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A NOTE ON LOCAL SENSITIVITY ANALYSIS FOR PARAMETRIC OPTIMIZATION PROBLEM*

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Abstract: This paper is concerned with the local sensitivity analysis for parametric optimization problem. By using an NCP function, the KKT system of parametric optimization problem can be reformulated as a system of semismooth equations. Based on the equations, local analytic formulas for the sensitivities of the objective function and primal and dual variables with respect to all parameters are obtained. This method can deal with the local sensitivities of all the optimal solutions which satisfy the linear independent constraint qualification. As a by-product of our analysis, we obtain a sufficient condition for the existence of the sensitivity solution to the problem we discussed. This new method is an improvement of the existed methods. Numerical examples are given to illustrate the method in the end.

Key words: *parametric optimization problem, sensitivity analysis, NCP function, semismooth equation*

Mathematics Subject Classification: *90C31, 90C05, 90C30*

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1 Introduction

This paper deals with the local sensitivity analysis for the following parametric optimization problem

$$\begin{array}{lll} \min\limits_{x} & z & = f(x,\sigma), \\ s.t. & & h(x,\sigma) = 0, \\ & & g(x,\sigma) \geq 0, \end{array} \tag{1.1}$$

where $f : R^n \times R^p \to R$, $h : R^n \times R^p \to R^l$, $g : R^n \times R^p \to R^m$, and $h(x,\sigma) = (h_1(x,\sigma),\ldots,h_l(x,a))^T$, $g(x,\sigma) = (g_1(x,\sigma),\ldots,g_m(x,\sigma))^T$ are functions over the feasible region $S(\sigma) = \{x|h(x,\sigma) = 0, g(x,\sigma) \geq 0\}$, $f,h,g \in C^2$.

Sensitivity analysis is a procedure to determine how and how much specific changes in the parameters of an optimization problem influence the optimal objective function value and the point or points where the optimum is attained. Many researchers have studied different version of this problem (see the ref.[1-8]). Some of them have dealt with the linear programming problem and discussed the effect of changes of (i) the cost coefficients, (ii) the right-hand sides of the constraints, or (iii) the constraint coefficients on either (a) the optimal value of the objective function or (b) the optimal solution. A similar analysis has been done

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for nonlinear problems(see ref.[3-8,16]), and the authors in ref.[7,8] also have considered the existence of the sensitivity solutions of the perturbed problems. But among the existing methods, there are still some limitations as follows: (a) There are only sensitivities analysis of the objective function value and the primal variables with respect to the parameters, but not the sensitivities of the dual variables with respect to the parameters; (b) There are diverse methods for obtaining each of the sensitivities for different cases, but there is no integrated approach providing all the sensitivities at once; (c) They assume the active constraints remain active and the inactive constraints remain inactive, which implies there is no need to distinguish between equality or inequality constraints. Recently, Castillo in [9] proposed a new approach to calculate the local sensitivity for mathematical programming problems. This method can get a simple linear system to calculate the sensitivity of the objective function and primal and dual variables with respect to all parameters in the case of regular nondegenerate solutions. But it can not deal with the case of regular degenerate solutions efficiently. One must compute all the possible combinations of the equalities and inequalities in the differentiation system of the Karush-Kuhn-Tucker systems and it is a difficult work.

So in this paper, we will aim to such works: (a) we need not assume any condition except the linear independent constraint qualification; (b) we transfer the KKT system of the parametric optimization problem into a system of semismooth equations by using an NCP function. (c) We get a system of linear equations for solving all the local sensitivities simultaneously in both the cases of regular nondegegerate solution and regular degenerate solution by using the semismooth properties.

The construction of this paper is as follows: In Section 2 , we transfer the KKT system of problem (1.1) into a system of semismooth equations and present the local sensitivity formulas for both the cases of regular nondegenerate and regular degenerate solutions. In Section 3, numerical examples are introduced to illustrate the new method of sensitivity analysis. In Section 4, we get some conclusions of this paper.

