d_0 \in F(x_0, y_0.u_0)$ such that $d_0 \in -intC$. This implies that there exists some neighborhood U of zero such that $d_0 + U \subset -intC$ and so

$$d_0 + U - C \subset -intC - C \subset -intC. \tag{4.15}$$

Since S is lower semicontinuous on $K \times D$, there exists $u_\alpha \in S(x_\alpha, y_\alpha)$ such that $u_\alpha \to u_0$. By the C-lower semicontinuity of F, there exists some neighborhood V of (x_0, y_0, u_0) such that

$$F(x', y', u') \cap (d_0 + U - C) \neq \emptyset, \quad \forall (x', y', u') \in V. \tag{4.16}$$

It follows from $(x_\alpha, y_\alpha, u_\alpha) \to (x_0, y_0, u_0)$ and (4.16) that there exists α_0 such that

$$F(x_\alpha, y_\alpha, u_\alpha) \cap (d_0 + U - C) \neq \emptyset, \quad \forall \alpha \geq \alpha_0. \tag{4.17}$$

By (4.15) and (4.17), we have

$$F(x_\alpha, y_\alpha, u_\alpha) \cap (-intC) \neq \emptyset, \quad \forall \alpha \geq \alpha_0,$$

which contradicts (4.12). Thus, $(x_0, y_0) \in V_w(F, G)$. If (4.14) holds, in a similar way, we can show that $(x_0, y_0) \in V_w(F, G)$. Therefore, $V_w(F, G)$ is closed. Noting that the compactness of $K_0 \times D_0$ and $V_w(F, G) \subset K_0 \times D_0$, we know that $V_w(F, G)$ is compact. This completes the proof. □

Remark 4.2. (i) Theorem 4.1 generalizes Theorem 5.1 of [7] from the set-valued vector equilibrium problem to the symmetric vector quasiequilibrium problem;

(ii) Theorem 4.1 generalizes Theorem 3.1 of [39] from the symmetric vector equilibrium problem to the symmetric vector quasiequilibrium problem;

(iii) Theorem 4.1 also generalizes the corresponding connected results presented in [13] and [23].

Now we give an example to illustrate Theorem 4.1.

Example 4.3. Let $X = Y = E = Z = R$, $C = P = [0, +\infty)$, and $K = D = [0, 1]$. For each $x \in K$, $y \in D$, $S(x, y) = [0, 1]$ and $T(x, y) = [0, 1]$. Define the set-valued mappings F and G by

$$F(x, y, z) = [3x + 2y - z, 11), \forall (x, y, z) \in K \times D \times K$$

and

$$G(y, x, z) = [x + 4y - 2z, 15], \forall (y, x, z) \in D \times K \times D,$$

respectively. It is easy to see that all assumptions of Theorem 4.1 are satisfied. Let H be the solution set of (SVQEP). Then it is easy to check that

$$H = \{\bar{x}, \bar{y}) \in K \times D : 3\bar{x} + 2\bar{y} \geq 1, \bar{x} + 4\bar{y} \geq 2\}$$

and H is a nonempty connected compact subset of $K \times D$.

If for any $(x, y) \in K \times D$, $S(x, y) \equiv K$ and $T(x, y) \equiv D$, then from Theorem 4.1, we have the following corollary.

Corollary 4.4. *Assume that*

(i) *For each $(x, y) \in K \times D$,$F(x, y, x) \subset C$ and $G(y, x, y) \subset P$;*

(ii) *For each $u \in K$, $F(\cdot, \cdot, u)$ is C-lower semicontinuous on $K \times D$ and for each $v \in D$, $G(\cdot, \cdot, v)$ is P-lower semicontinuous on $D \times K$;*

(iii) *For each $(x, y, u) \in K \times D \times K$, $F(x, y, \cdot)$ is C-quasiconvex, C-arcwise connected on K and $F(\cdot, \cdot, u)$ is C-concave on $K \times D$ and for each $(x, y, v) \in D \times K \times D$, $G(x, y, \cdot)$ is P-quasiconvex, P-arcwise connected on D and $G(\cdot, \cdot, v)$ is P-concave on $D \times K$;*

(iv) *$\{F(x, y, u) : x \in K, y \in D, u \in K\}$ is a bounded subset in E and $\{G(x, y, u) : x \in D, y \in K, u \in D\}$ is a bounded subset in Z;*

(v) *If* $K \times D$ *is not compact, then there exists a nonempty compact subset* $K_0 \times D_0$ *of* $K \times D$ *and a nonempty compact convex subset* $K_1 \times D_1$ *of* $K \times D$ *such that for each* $(x, y) \in (K \times D) \backslash (K_0 \times D_0)$*, there exists a* $\bar{x} \in K_1$ *such that* $F(x, y, \bar{x}) \cap (-intC) \neq \emptyset$ *or a* $\bar{y} \in D_1$ *such that* $G(y, x, \bar{y}) \cap (-intP) \neq \emptyset$.

Then $V_w(F, G)$ *is nonempty connected compact.*

Remark 4.5. Corollary 4.4 improves Theorem 3.1 of [39] in the following three aspects:

(i) The lower semicontinuity is relaxed to the C-lower semicontinuity;

(ii) The C-convex is relaxed to the C-arcwise connected;

(iii) The condition (v) of Corallary 4.4 is weaker than the condition (v) of Theorem 3.1 in [39].

Remark 4.6. In the corollary 4.4, the condition (iii) can be replaced by the following condition:

(iii') For each $(x, y, u) \in K \times D \times K$, $F(x, y, \cdot)$ is C-convex on K and $F(\cdot, \cdot, u)$ is C-concave on $K \times D$ and for each $(x, y, v) \in D \times K \times D$, $G(x, y, \cdot)$ is P-convex on D and $G(\cdot, \cdot, v)$ is P-concave on $D \times K$.

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