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A COMBINED HOMOTOPY INFEASIBLE INTERIOR POINT METHOD FOR NONCONVEX PROGRAMMING*

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Dedicated to the memory of Professor Paul Tseng.

Abstract: In this paper, a new homotopy method is proposed for solving nonconvex programming problems with both equality and inequality constraints. Existence and global convergence of the homotopy path with probability one to a solution of the KKT system are proven under a normal cone condition, which concerns only with the inequality constraints, as well as a stronger positive linear independence assumption. It requires only an interior point and not necessarily feasible point as the starting point, so it is more practical than the combined homotopy interior point method, which needs a feasible interior starting point. Simple numerical examples are given to show the feasibility and efficiency of the proposed method.

Key words: *nonconvex programming, interior point method, infeasible interior point method, homotopy method, global convergence*

Mathematics Subject Classification: *49M05, 90C30*

1 Introduction

We consider the following nonlinear programming problem

$$\begin{array}{ll} \min & f(x), \\ \text{s.t.} & h(x) = 0, \\ & g(x) \leq 0, \end{array} \tag{1.1}$$

where $x \in R^n$, $f(x) : R^n \to R$, $h(x) : R^n \to R^l$ and $g(x) : R^n \to R^m$ are sufficiently smooth. (1.1) is a convex programming problem if $f(x)$ and $g_i(x)$ $(i = 1, \ldots, m)$ are convex, and $h(x)$ is linear. Denote $\Omega = \{x \mid g(x) \leq 0,\ h(x) = 0\}$.

The interior point methods (IPMs) are efficient in both theory and practice for solving nonlinear programming problems (see, e.g., [4, 5, 9, 12, 14, 20, 25, 33, 34, 36]). The term "interior point method" appears to have originated in the book [11], in which, it was suggested to minimize a sequence of unconstrained subproblems

$$\min f(x) - \mu \sum_{i=1}^{m} \log(-g_i(x))$$

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parameterized by a scalar parameter μ, to solve an original constrained optimization. And existence of a unique differentiable path $x(\mu)$ for positive μ in a neighborhood of $\mu = 0$ was proven under Mangasarian-Fromowich constraint qualification (MFCQ), strict complementarity and second-order sufficient conditions. Due to the reasons (e.g., ill-conditioning of the Hessian matrix of the unconstrained subproblems in the limit) as discussed in [17, 31], IPMs were not paid enough attention until Karmarkar published his new polynomial algorithm for linear programming in 1984 (see, e.g., [18]). In the past three decades, the success of IPMs for linear and convex nonlinear programming stimulated renewed interest in them for the nonconvex case. Using globalization strategies like line search and trust region to solve a sequence of subproblems with decreasing parameters μ_k, the variants of classic IPMs in [11], with different merit functions, were proposed for nonconvex programming. Global convergence results of some IPMs for nonconvex programming have been established, see, for example, [1, 6, 8, 15, 22, 34]. Unlike IPMs for linear or convex nonlinear programming, however, without existence and global convergence of smooth path, the solution of the previous subproblem may not supply a good initial guess of a solution of the current subproblem for some parameter μ_k, and hence a crucial advantage of the interior point method is lost.

Modern homotopy method, whose probability-one global convergence can be proven under fairly weak conditions by using differential topology tools, is a powerful global convergent method for solving nonlinear problems. See [7, 19, 26] for pioneer works and [2, 13] for survey monographs. Some homotopy methods, interior point or noninterior point, are presented for solving nonlinear programming, see, for example, [9, 10, 20, 27, 29, 32, 36]. Their probability-one convergence was proven for nonconvex programming without uniformly positive definiteness or even uniform nonsingularity of the Hessian of the Lagrangian function.

The first homotopy method for nonlinear programming was proposed for the solution of convex programming in [13]. For a given $x^{(0)}$ satisfying $g(x^{(0)}) < 0$, the homotopy equation

$$H(x, y, \mu) = \begin{pmatrix} \mu \nabla f(x) + (1-\mu)(x - x^{(0)}) + \sum_{j=1}^{m} y_j^{+} \nabla g_j(x) \\ y_j^{-} + g_j(x), j = 1, \ldots, m \end{pmatrix} = 0,$$

was constructed, where $\mu \in [0, 1]$, $y = (y_1, \ldots, y_m)^T \in R^m$, $y_j^{+} = [\max\{0, y_j\}]^3$, $y_j^{-} = [\max\{0, -y_j\}]^3$, $j = 1, \ldots, m$, $\nabla f(x)$ and $\nabla g_j(x)$ are the gradients of $f(x)$ and $g_j(x)$ respectively. The homotopy equation is equivalent to the Karush-Kuhn-Tucker (KKT) conditions of the parameterized and constrained optimization problem

$$\min \ \mu f(x) + (1-\mu)\|x - x^{(0)}\|^2/2, \text{ s.t. } g(x) \le 0.$$

At $\mu = 1$, it is equivalent to the KKT conditions for the original optimization. Existence and global convergence of a homotopy path determined by the homotopy equation, were proven under the nonemptiness and boundedness of the convex feasible set and the regularity of the homotopy map. Hence, tracing the path from $\mu = 0$ to $\mu = 1$, a solution of KKT system can be obtained.

