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A PROXIMAL ALTERNATING DIRECTION METHOD FOR WEAKLY COUPLED VARIATIONAL INEQUALITIES*

Lingling Xu and Deren Han

Abstract: In this paper, we present a proximal alternating direction method (PADM) to solve a class of weakly coupled variational inequalities. The algorithm generates the iterate via a simple correction step, where the descent direction is based on the PADM. Under some mild assumptions, we prove the convergence of the generated sequence. We also report some preliminary numerical results, illustrating that the method is efficient.

Key words: *proximal point method, alternating direction method, weakly coupled variational inequality, Nash equilibrium*

Mathematics Subject Classification: *65K05, 65K10, 49J40*

1 Introduction

Let $X \subseteq R^n$ and $Y \subseteq R^m$ be two nonempty, closed and convex sets, $G \in R^{l\times m}$ be a given matrix, and $h : R^n \to R^l$ and $g : R^{m+n} \to R^t$ be two continuous mappings. In this paper, we consider the following weakly coupled variational inequality (VI) consisting of finding $(x^*, y^*) \in \Omega$, such that

$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{pmatrix} h(x^*) + Gy^* \\ g(x^*, y^*) \end{pmatrix} \geq 0, \ \forall (x, y) \in \Omega, \tag{1.1}$$

where

$$\Omega = \{(x, y) \ |Ax + By = b, \ x \in X, \quad y \in Y\}, \tag{1.2}$$

$A \in R^{r\times n}, \ B \in R^{r\times m}$ are two given matrices, and $b \in R^r$ is a given vector.

Weakly coupled variational inequality problems have wide applications in various fields, see [2],[6]-[8]. Specially, consider the two-person game:

$$\begin{array}{rlcrl} & \min \theta_1(x, \ y) & | & & \min \theta_2(x, \ y) \\ s.t. & Ax + By = b, \quad x \in X & | & s.t. & Ax + By = b, \quad y \in Y, \end{array}$$

where θ_1 and $\theta_2 : R^{m+n} \to R$ are differentiable convex functions; $X \subseteq R^n$, $Y \subseteq R^m$ are given closed convex set; $A \in R^{r\times n}$ and $B \in R^{r\times m}$. Calculating a normalized equilibrium of the game is equivalent to solving a VI:

$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}^T \begin{pmatrix} \nabla_x \theta_1(x^*, y^*) \\ \nabla_y \theta_2(x^*, y^*) \end{pmatrix} \geq 0, \ \forall (x, y) \in \Omega, \tag{1.3}$$

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where Ω is defined as (1.2). If $\theta_1(x, y)$ or $\theta_2(x, y)$ is weakly coupled, i.e., if the function can be expressed as $f(x) + s(y) + \|Px + Gy\|_2^2$, then (1.3) has the same structure as (1.1).

By attaching a Lagrange multiplier vector $\lambda \in R^r$ to the linear constraints $Ax + By = b$, problem (1.1) can be rewritten in the following form:

$$\text{Find } \omega^* \in W^*, \quad \text{such that } (\omega - \omega^*)^T F(\omega^*) \geq 0, \quad \forall\, \omega \in W, \tag{1.4}$$

where $\omega = (x, y, \lambda)$; $W = X \times Y \times R^r$; and W^* denotes the solution set of (1.4), and

$$F(\omega) = F(x, y, \lambda) = \begin{pmatrix} h(x) + Gy - A^T\lambda \\ g(x, y) - B^T\lambda \\ Ax + By - b \end{pmatrix}.$$

When $G = 0$ and $g(x, y) = g(y)$, problem (1.1) reduces to the variational inequality with separable structure. A powerful tool for solving such a problem is the alternating direction method (ADM for short); see e.g., [2], [13], [15], [16], [17], [19], [22]. For given current iterate (x_k, y_k, λ_k), ADM generates $\tilde{x}_k \in X$ and $\tilde{y}_k \in Y$ by solving the following subproblems

$$(x - \widetilde{x}_k)^T \{(h(\widetilde{x}_k) + Gy_k - A^T(\lambda_k - H(A\widetilde{x}_k + By_k - b))\} \geq 0, \ \forall x \in X \tag{1.5}$$

and

$$(y - \widetilde{y}_k)^T \{g(\widetilde{x}_k, \widetilde{y}_k) - B^T(\lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b))\} \geq 0, \ \forall y \in Y, \tag{1.6}$$

respectively; and sets

$$\widetilde{\lambda}_k = \lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b),$$

where $H \in R^{r \times r}$ is a selected symmetric positive definite matrix. The point $(\widetilde{x}_k, \widetilde{y}_k, \widetilde{\lambda}_k)$ can be used as the next iterate directly, or be used to produce the next iterate through a simple correction step.