A few words about our notation. For the given parameter $\bar{\sigma}$, x^* is the corresponding optimal solution, λ^* is the dual variable with respect to $h(x^*,\sigma)=0$ and μ^* is the dual variables with respect to $g(x^*,\sigma)\geq 0$, $z^*=f(x^*,\bar{\sigma})$ is the optimal objective function value. We define the index sets associated with the active and inactive constraints in the usual way:

$$\begin{aligned}&E=\{j=1,\ldots,l|h_j(x^*,\bar{\sigma})=0\},\quad I=\{j=1,\ldots,m|g_j(x^*,\bar{\sigma})=0\},\\&N=\{1,\ldots,m\}\setminus I,\quad I_0=\{j\in I|\mu_j^*=0\},\quad I_+=I\setminus I_0=\{j\in I|\mu_j^*>0\}.\end{aligned}$$

In this paper, we only consider the sensitivity analysis of the optimal solution which satisfies the linear independent constraint qualification(LICQ). And the optimal point $(x^*,\lambda^*,\mu^*,z^*)$ can be classified as follows:

Regular nondegenerate optimal solution. The solution $(x^*,\lambda^*,\mu^*,z^*)$ satisfies the linear independent constraint qualification(LICQ) and $I_0=\emptyset$.

Regular degenerate optimal solution. The solution $(x^*,\lambda^*,\mu^*,z^*)$ satisfies the linear independent constraint qualification(LICQ) and $I_0\neq\emptyset$.

Note that we deal with local sensitivity, that is, changes produced by differential changes, and the formulas derived from the KKT system of the parametric optimization problem. If the condition of LICQ is removed, the KKT conditions do not characterize adequately this case. It is because there are infinite Lagrange value combinations that hold, see ref.[9]. However, our method can also provide the sensitivities formulas if the values of the Lagrange multipliers are given. So in this paper, we only consider the most common situation when we have a regular point.

$\boxed{2}$ Sensitivity Analysis Based On Semismooth Equations

In this section, we derive local sensitivity formulas of $(x^*, \lambda^*, \mu^*, z^*)$ for problem (1.1) under small perturbations in the parameter σ around $\bar{\sigma}$.

$\boxed{2.1}$ Semismooth Equations Equivalent to the KKT Systems of Problem (1.1)

Let x^* be a local optimal solution and we assume the linear independent constraint qualification is satisfied in problem (1.1). From the first-order optimal condition, there exists a pair of vectors $\lambda^* \in R^l, \mu^* \in R^m$ satisfying the following KKT systems:

$$\begin{array}{l} \nabla_x f(x^*, \bar{\sigma}) - \sum_{k=1}^{l} \lambda_k^* \nabla_x h_k(x^*, \bar{\sigma}) - \sum_{j=1}^{m} \mu_j^* \nabla_x g_j(x^*, \bar{\sigma}) = 0_n, \\ h_k(x^*, \bar{\sigma}) = 0, k = 1, 2, \ldots, l, \\ g_j(x^*, \bar{\sigma}) \geq 0, j = 1, 2 \ldots, m, \\ \mu_j^* g_j(x^*, \bar{\sigma}) = 0, j = 1, 2, \ldots, m, \\ \mu_j^* \geq 0, j = 1, 2, \ldots, m. \end{array} \tag{2.1}$$

To obtain the sensitivity analysis, we perturb or modify $x^*, \sigma, \lambda^*, \mu^*, z^*$ in such a way that the KKT systems (2.1) still hold. Thus we will usually differentiate the objective function in (1.1) and the KKT systems (2.1) directly (see ref.[10]). But in this kind of method, one must deal with a lot compositions of equalities and inequalities in the case of regular degenerate solution, the calculation and analysis become much complex. So in this paper, we will transfer the KKT systems of problem (1.1) into a system of semismooth equations by using an NCP function, then we can get the united formulas to calculate the local sensitivity for both the cases of regular nondegenerate solution and regular degenerate solution by using the semismooth properties.