In [9, 10], a combined homotopy interior point (CHIP) method was proposed to solve nonconvex programming with only inequality constraints. The combined homotopy is as follows

$$H(x, y, \mu) = \begin{pmatrix} (1-\mu)(\nabla f(x) + \nabla g(x) y) + \mu(x - x^{(0)}) \\ Y g(x) - \mu Y^{(0)} g(x^{(0)}) \end{pmatrix},$$

where $y \in R_{+}^{m}$, $(x^{(0)}, y^{(0)}) \in \Omega^0 \times R_{++}^{m}$, $\mu \in [0, 1]$, Y and $Y^{(0)}$ are diagonal matrices with i-th diagonal elements y_i and $y_i^{(0)}$ respectively, and $\Omega^0 = \{x \in R^n \mid g(x) < 0\}$. Existence and global convergence to a KKT point of a smooth homotopy path emanating from a random initial interior point, were proven under the nonemptiness and boundedness of Ω^0, linear independence constraint qualification (LICQ) and a so-called "normal cone condition". The CHIP method was implemented by some predictor-corrector procedure for numerically tracing the homotopy path.

The normal cone condition is somewhat restricted. To solve more general nonconvex programming problem, modified combined homotopies were constructed and the normal cone condition was weaken to a quasi normal cone condition (in [21]) and a pseudo cone condition (in [35]) respectively, and similar global convergence results were given. In [29], by using the KKT conditions of a constrained parametric optimization problem, a new homotopy was constructed. Global convergence of a homotopy path to a KKT point was proven under assumption, in addition to the nonemptiness and boundedness of a parameterized feasible set and Arrow-Hurwicz-Uzawa constraint qualification, that the homotopy path does not go to infinity near the starting hyperplane. In [36], the polynomial complexity of a CHIP method for convex programming was given. In [27], global convergence to a local minimum point was proven under some additional assumptions.

In [20], a CHIP method for solving nonconvex programming with both equality and inequality constraints, as in (1.1), was proposed. The homotopy equation was defined as

$$H(x,y,z,\mu) = \begin{pmatrix} (1-\mu)(\nabla f(x) + \nabla g(x)y) + \nabla h(x)z + \mu(x - x^{(0)}) \\ Yg(x) - \mu Y^{(0)} g(x^{(0)}) \\ h(x) \end{pmatrix} = 0, \qquad (1.2)$$

where $(y,z) \in R^m_+ \times R^l$ is a Lagrange multiplier, $(x^{(0)}, y^{(0)}) \in \Omega^0 \times R^m_{++}$, $\mu \in [0,1]$ and $\Omega^0 = \{x \in R^n \mid g(x) < 0, h(x) = 0\}$. Existence and global convergence to a KKT point of a smooth homotopy path were proven under the nonemptiness and boundedness of Ω^0, LICQ and a normal cone condition on both inequality and equality constraints: $\forall x \in \partial\Omega = \Omega \backslash \Omega^0$ (a gap, should be "$\forall x \in \Omega$"),

$$\left\{ x + \sum_{i \in I(x)} y_i \nabla g_i(x) + \nabla h(x) z \;\middle|\; y_i \geq 0,\ i \in I(x),\ z \in R^l \right\} \cap \Omega = \{x\}, \qquad (1.3)$$

where $I(x) = \{i \mid g_i(x) = 0,\ i = 1, \ldots, m\}$. Due to the last component, $h(x) = 0$, the homotopy equation (1.2) needs an interior starting point $x^{(0)}$ which satisfies also equality constraints. Because finding a point satisfying both equality and strict inequality constraints may be as difficult as the solving original problem, the CHIP method is not convenient to use.

In this paper, a new homotopy method, called combined homotopy infeasible interior point (CHIIP) method, was proposed for (1.1). It requires only the starting point to be an interior point and not to be a feasible point, so it is more practical than CHIP method for (1.1). Existence and global convergence of the homotopy path with probability one to a solution of the KKT system are proven under a normal cone condition concerning only with the inequality constraints, as well as a stronger positive linear independence assumption.