Per iteration, the main task of the ADM is to solve the subproblems. In order to make the subproblems easily solvable, some proximal alternating directions methods (PADM) were proposed; see e.g., [4], [17], [18], [20], [23]. The PADM generates the next iterate via solving

$$(x - \widetilde{x}_k)^T \{(h(\widetilde{x}_k) + Gy_k - A^T(\lambda_k - H(A\widetilde{x}_k + By_k - b)) + R(\widetilde{x}_k - x_k)\} \geq 0, \ \forall x \in X,$$
$$(y - \widetilde{y}_k)^T \{g(\widetilde{x}_k, \widetilde{y}_k) - B^T(\lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b)) + S(\widetilde{y}_k - y_k)\} \geq 0, \ \forall y \in Y,$$

and updating the multiplier via

$$\widetilde{\lambda}_k = \lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b),$$

where R and S are symmetric positive definite matrices. Note that the mappings involved in the subproblems are strongly monotone, provided that the original mappings are monotone. As a consequence, these subproblems are easier than those in (1.4), and some efficient numerical methods are ready to solve them.

ADM and PADM have been studied extensively for separable optimization problems and variational inequalities, and attract more and more attention in application fields. However, there are few results for the more general problem considered in this paper, i.e., a weakly coupled VI. In this paper, we utilize the PADM to solve the weakly coupled VIs with the structure as (1.1). In Section 2, we present the PADM and in Section 3, we give some contractive properties, which are basis for our convergence analysis. In Section 4, we prove the convergence of this method. We report some numerical results in Section 5 and conclude the paper with some conclusions in Section 6. Throughout the paper we denote $\|x\| = \sqrt{x^T x}$ as the Euclidean-norm and $\|x\|_M = \sqrt{x^T M x}$ as M-norm for given symmetric positive definite matrix M.

2 The Method

For convenience, we denote

$$M := \begin{pmatrix} I & & \\ & B^T HB + Q & \\ & & H^{-1} \end{pmatrix}, \tag{2.1}$$

which is symmetric positive definite due to the fact that the matrices Q and H are both symmetric positive definite.

Let

$$F(u) := \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} := \begin{pmatrix} h(x) + Gy \\ g(x,y) \end{pmatrix}. \tag{2.2}$$

In the sequel, we make the following assumption:

Assumption:

(1) F is a monotone operator on Ω;

(2) The solution set W^* is nonempty.

We are now ready to describe our proximal alternating directions method.

Proximal Alternating Directions Method(PADM):

Step 1. Given $\varepsilon > 0,\ \omega_0 = (x_0, y_0, \lambda_0)$, set $k = 0$.

Step 2. Find $\widetilde{x}_k$ such that

$$(x - \widetilde{x}_k)^T \{f_k(\widetilde{x}_k) + (\widetilde{x}_k - x_k)\} \geq 0,\ \forall x \in X, \tag{2.3}$$

with fixed y_k and λ_k, where

$$f_k(x) := h(x) + Gy_k - A^T(\lambda_k - H(Ax + By_k - b)).$$

Step 3. Find $\widetilde{y}_k$ such that

$$(y - \widetilde{y}_k)^T \{g_k(\widetilde{y}_k) + Q(\widetilde{y}_k - y_k)\} \geq 0, \quad \forall y \in Y, \tag{2.4}$$

with fixed $\widetilde{x}_k$ and λ_k, where

$$g_k(y) := g(\widetilde{x}_k, y) - B^T(\lambda_k - H(A\widetilde{x}_k + By - b)).$$

Step 4. Compute $\tilde{\lambda}_k$ via

$$\widetilde{\lambda}_k = \lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b). \tag{2.5}$$