The rest of this paper is organized as follows. In section 2, the homotopy is constructed, existence and global convergence of the homotopy path are proven. In section 3, the CHIIP method is given by numerically tracing the homotopy path, and numerical results are given to show its convenience and effectiveness.

Throughout this paper, we use the following notations: For any $v \in R^n$, Diag(v) denotes the diagonal matrix with its i-th diagonal element given by v_i. We denote by R^n_+ (R^n_{++}) the set of vectors with n nonnegative (positive) components. Finally, we denote by $\Omega_1 =$

$\{x \mid g(x) \leq 0\}$, $\Omega_1^0 = \{x \mid g(x) < 0\}$, $\partial\Omega_1 = \Omega_1 \setminus \Omega_1^0$, $I(x) = \{i \mid g_i(x) = 0,\ i = 1, \ldots, m\}$. e represents a vector whose entries are all ones.

$\boxed{2}$ The Homotopy and Homotopy Path

Let x be a local solution of the problem (1.1). If MFCQ holds at x, then there exists (y, z) such that

$$\begin{array}{l} \nabla f(x) + \nabla g(x) y + \nabla h(x) z = 0, \\ h(x) = 0, \\ Y g(x) = 0,\ y \geq 0,\ g(x) \leq 0, \end{array} \tag{2.1}$$

where $Y = \mathrm{Diag}(y)$. The system (2.1) is called the KKT system or first order optimality conditions for the problem (1.1).

For simplicity, we give the following definition.

Definition 2.1 ([3, 24])**.** *$\left\{\{\nabla g_i(x)\}_{i \in I(x)}, \{\nabla h_j(x)\}_{j=1}^l\right\}$ is called positive-linearly independent at x if*

$$\sum_{i \in I(x)} \alpha_i \nabla g_i(x) + \sum_{j=1}^{l} \beta_j \nabla h_j(x) = 0,\ \alpha_i \in R_+,\ \beta_j \in R \Rightarrow \alpha_i = \beta_j = 0.$$

Otherwise, $\left\{\{\nabla g_i(x)\}_{i \in I(x)}, \{\nabla h_j(x)\}_{j=1}^l\right\}$ is called positive-linearly dependent.

Remark. For any given feasible point x, MFCQ holds at x if and only if $\left\{\{\nabla g_i(x)\}_{i \in I(x)}, \{\nabla h_j(x)\}_{j=1}^l\right\}$ is positive-linearly independent. For details, see [24].

Definition 2.2. *A point x is called PLI-regular if $\left\{\{\nabla g_i(x)\}_{i \in I(x)}, \{\nabla h_j(x)\}_{j=1}^l\right\}$ is positive-linearly independent. Otherwise, x is called PLI-nonregular.*

The following hypotheses will be used in this paper:

C1. *Ω_1^0 is nonempty and bounded;*

C2 (The normal cone condition). $\forall x \in \partial\Omega_1$,

$$\left\{ x + \sum_{i \in I(x)} y_i \nabla g_i(x) \;\middle|\; y_i \geq 0 \right\} \cap \Omega_1 = \{x\}.$$

To solve KKT system (2.1), we construct the following homotopy

$$H(w, \mu) = \begin{pmatrix} (1-\mu)(\nabla f(x) + \nabla g(x) y + \nabla h(x) z) + \mu(x - x^{(0)}) \\ Y g(x) + \mu e \\ h(x) - \mu z \end{pmatrix}, \tag{2.2}$$

where $w = (x, y, z) \in \hat{\Omega} = \Omega_1 \times R_+^m \times R^l$, $\mu \in [0, 1]$, $x^{(0)} \in \Omega_1^0$. For any given $x^{(0)} \in \Omega_1^0$, we denote the zero point set of $H(w, \mu)$ as

$$H^{-1}(0) = \left\{(w, \mu) \in \Omega_1 \times R_+^m \times R^l \times (0, 1] \mid H(w, \mu) = 0\right\}.$$

It follows from $x^{(0)} \in \Omega_1^0$ that $H(w, 1) = 0$ has only one simple solution

$$w^{(0)} = \left(x^{(0)}, y^{(0)}, z^{(0)}\right) = \left(x^{(0)}, -\left[G(x^{(0)})\right]^{-1} e, h(x^{(0)})\right),$$

where, and from here on, $G(x) = \text{Diag}(g(x))$. When $\mu = 0$, the homotopy equation $H(w,\mu) = 0$ turns to the KKT system.