Step 5. Generate the next iterate via

$$\omega_{k+1} := \omega_k - \alpha_k(\omega_k - \widetilde{\omega}_k),$$

with

$$\alpha_k := \frac{\phi(\omega_k,\ \widetilde{\omega}_k)}{\|\omega_k - \widetilde{\omega}_k\|_M^2}, \tag{2.6}$$

and

$$\phi(\omega_k,\ \widetilde{\omega}_k) := \frac{1}{4}\|x_k - \widetilde{x}_k\|^2 + \frac{1}{2}\|\lambda_k - \widetilde{\lambda}_k\|^2_{H^{-1}} + \frac{1}{2}\|y_k - \widetilde{y}_k\|^2_{(2Q+B^THB-G^TG)} + \frac{1}{2}\|A\widetilde{x}_k + By_k - b\|^2_H. \tag{2.7}$$

Step 6. If $\|\omega_k - \widetilde{\omega}_k\| < \varepsilon$, stop; otherwise set $k := k+1$, and go to Step 2.

Remark 1: In this algorithm, for convenience of presentation, we only consider the case that the proximal stepsize, i.e., the coefficient of the proximal term, is a constant 1. In fact, we can set the stepsize to be δ_k, where $\tau \geq \delta_k \geq \gamma$, where $\tau > 0$ and $\gamma > 0$ are two positive constants. That is, if we change the subproblems in Step 2 and Step 3 to the following problems

$$(x - \widetilde{x}_k)^T\{f_k(\widetilde{x}_k) + \delta_k(\widetilde{x}_k - x_k)\} \geq 0,\ \forall x \in X,$$

and

$$(y - \widetilde{y}_k)^T\{g_k(\widetilde{y}_k) + \delta_k Q(\widetilde{y}_k - y_k)\} \geq 0, \quad \forall y \in Y,$$

respectively, we can obtain the same convergence results easily by similar analysis. In real applications, using variable stepsize suitably can enhance the performance of the algorithm, and how to update δ_k at each iteration deserves our further research.

Remark 2: In the definition of $\phi(\omega_k,\ \widetilde{\omega}_k)$, $2Q+B^THB-G^TG$ should be symmetric positive definite. In fact, it is not difficult to choose the matrices Q and H to make it symmetric positive definite.

Remark 3: Here we prefer to use the decent direction $\omega_k - \widetilde{\omega}_k$ to get the next iterate. We call it a correction step. For more detail on corrected step, we prefer [16] and [23].

Remark 4: It is easy to prove that $\alpha_k \geq \frac{1}{4}$ if we choose Q such that $Q - G^TG$ is semidefinite positive.

3 Contractive Properties

In this section, we list some contractive properties of the sequence generated by the proposed method.

Lemma 3.1. *Let $\widetilde{\omega}_k = (\widetilde{x}_k, \widetilde{y}_k, \widetilde{\lambda}_k)$ be generated by (2.3)-(2.5). Then for any $\omega^* \in W^*$, we have*

$$(\lambda_k - \widetilde{\lambda}_k)^T H^{-1}(\widetilde{\lambda}_k - \lambda^*) + (\widetilde{x}_k - x^*)^T(x_k - \widetilde{x}_k) + (\widetilde{y}_k - y^*)^T Q(y_k - \widetilde{y}_k) \geq (\widetilde{x}_k - x^*)^T(G(y_k - \widetilde{y}_k) + A^T H(By_k - B\widetilde{y}_k)). \tag{3.1}$$

Proof. Since $x^* \in X,\ y^* \in Y$, it follows from (2.3) and (2.4) that

$$(x^* - \widetilde{x}_k)^T(f(\widetilde{x}_k, y_k) - A^T(\lambda_k - H(A\widetilde{x}_k + By_k - b)) + \widetilde{x}_k - x_k) \geq 0 \tag{3.2}$$

and

$$(y^* - \widetilde{y}_k)^T(g(\widetilde{x}_k, \widetilde{y}_k) - B^T(\lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b)) + Q(\widetilde{y}_k - y_k)) \geq 0, \tag{3.3}$$

where $f(x,y)$ is defined as (2.2). On the other hand, since $(x^*,\ y^*,\ \lambda^*) \in W^*,\ \widetilde{x}_k \in X,\ \widetilde{y}_k \in Y$, we also have