The following main theorem ensures that the homotopy (2.2) does work.

Theorem 2.3. *Suppose that $f(x)$, $h(x)$ and $g(x)$ are three times continuously differentiable, and conditions C1, C2 hold. Let $H(x,y,z,\mu)$ be defined as (2.2), then for almost all $x^{(0)} \in \Omega_1^0$, $H^{-1}(0)$ contains a smooth curve $\Gamma_{x^{(0)}}$, called the homotopy path, starting from $(w^{(0)},1)$. It approaches to the boundary of $\hat{\Omega} \times (0,1]$. If the x-component of any accumulation point of any sequence of points on $\Gamma_{x^{(0)}}$ is PLI-regular, then $\Gamma_{x^{(0)}}$ terminates in or approaches to the hyperplane $\mu = 0$. And if $(x^*,y^*,z^*,0)$ is a limit point of $\Gamma_{x^{(0)}}$ on the hyperplane $\mu = 0$, then x^* is a KKT point for the problem (1.1).*

Proof. Let $\tilde{H}(w,x^{(0)},\mu)$ be the same map with $H(w,\mu)$ but taking $x^{(0)}$ as a variate. Consider the following submatrix of the Jacobian $D\tilde{H}(w,x^{(0)},\mu)$ of $\tilde{H}(w,x^{(0)},\mu)$:

$$\frac{\partial \tilde{H}(w,x^{(0)},\mu)}{\partial(y,z,x^{(0)})} = \begin{bmatrix} * & * & -\mu I \\ G(x) & 0 & 0 \\ 0 & -\mu I & 0 \end{bmatrix}.$$

For any $x^{(0)} \in \Omega_1^0$ and $(w,\mu) \in H^{-1}(0)$, from $\mu > 0$ and $Yg(x) + \mu e = 0$, we know that $G(x)$ is nonsingular. It follows that $\frac{\partial \tilde{H}(w,x^{(0)},\mu)}{\partial(y,z,x^{(0)})}$ is a nonsingular matrix, and hence, $D\tilde{H}(w,x^{(0)},\mu)$ is a matrix of full row rank. That is, 0 is a regular value of $\tilde{H}(w,x^{(0)},\mu)$.

From the parameterized Sard theorem (Th. 2.1, [7]), we know that for almost all $x^{(0)} \in \Omega_1^0$, 0 is a regular value of $H(w,\mu) : \hat{\Omega}^0 \times (0,1) \to R^{n+l+m}$. If 0 is a regular value of $H(w,\mu)$, from the implicit function theorem, the nonsingularity of

$$\frac{\partial H(w^{(0)},1)}{\partial w} = \begin{pmatrix} I & 0 & 0 \\ * & G(x^{(0)}) & 0 \\ * & 0 & -I \end{pmatrix},$$

and the fact that $H(w^{(0)},1) = 0$, there must be a smooth curve $\Gamma_{x^{(0)}}$ starting from $(w^{(0)},1)$ and going into $\hat{\Omega}^0 \times (0,1)$. $\Gamma_{x^{(0)}}$ must terminate in or approach to the boundary of $\hat{\Omega} \times [0,1]$.

Let $(w^*,\mu_*) = (x^*,y^*,z^*,\mu_*) \in \partial(\hat{\Omega} \times [0,1])$ be an ending limit point of $\Gamma_{x^{(0)}}$. Only the following five cases are possible:

(i) $w^* \in \Omega_1 \times R_+^m \times R^l$, $\mu_* = 1$, $\|(y^*,z^*)\| < \infty$;
(ii) $w^* \in \Omega_1 \times R_+^m \times R^l$, $\mu_* \in [0,1]$, $\|(y^*,z^*)\| = \infty$;
(iii) $w^* \in \Omega_1 \times \partial R_+^m \times R^l$, $\mu_* \in (0,1)$, $\|(y^*,z^*)\| < \infty$;
(iv) $w^* \in \partial\Omega_1 \times R_{++}^m \times R^l$, $\mu_* \in (0,1)$, $\|(y^*,z^*)\| < \infty$;
(v) $w^* \in \Omega_1 \times R_+^m \times R^l$, $\mu_* = 0$, $\|(y^*,z^*)\| < \infty$.

Because $(w^{(0)},1)$ is the unique solution of $H(w,1) = 0$, and $\frac{\partial H(w^{(0)},1)}{\partial w}$ is nonsingular, the case (i) is impossible.

If the case (ii) happens, by the condition C1, there exists a sequence of points $\{(x^{(k)},y^{(k)},z^{(k)},\mu_k)\}$ on $\Gamma_{x^{(0)}}$ such that $x^{(k)} \to x^*$, $\|(y^{(k)},z^{(k)})\| \to \infty$, $\mu_k \to \mu_*$, as $k \to \infty$. And only the following three subcases are possible: (a) $\mu_* = 1$; (b) $\mu_* \in (0,1)$; (c) $\mu_* = 0$.

(a) $\mu_* = 1$.

By the first equality of the homotopy equation, we have that

$$(1-\mu_k)\left(\nabla f(x^{(k)}) + \nabla g(x^{(k)})y^{(k)} + \nabla h(x^{(k)})z^{(k)}\right) + \mu_k\left(x^{(k)} - x^{(0)}\right) = 0. \tag{2.3}$$

From the third equality of the homotopy equation, we have that $z^{(k)} \to h(x^*)$, as $k \to \infty$. This implies that $\{z^{(k)}\}$ is bounded, hence $\|y^{(k)}\| \to \infty$ and $x^* \in \partial\Omega_1$.

If $\|(1-\mu_k)y^{(k)}\| < \infty$, suppose without loss of generality that $(1-\mu_k)y^{(k)} \to \bar{y}$, then $\bar{y}_i = 0$ for $i \notin I(x^*)$ from the second equality of the homotopy equation. By taking the limit in (2.3), we get

$$\begin{aligned} x^{(0)} &= x^* + \lim_{k\to\infty}(1-\mu_k)\left(\nabla f(x^{(k)}) + \nabla g(x^{(k)})y^{(k)} + \nabla h(x^{(k)})z^{(k)}\right) \\ &= x^* + \lim_{k\to\infty}\sum_{i\in I(x^*)}(1-\mu_k)y_i^{(k)}\nabla g_i(x^{(k)}) \\ &= x^* + \sum_{i\in I(x^*)}\bar{y}_i\nabla g_i(x^*), \end{aligned}$$

which contradicts with the condition C2.

For the case $\|(1-\mu_k)y^{(k)}\| \to \infty$, the discussion is the same with the case (b), which will be done below.

(b) $\mu_* \in (0,1)$.

As in the proof of (a), we have that $\{z^{(k)}\}$ is bounded, $x^* \in \partial\Omega_1$, and $\|(1-\mu_k)y^{(k)}\| \to \infty$. Suppose without loss of generality that $(1-\mu_k)y^{(k)}/\|(1-\mu_k)y^{(k)}\| \to \alpha^*$ with $\|\alpha^*\| = 1$ and $\alpha_i^* = 0$ for $i \notin I(x^*)$. Divide (2.3) by $\|(1-\mu_k)y^{(k)}\|$ and take the limit, we have that

$$\sum_{i\in I(x^*)}\alpha_i^*\nabla g_i(x^*) = 0,$$

which contradicts with the condition that x^* is PLI-regular.

(c) $\mu_* = 0$.

Suppose without loss of generality that $(y^{(k)}, z^{(k)})/\|(y^{(k)}, z^{(k)})\| \to (\bar{\alpha}, \bar{\beta})$ with $\|(\bar{\alpha}, \bar{\beta})\| = 1$ and $\bar{\alpha}_i = 0$ for $i \notin I(x^*)$. Divide the both sides of (2.3) by $\|(y^{(k)}, z^{(k)})\|$ and take the limit, we have that

$$\sum_{i\in I(x^*)}\bar{\alpha}_i\nabla g_i(x^*) + \sum_{i=1}^{l}\bar{\beta}_i\nabla h_i(x^*) = 0,$$

which contradicts with the condition that x^* is PLI-regular.

From (a), (b) and (c), we conclude that the case (ii) is impossible.

From $G(x^*)y^* + \mu_* e = 0$, we know that $\mu_* > 0$ and $y^* \in \partial R_+^m$, i.e., $y_i^* = 0$ for some $i \in \{1, \ldots, m\}$, can not happen simultaneously, which implies that the case (iii) is impossible. If $y^* > 0$ and $\mu_* > 0$, from $G(x^*)y^* + \mu_* e = 0$, we have that $g(x^*) < 0$, which implies that the case (iv) is impossible. As a result, the case (v) is the only possible case. Hence, (x^*, y^*, z^*) is a solution of the KKT system (2.1). This completes the proof. □

Remark. a) Different with [20], the normal cone condition C2 concerns only with inequality constraints $g(x) \leq 0$ and does not concern with the equality constraints $h(x) = 0$. Specially, it is satisfied if only inequality constraints are convex.

b) On the other hand, a posteriori condition that the x-component of any limit point of any sequence of points on $\Gamma_{x^{(0)}}$ is PLI-regular, which is not satisfying, is used in Theorem 2.1.