$$(\widetilde{x}_k - x^*)^T (f(x^*, y^*) - A^T \lambda^*) \geq 0 \tag{3.4}$$

and

$$(\widetilde{y}_k - y^*)^T (g(x^*, y^*) - A^T \lambda^*) \geq 0. \tag{3.5}$$

Adding (3.2) and (3.4), (3.3) and (3.5), we get

$$\begin{aligned}&(\widetilde{x}_k - x^*)^T (f(x^*, y^*) - f(\widetilde{x}_k, y_k))\\ &- (\widetilde{x}_k - x^*)^T (A^T \lambda^* - A^T(\lambda_k - H(A\widetilde{x}_k + By_k - b)) - (x_k - \widetilde{x}_k)) \geq 0\end{aligned}$$

and

$$\begin{aligned}&(\widetilde{y}_k - y^*)^T (g(x^*, y^*) - g(\widetilde{x}_k, \widetilde{y}_k))\\ &- (\widetilde{y}_k - y^*)^T (B^T \lambda^* - B^T(\lambda_k - H(A\widetilde{x}_k + B\widetilde{y}_k - b)) - Q(y_k - \widetilde{y}_k)) \geq 0\end{aligned}$$

respectively. Recall that $(f(x,\ y),\ g(x,\ y))^T$ is monotone on Ω, i.e.,

$$(\widetilde{x}_k - x^*)^T (f(\widetilde{x}_k, \widetilde{y}_k) - f(x^*, y^*)) \geq 0,$$

and

$$(\widetilde{y}_k - y^*)^T (g(\widetilde{x}_k, \widetilde{y}_k) - g(x^*, y^*)) \geq 0,$$

and $f(\widetilde{x}_k, y_k) - f(\widetilde{x}_k, \widetilde{y}_k) = G(y_k - \widetilde{y}_k)$, it follows that

$$\begin{aligned}&(\widetilde{x}_k - x^*)^T A^T (\widetilde{\lambda}_k - \lambda^*) + (\widetilde{x}_k - x^*)^T (x_k - \widetilde{x}_k)\\ &\geq (\widetilde{x}_k - x^*)^T (G(y_k - \widetilde{y}_k) + A^T H(By_k - B\widetilde{y}_k))\end{aligned} \tag{3.6}$$

and

$$(\widetilde{y}_k - y^*)^T B^T (\widetilde{\lambda}_k - \lambda^*) + (\widetilde{y}_k - y^*)^T Q(y_k - \widetilde{y}_k) \geq 0. \tag{3.7}$$

Adding (3.6) and (3.7) and using $Ax^* + By^* = b$ and (2.5), we get (3.1). □

Lemma 3.2. *Let $\widetilde{\omega}_k = (\widetilde{x}_k, \widetilde{y}_k, \widetilde{\lambda}_k)$ be generated by (2.3)-(2.5), then for any $\omega^* \in W^*$, we have*

$$(\widetilde{\omega}_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k) \geq (\lambda_k - \widetilde{\lambda}_k)^T (By_k - B\widetilde{y}_k) + (\widetilde{x}_k - x^*)^T G(y_k - \widetilde{y}_k), \tag{3.8}$$

which implies

$$(\omega_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k) \geq \phi(\omega_k,\ \widetilde{\omega}_k),$$

where M and $\phi(\omega_k,\ \widetilde{\omega}_k)$ are defined in (2.1) and (2.7) respectively.

Proof. Adding $(\widetilde{y}_k - y^*)^T B^T H(By_k - B\widetilde{y}_k)$ to both sides of (3.1), we get

$$\begin{aligned}&(\widetilde{\omega}_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k)\\ &\geq (\widetilde{x}_k - x^*)^T G(y_k - \widetilde{y}_k) + (\widetilde{x}_k - x^*)^T A^T H(By_k - B\widetilde{y}_k))\\ &\quad + (\widetilde{y}_k - y^*)^T B^T H(By_k - B\widetilde{y}_k)\\ &= (\lambda_k - \widetilde{\lambda}_k)^T (By_k - B\widetilde{y}_k) + (\widetilde{x}_k - x^*)^T G(y_k - \widetilde{y}_k).\end{aligned}$$

Consequently,

$$
\begin{aligned}
&(\omega_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k)\\
&= (\omega_k - \widetilde{\omega}_k + \widetilde{\omega}_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k)\\
&= \|\omega_k - \widetilde{\omega}_k\|_M^2 + (\widetilde{\omega}_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k)\\
&\geq \|\omega_k - \widetilde{\omega}_k\|_M^2 + (\lambda_k - \widetilde{\lambda}_k)^T(By_k - B\widetilde{y}_k) + (\widetilde{x}_k - x^*)^T G(y_k - \widetilde{y}_k)\\
&= \frac{1}{2}\|x_k - \widetilde{x}_k\|^2 + \frac{1}{2}\|y_k - \widetilde{y}_k\|^2_{(Q+B^T HB)} + \frac{1}{2}\|\lambda_k - \widetilde{\lambda}_k\|^2_{H^{-1}}\\
&\quad + \frac{1}{2}(\|x_k - x^*\|^2 + \|x^* - \widetilde{x}_k\|^2) + (x_k - x^*)^T(x^* - \widetilde{x}_k)\\
&\quad + \frac{1}{2}(y_k - \widetilde{y}_k)^T B^T HB(y_k - \widetilde{y}_k) + \frac{1}{2}\|\lambda_k - \widetilde{\lambda}_k\|^2_{H^{-1}} + (\lambda_k - \widetilde{\lambda}_k)^T(By_k - B\widetilde{y}_k)\\
&\quad + \frac{1}{2}(y_k - \widetilde{y}_k)^T Q(y_k - \widetilde{y}_k) + (\widetilde{x}_k - x^*)^T G(y_k - \widetilde{y}_k)\\
&\geq \frac{1}{2}\|x_k - \widetilde{x}_k\|^2 + \frac{1}{2}\|y_k - \widetilde{y}_k\|^2_{(Q+B^T HB)} + \frac{1}{2}\|\lambda_k - \widetilde{\lambda}_k\|^2_{H^{-1}}\\
&\quad + \frac{1}{2}(\|x_k - x^*\|^2 + \|x^* - \widetilde{x}_k\|^2) + (x_k - x^*)^T(x^* - \widetilde{x}_k) + \frac{1}{2}\|A\widetilde{x}_k + By_k - b\|_H^2\\
&\quad + \frac{1}{2}(y_k - \widetilde{y}_k)^T Q(y_k - \widetilde{y}_k) - \frac{1}{2}(\|\widetilde{x}_k - x^*\|^2 + \|G(y_k - \widetilde{y}_k)\|^2)\\
&\geq \frac{1}{2}\|x_k - \widetilde{x}_k\|^2 + \frac{1}{2}\|x_k - x^*\|^2 - \frac{1}{4}\|x_k - \widetilde{x}_k\|^2 + \frac{1}{2}\|\lambda_k - \widetilde{\lambda}_k\|^2_{H^{-1}}\\
&\quad + \frac{1}{2}(y_k - \widetilde{y}_k)^T(2Q + B^T HB - G^T G)(y_k - \widetilde{y}_k) + \frac{1}{2}\|A\widetilde{x}_k + By_k - b\|_H^2\\
&\geq \frac{1}{4}\|x_k - \widetilde{x}_k\|^2 + \frac{1}{2}\|\lambda_k - \widetilde{\lambda}_k\|^2_{H^{-1}} + \frac{1}{2}\|y_k - \widetilde{y}_k\|^2_{(2Q+B^T HB - G^T G)}\\
&\quad + \frac{1}{2}\|A\widetilde{x}_k + By_k - b\|_H^2\\
&= \phi(\omega_k,\ \widetilde{\omega}_k)
\end{aligned}
$$

which completes the proof. □

It is obvious that $-(\omega_k - \widetilde{\omega}_k)$ is a descent direction of the unknown function $\frac{1}{2}\|\omega - \omega^*\|_M^2$ at ω_k. Along this direction, we can choose an approximate step size to make $\frac{1}{2}\|\omega - \omega^*\|_M^2$ decrease.