3 Implementation of the CHIIP Method and Numerical Tests

For numerically tracing the homotopy path $\Gamma_{x^{(0)}}$, a predictor-corrector algorithm is given below. The first predictor step is taken by computing the tangent direction, and the midway predictor steps are taken by using secant directions. The corrector steps proceed with Newton iterations for solving the extended system $H(w, \mu) = 0$ by adding a linear equation $(d^{(k)})^T((w, \mu) - (w^{(k+1,0)}, \mu_{k+1,0})) = 0$, where $d^{(k)}$ is the predicting direction and $(w^{(k+1,0)}, \mu_{k+1,0})$ is the predicted point at the k-th step. At each predictor step and corrector step, we need to check if the computed point is in Ω_1^0 or not. If not, we take some damped step. Detailed descriptions on predictor-corrector algorithms with discussions on their convergence were given in [2, 13, 30].

Algorithm 3.1 (The CHIIP method for solving NLP)

Step 1. *Give the accuracy parameters $\epsilon_1 \geq \epsilon_2 > 0$, the initial steplength $s_0 > 0$, the initial point $(w^{(0)}, 1)$, the steplength adjusting parameters $0 < \theta_1 < \theta_2 < \theta_3 < 1 < \theta_4 < \theta_5$, the maximum number K of the corrector steps, the threshold value $0 < \theta_\alpha < 1$ for the angle between two neighboring predictor directions, and the threshold value $0 < \theta_\mu < 1$ for starting the end game.*

Step 2. *The first predictor step.*

Set $s = s_0$, $\epsilon = \epsilon_1$, $k = 0$;

Compute $d^{(0)} \in R^{n+m+l+1}$, such that $DH(w^{(0)}, 1)d^{(0)} = 0$, $\|d^{(0)}\| = 1$ and $d^{(0)}_{n+m+l+1} < 0$;

Set $d^{(-1)} = d^{(0)}$;

Determine the smallest nonnegative integer i such that $(w^{(0)}, \mu_0) + \theta_3^i s d^{(0)} \in \hat{\Omega}^0 \times (0, 1)$, set $s = \theta_3^i s$ and $(w^{(1,0)}, \mu_{1,0}) = (w^{(0)}, \mu_0) + s d^{(0)}$.

Step 3. *The corrector step.*

Set $j = 0$;

Repeat

Compute the Newton step $\bar{d}$ by solving $\begin{cases} DH_{k+1,j}\bar{d} = -H_{k+1,j}, \\ (d^{(k)})^T \bar{d} = 0; \end{cases}$

Determine the smallest nonnegative integer i such that $(w^{(k+1,j)}, \mu_{k+1,j}) + \theta_3^i \bar{d} \in \hat{\Omega}^0 \times (0, 1)$, set $(w^{(k+1,j+1)}, \mu_{k+1,j+1}) = (w^{(k+1,j)}, \mu_{k+1,j}) + \theta_3^i \bar{d}$;

$j = j + 1$,

Until $\|H_{k+1,j}\|_\infty \leq \epsilon$ or $j = K$.

Step 4. *The steplength adjusting.*

If $j = K$ and $\|H_{k+1,j}\|_\infty > \epsilon$,

Set $s = \theta_2 s$ and $(w^{(k+1,0)}, \mu_{k+1,0}) = (w^{(k)}, \mu_k) + s d^{(k)}$;

go to Step 3;

else

$(w^{(k+1)}, \mu_{k+1}) = (w^{(k+1,j)}, \mu_{k+1,j})$;

Adjust the steplength s as follows:

i. If $d^{(k)T} d^{(k-1)} < \theta_\alpha$, set $s = \theta_1 s$;

ii. If $j > 4$, set $s = \theta_2 s$;

iii. If $j = 2$, set $s = \theta_4 s$;

iv. If $j < 2$, set $s = \theta_5 s$.

If $\mu_{k+1} < \theta_\mu$, go to Step 6;

If $\|H(w^{(k+1)}, 0)\|_\infty \leq \epsilon_2$, terminate the algorithm ($w^* = w^{(k+1)}$ **is the computed solution of the KKT system)***;*

Set $\epsilon = \min\{\mu_{k+1}, \epsilon_1\}$, $k = k+1$.