Theorem 3.3. *Let $\widetilde{\omega}_k = (\widetilde{x}_k, \widetilde{y}_k, \widetilde{\lambda}_k)$ be generated by (2.3)-(2.5), and*

$$\omega_{k+1} = \omega_k - \alpha_k(\omega_k - \widetilde{\omega}_k),$$

where α_k is defined as (2.6). Then for any $\omega^* \in W^*$,

$$\|\omega_{k+1} - \omega^*\|_M^2 \leq \|\omega_k - \omega^*\|_M^2 - \alpha_k \phi(\omega_k,\ \widetilde{\omega}_k).$$

Proof. Let

$$\omega_{k+1}(\alpha) = \omega_k - \alpha(\omega_k - \widetilde{\omega}_k),$$

then

$$
\begin{aligned}
&\|\omega_{k+1}(\alpha) - \omega^*\|_M^2 = \|\omega_k - \alpha(\omega_k - \widetilde{\omega}_k) - \omega^*\|_M^2\\
&= \|\omega_k - \omega^*\|_M^2 + \alpha^2\|\omega_k - \widetilde{\omega}_k\|_M^2 - 2\alpha(\omega_k - \omega^*)^T M(\omega_k - \widetilde{\omega}_k)\\
&\leq \|\omega_k - \omega^*\|_M^2 + \alpha^2\|\omega_k - \widetilde{\omega}_k\|_M^2 - 2\alpha\phi(\omega_k,\ \widetilde{\omega}_k).
\end{aligned}
$$

So

$$\|\omega_k - \omega^*\|_M^2 - \|\omega_{k+1}(\alpha) - \omega^*\|_M^2 \geq -\alpha^2 \|\omega_k - \widetilde{\omega}_k\|_M^2 + 2\alpha\phi(\omega_k,\ \widetilde{\omega}_k). \tag{3.9}$$

Define

$$\psi(\alpha) = -\alpha^2 \|\omega_k - \widetilde{\omega}_k\|_M^2 + 2\alpha\phi(\omega_k,\ \widetilde{\omega}_k).$$

When

$$\alpha = \frac{\phi(\omega_k,\ \widetilde{\omega}_k)}{\|\omega_k - \widetilde{\omega}_k\|_M^2}, \tag{3.10}$$

$\psi(\alpha)$ attains its maximum. Substituting (3.10) into (3.9), we can easily get

$$\|\omega_{k+1} - \omega^*\|_M^2 \leq \|\omega_k - \omega^*\|_M^2 - \alpha^*\phi(\omega_k,\ \widetilde{\omega}_k).$$

□

From Theorem 3.3, we can immediately draw the following conclusions:

Corollary 3.4. *Let $\widetilde{\omega}_k = (\widetilde{x}_k, \widetilde{y}_k, \widetilde{\lambda}_k)$ be generated by (2.3)-(2.5) and for any $\omega^* \in W^*$, we have*

(1) *The sequence $\|\omega_k - \omega^*\|_M$ is non-increasing.*

(2)
$$\lim_{k\to+\infty} \phi(\omega_k,\ \widetilde{\omega}_k) = 0,$$

which implies

$$\lim_{k\to+\infty} \|\omega_k - \widetilde{\omega}_k\|_M = 0.$$

(3) *The sequences $\{\omega_k\}$ and $\{\widetilde{\omega}_k\}$ are both bounded.*

4 Convergence

Now we give the convergence result.

Theorem 4.1. *Let $\{\omega_k\}$ be the sequence generated by the proposed algorithm, then $\{\omega_k\}$ converges to some $\omega^* \in W^*$.*

Proof. According to Corollary 3.4, we know that

$$\lim_{k\to+\infty} \|x_k - \widetilde{x}_k\| = \lim_{k\to+\infty} \|y_k - \widetilde{y}_k\| = \lim_{k\to+\infty} \|\lambda_k - \widetilde{\lambda}_k\| = 0.$$

From (2.3), (2.4) and (2.5), we have

$$\lim_{k\to+\infty}(x - \widetilde{x}_k)^T (f(\widetilde{x}_k, \widetilde{y}_k) - A^T\widetilde{\lambda}_k) \geq 0, \quad x \in X, \tag{4.1}$$

$$\lim_{k\to+\infty}(y - \widetilde{y}_k)^T (g(\widetilde{x}_k, \widetilde{y}_k) - B^T\widetilde{\lambda}_k) \geq 0, \ \ y \in Y. \tag{4.2}$$