Step 5. *The midway predictor step.*

Let $d^{(k)} = ((w^{(k)}, \mu_k) - (w^{(k-1)}, \mu_{k-1}))/\|(w^{(k)}, \mu_k) - (w^{(k-1)}, \mu_{k-1})\|$;

Determine the smallest nonnegative integer i *such that* $(w^{(k)}, \mu_k) + \theta_3^i s d^{(k)} \in \hat{\Omega}^0 \times (0,1)$, *set* $s = \theta_3^i s$ *and* $(w^{(k+1,0)}, \mu_{k+1,0}) = (w^{(k)}, \mu_k) + sd^{(k)}$.

Step 6. *The end game.*

Set $j = 0$ *and* $w^{(k+1,0)} = w^{(k+1)}$;

Repeat

Compute the Newton step $d_{end} \in R^{n+m+l}$ *by solving the equation* $\frac{\partial H}{\partial w}(w^{(k+1,j)}, 0) d_{end} = -H(w^{(k+1,j)}, 0)$. *Set* $w^{(k+1,j+1)} = w^{(k+1,j)} + d_{end}$, $j = j+1$,

Until $\|H(w^{(k+1,j)}, 0)\|_\infty \leq \epsilon_2$ *or* $j = K$;

If $j < K$, *set* $w^{(k+1)} = w^{(k+1,j)}$;

Set $\theta_\mu = 0.1\theta_\mu$.

In Algorithm 3.1, for convenience, we used $H_{k,j}$ and $DH_{k,j}$ to denote $H(w^{(k,j)}, \mu_{k,j})$ and $DH(w^{(k,j)}, \mu_{k,j})$ respectively.

Algorithm 3.1 was implemented by programming in Matlab language and preliminary numerical tests were done. Some examples of nonconvex programming problems and numerical results, with comparison with the CHIP method and LOQO 6.01 (student edition), are listed below.

Example 3.1 is taken from [28]. Example 3.2 and 3.3 are taken from [16]. Example 3.4 from [23] is a problem with variable dimension and number of constraints.

In Table 1–4, the initial point $x^{(0)}$, the final approximate solution x^*, the corresponding objective value $f(x^*)$, and the number N of overall iterations are listed. In Table 4, the integer M is listed, but the computed solution is not listed due to the large number of variables.

For $M = 500$, we can not compute Example 3.4 by LOQO 6.01 student edition because of its limitations: 300 variables, 300 constraints and objectives, and 500 iterations, so the results are indicated by "$--$".

For all examples, we set parameters in Algorithm 3.1 as: $K = 6$, $h_0 = 0.7$, $\theta_1 = 0.4$, $\theta_2 = 0.6$, $\theta_3 = 0.9$, $\theta_4 = 1.4$, $\theta_5 = 2.5$, $\theta_\mu = 0.1$, $\theta_\alpha = \sqrt{2}/2$, $\epsilon_1 = 1\mathrm{e}-2$ and $\epsilon_2 = 1\mathrm{e}-6$.

Example 3.1 ([28]).

$$
\begin{array}{rl}
\min & x_1, \\
\text{s.t.} & x_1^2 - x_2 - 1 = 0, \\
& x_1 - x_3 - 1 = 0, \\
& x_2 \geq 0,\ x_3 \geq 0.
\end{array}
$$

Example 3.2 ([16]).

$$
\begin{array}{rl}
\min & e^{x_1 x_2 x_3 x_4 x_5} - 0.5(x_1^3 + x_2^3 + 1)^2, \\
\text{s.t.} & x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 10, \\
& x_2 x_3 - 5 x_4 x_5 = 0, \\
& x_1^3 + x_2^3 = -1, \\
& l \leq x \leq u,
\end{array}
$$

where $l = (-2.3, -2.3, -3.2, -3.2, -3.2)^T$, $u = -l$.

Table 1: Numerical results of Example 3.1.

$x^{(0)}$	method	N	x^*	$f(x^*)$
	CHIPM	—	—	—
(-6,10,10)	CHIIPM	100	(1.00,-0.00,-0.00)	1.0000
	LOQO6.01	469	(1.00,0.00,0.00)	1.0000
	CHIPM	—	—	—
(-2,3,1)	CHIIPM	13	(1.00,0.00,0.00)	1.0000
	LOQO6.01	74	(1.00,0.00,0.00)	1.0000

The failure of a method is indicated by the dash

Table 2: Numerical results of Example 3.2.