Since $\{\widetilde{\omega}_k\}$ is bounded from Corollary 3.4, it has at least one cluster point. Let ω^* be the cluster point of $\{\widetilde{\omega}_k\}$, and let $\{\widetilde{\omega}_{k_i}\}$ be the subsequence converging to it, i.e.,

$$\lim_{k_i\to+\infty} \widetilde{\omega}_{k_i} = \omega^*.$$

It follows from (4.1) and (4.2) that

$$\lim_{k_i\to+\infty}(x-\widetilde{x}_{k_i})^T(f(\widetilde{x}_{k_i},\widetilde{y}_{k_i})-A^T\widetilde{\lambda}_{k_i})\geq 0,\quad x\in X,$$
$$\lim_{k_i\to+\infty}(y-\widetilde{y}_{k_i})^T(g(\widetilde{x}_{k_i},\widetilde{y}_{k_i})-B^T\widetilde{\lambda}_{k_i})\geq 0,\quad y\in Y,$$
$$\lim_{k_i\to+\infty}(A\widetilde{x}_{k_i}+B\widetilde{y}_{k_i}-b)=0.$$

Consequently,

$$(x-x^*)^T(f(x^*,y^*)-A^T\lambda^*)\geq 0,\quad x\in X,$$
$$(y-y^*)^T(g(x^*,y^*)-B^T\lambda^*)\geq 0,\ y\in Y,$$
$$Ax^*+By^*-b=0.$$

which implies $\omega^*\in W^*$. Since $\lim_{k\to+\infty}\|\omega_k-\widetilde{\omega}_k\|_M=0$ and $\lim_{k_i\to+\infty}\widetilde{\omega}_{k_i}=\omega^*$, for any $\varepsilon>0$, there exists an integer N such that $\|\omega_N-\widetilde{\omega}_N\|_M<\frac{\varepsilon}{2}$ and $\|\widetilde{\omega}_N-\omega^*\|_M<\frac{\varepsilon}{2}$. For any $k>N$,

$$\|\omega_k-\omega^*\|_M\leq\|\omega_N-\omega^*\|_M\leq\|\omega_N-\widetilde{\omega}_N\|_M+\|\widetilde{\omega}_N-\omega^*\|_M\leq\varepsilon,$$

which implies that the sequence $\{\omega_k\}$ converges to $\omega^*\in W^*$. □

5 Numerical Experiments

In this section, we implement the PADM to solve some weakly coupled VIs. We code our algorithm in Matlab and all the tests were run on a lenovo notebook.

Example 1: The first example is a GNEP taken from [10], which consists of 2 players. The first player controls the two-dimensional decision variable $x=(x_1,x_2)$, whereas the second player has a one-dimensional decision variable $y\in R$. The optimization problems of the players are given by

$$\begin{array}{ll}\min & \theta_1(x,y)=x_1^2+x_1x_2+x_2^2+(x_1+x_2)y-25x_1-38x_2\\ s.t. & x_1+2x_2-y\leq 14\\ & 3x_1+2x_2+y\leq 30\\ & x_1\geq 0,\quad x_2\geq 0\end{array}$$

and

$$\begin{array}{ll}\min & \theta_2(x,y)=y^2+(x_1+x_2)y-25y\\ s.t. & x_1+2x_2-y\leq 14\\ & 3x_1+2x_2+y\leq 30\\ & y\geq 0,\end{array}$$

respectively. This problem has an infinite number of solutions given by

$$\{(t,\ 11-t,\ 8-t)\mid t\in[0,2]\}$$

and the normalized equilibrium is $(0,11,8)$. It is obvious that the inequality constraints are active at all the solutions of the problem. Therefore the problem with equality constraints has the same solution. We consider the following GNEP:

$$\begin{array}{ll} \min & \theta_1(x,y) = x_1^2 + x_1x_2 + x_2^2 + (x_1+x_2)y - 25x_1 - 38x_2 \\ s.t. & x_1 + 2x_2 - y = 14 \\ & 3x_1 + 2x_2 + y = 30 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array}$$

and

$$\begin{array}{ll} \min & \theta_2(x,y) = y^2 + (x_1+x_2)y - 25y \\ s.t. & x_1 + 2x_2 - y = 14 \\ & 3x_1 + 2x_2 + y = 30 \\ & y \geq 0. \end{array}$$