$x^{(0)}$	method	N	x^*	$f(x^*)$
	CHIPM	—	—	—
(-1,1,2,0,-2)	CHIIPM	27	(-1.71,1.59,1.82,-0.76,-0.76)	0.0539
	LOQO6.01	35	(-0.69,-0.86,2.78,-0.69,0.69)	0.4388
	CHIPM	—	—	—
(1.5,-1.5,-2.2,2,2.4)	CHIIPM	53	(-0.69,-0.86,-2.78,0.69,0.69)	0.4388
	LOQO6.01	117	(-0.69,-0.86,-2.78,0.69,0.69)	0.4388

The failure of a method is indicated by the dash

Example 3.3 ([16]).

$$\begin{array}{rl} \min & -(0.063x_4x_7 - 5.04x_1 - 3.36x_5 - 0.035x_2 - 10x_3), \\ \text{s.t.} & x_5 = 1.22x_4 - x_1, \\ & x_8 = (x_2 + x_5)/x_1, \\ & x_4 = 0.01x_1(112 + 13.167x_8 - 0.6667x_8^2), \\ & x_7 = 86.35 + 1.098x_8 - 0.038x_8^2 + 0.325(x_6 - 89), \\ & x_{10} = 3x_7 - 133, \\ & x_9 = 35.82 - 0.222x_{10}, \\ & x_6 = 98000x_3/(x_4x_9 + 1000x_3), \\ & l \le x \le u, \end{array}$$

where $l = (1\mathrm{e}-5, 1\mathrm{e}-5, 1\mathrm{e}-5, 0, 0, 85, 90, 3, 0.01, 145)^T$, $u = (2000, 16000, 120, 5000, 2000, 93, 95, 12, 4, 162)^T$.

Example 3.4 ([23]). Let $\alpha = 350$, $h = 1/M$,

$$\begin{array}{rl} \min\limits_{t_i, v_i, u_i} & 0.5h \sum\limits_{i=0}^{M-1} (u_{i+1}^2 + u_i^2 + \alpha(\cos t_{i+1} + \cos t_i)), \\ \text{s.t.} & v_{i+1} - v_i - 0.5h(\sin t_{i+1} + \sin t_i) = 0,\ i = 0, \ldots, M-1, \\ & t_{i+1} - t_i - 0.5hu_{i+1} - 0.5hu_i = 0,\ i = 0, \ldots, M-1, \\ & -1 \le t_i \le 1,\ i = 1, \ldots, M-1, \\ & -0.05 \le v_i \le 0.05,\ i = 1, \ldots, M-1, \end{array}$$

where M is a positive integer, v_0, v_M, t_0 and t_M are all 0.

From the numerical results, we see that different starting points affect the efficiency of the methods. For given starting points, simple experiments show that our algorithm is

Table 3: Numerical results of Example 3.3.

$x^{(0)}$	method	N	$f(x^*)$
	CHIPM	—	—
(1,10,1,1,10,90,93,8,1,155)	CHIIPM	200	-1162.0269
	LOQO6.01	271	-686.485
	CHIPM	—	—
(10,10,1,10,10,90,94,10,2,150)	CHIIPM	206	-1162.0269
	LOQO6.01	254	-481.391
CHIIPM	x^*=(1728.37,16000.00,98.13,3056.04,2000.00,90.61,94.18,10.41,2.61,149.56)		
LOQO 6.01	x^*_{-686}=(1004.95,10595.8,54.43,1757.11,1138.73,91.45,94.79,11.68,2.2,151.36)		
LOQO 6.01	x^*_{-481}=(768.37,6388.8,44.98,1359.27,889.94,89.88,93.63,9.47,2.99,147.88)		

The failure of a method is indicated by the dash

Table 4: Numerical results of Example 3.4.

M	$x^{(0)}$	method	N	$f(x^*)$
	$v_i = 0.05\cos ih,\ \ i = 1,\dots,M-1$	CHIPM	—	—
50	$t_i = 0.05\cos ih,\ \ i = 1,\dots,M-1$	CHIIPM	467	344.8687
	$u_i = -50,\ \ i = 0,\dots,M$	LOQO6.01	490	344.8687
	$v_i = 0.05\cos ih,\ \ i = 1,\dots,M-1$	CHIPM	—	—
100	$t_i = 0.05\cos ih,\ \ i = 1,\dots,M-1$	CHIIPM	634	344.8775
	$u_i = 60,\ \ i = 0,\dots,M$	LOQO6.01	440	348.151
	$v_i = 0.05\cos ih,\ \ i = 1,\dots,M-1$	CHIPM	—	—
500	$t_i = 0.5\cos ih,\ \ i = 1,\dots,M-1$	CHIIPM	1096	344.8763
	$u_i = -45,\ \ i = 0,\dots,M$	LOQO6.01	$--$	$--$

The failure of a method is indicated by the dash

feasible and effective. We will continue work on more efficient linear equation solver for Newton corrector and strategy on the ill-conditioning.

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