Let $\nabla_x \theta_1(x,y) = h(x) + Gy,\ \nabla_y \theta_2(x,y) = g(x,y)$, where

$$h(x) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} x - \begin{pmatrix} 25 \\ 38 \end{pmatrix},\ Gy = \begin{pmatrix} 1 \\ 1 \end{pmatrix} y$$

and $g(x,y) = 2y + (1,1)x - 25$ in this problem. In the experiments, we set $Q = 10$, the maximal number of iterations is 1000 and the stopping criterion is $\|\omega_k - \widetilde{\omega}_k\| \leq 10^{-6}$. Because of the nonnegativity of all the variables, solving subproblems (2.3) and (2.4) reduces to solving a linear complementarity problems (LCP). In our tests the LCP is solved by the MATLAB code pathlcp.m [11]. $x = (1,1), y = 1$ and $\lambda = (1,1)$ are used as the starting point. Via 24 iterations, we get a solution $(0,11,8)$, and $\|\omega_k - \widetilde{\omega}_k\| = 8.9899e - 007$. If we adjust the step size α_k to $\gamma\alpha_k$ as in [23] by setting $\gamma = 1.5$, the iteration number reduces to 19. In addition, we test the problem with different starting points generated by Matlab randomly, the solution $(0,11,8)$ can be obtained in about 19 iterations.

Example 2: Consider the following GNEP:

$$\begin{array}{ll} \min & \theta_1(x,y) = x^2 + \frac{8}{3}xy - 34y \\ s.t. & x + y = 15 \\ & 0 \leq x \leq 10 \end{array}$$

and

$$\begin{array}{ll} \min & \theta_2(x,y) = y^2 + \frac{5}{4}xy - 24.25y \\ s.t. & x + y = 15 \\ & 0 \leq y \leq 10. \end{array}$$

This example with inequality constraints is taken from [14]. Here we utilize PADM to solve the problem with equality constraints. During this experiments, we still set $Q = 10$, the maximal number of iterations as 1000 and the stopping criterion is $\|\omega_k - \widetilde{\omega}_k\| = 10^{-6}$. Using $x = 1, y = 1, \lambda = 1$ as starting point, we get a solution $(10,5)$ via 24 iterations. If we use different starting point generated by Matlab randomly, the average number of iterations is also 24.

6 Conclusions

In this paper, we have presented a proximal alternating directions method to solve a class of VIs , where the operator is weakly coupled of all the variables. Under the mild conditions that the solution set is nonempty and the involving mapping is monotone, the same conditions for separable variational inequalities, we proved the global convergence of the algorithm. Our numerical experiments, although were very preliminary, verified our theoretical results.

The weakly coupled variational inequalities include some interesting problems as special cases, e.g., the generalized Nash equilibrium problems, which attracted a lot of attention of the researchers. In [9], a Gauss-Seidel-like method was proposed for potential games, and the authors proved that the generated sequence converges to a solution, provided that the sequence has a cluster point. The conditions that guarantee the generated sequence has a cluster point are still lack. In [2], the authors presented an augmented Lagrangian-type algorithm for solving convex separable minimization problems of the form

$$\{\min f(x) + g(y) : Ax - By = 0\}.$$

But the algorithm did not work for every convex and continuous differentiable function $\phi(x, y)$ in the following two-person game:

$$\begin{array}{lll|lll} \min & f(x) + \phi(x,y) & & & \min & g(y) + \phi(x,y) \\ s.t. & Ax - By = 0, & & & s.t. & Ax - By = 0, \end{array}$$

Our results, although are very preliminary, provided a simple while interesting structure on the mappings that guaranteed global convergence of the method. What general conditions should $\phi(x, y)$ satisfy so that the GNEP can be solved? Can the same kind of results be obtained in Hilbert setting ? How to generalize the method to three or more variables? All these problems deserve further research.

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Lingling Xu
School of Mathematical Sciences, Jiangsu Key Labratory for NSLSCS
Nanjing Normal University, Nanjing 210023, P.R. China
E-mail address: xulingling@njnu.edu.cn

Deren Han
School of Mathematical Sciences, Jiangsu Key Labratory for NSLSCS
Nanjing Normal University, Nanjing 210023, P.R. China
E-mail address: handeren@njnu.edu.cn