Let $\varphi : R^2 \to R$ be Fischer-Burmeister function (see ref.[10,11]), the definition is

$$\varphi(a, b) = \sqrt{a^2 + b^2} - (a + b). \tag{2.2}$$

It is easily to see that $\varphi(a, b) = 0$ if and only if

$$a \geq 0, b \geq 0, a^T b = 0.$$

A function with this property is called an NCP function, see ref.[11,13]. For the sake of convenience, we denote the Fischer-Burmeister function as FB function. From ref.[10,12], we know that the FB function is differentiable everywhere except at the point $(0, 0)$ and it is semismooth at $(0, 0)$. The definition and properties of semismooth function can be found in ref.[12-15].

By using the FB function, we transfer the KKT systems of problem (1.1) into a system of semismooth equations

$$\begin{array}{l} \nabla_x f(x^*, \bar{\sigma}) - \sum_{k=1}^{l} \lambda_k^* \nabla_x h_k(x^*, \bar{\sigma}) - \sum_{j=1}^{m} \mu_j^* \nabla_x g_j(x^*, \bar{\sigma}) = 0_n, \\ h_k(x^*, \bar{\sigma}) = 0, k = 1, 2, \ldots, l, \\ \phi(x^*, \mu^*, \bar{\sigma}) = 0_m. \end{array} \tag{2.3}$$

where

$$\begin{array}{l} \phi(x^*, \mu^*, \bar{\sigma}) = (\phi_1(x^*, \mu^*, \bar{\sigma}), \phi_2(x^*, \mu^*, \bar{\sigma}) \ldots, \phi_m(x^*, \mu^*, \bar{\sigma}))^T, \\ \phi_j(x^*, \mu^*, \bar{\sigma}) = \varphi(\mu_j^*, g_j(x^*, \bar{\sigma})) = \mu_j^* + g_j(x^*, \bar{\sigma}) - \sqrt{(\mu_j^*)^2 + (g_j(x^*, \bar{\sigma}))^2}, j = 1, 2, \ldots, m. \end{array}$$

$\boxed{2.2}$ Local Sensitivity Formulas for the Case of Regular Nondegenerate Solution

In this case, $I_0 = \emptyset$. From the properties of the FB function, we know that $\phi_j(x^*, \mu^*, \sigma)$ is continuously differentiable. So we differentiate the object function in (1.1) and the KKT system (2.3) as follows:

$$\begin{array}{l}(\nabla_x f(x^*,\bar\sigma))^T dx + (\nabla_\sigma f(x^*,\bar\sigma))^T d\sigma - dz = 0,\\ (\nabla_{xx} f(x^*,\bar\sigma) - \sum_{k=1}^{l}\lambda_k^*\nabla_{xx}h_k(x^*,\bar\sigma) - \sum_{j=1}^{m}\mu_j^*\nabla_{xx}g_j(x^*,\bar\sigma))dx\\ +(\nabla_{x\sigma} f(x^*,\bar\sigma) - \sum_{k=1}^{l}\lambda_k^*\nabla_{x\sigma}h_k(x^*,\bar\sigma) - \sum_{j=1}^{m}\mu_j^*\nabla_{x\sigma}g_j(x^*,\bar\sigma))d\sigma\\ -\nabla_x h(x^*,\bar\sigma)d\lambda - \nabla_x g(x^*,\bar\sigma)d\mu = 0_n,\\ (\nabla_x h(x^*,\bar\sigma))^T dx + (\nabla_\sigma h(x^*,\bar\sigma))^T d\sigma = 0_l,\\ (\nabla_x \phi(x^*,\mu^*,\bar\sigma))^T dx + (\nabla_\mu \phi(x^*,\mu^*,\bar\sigma))^T d\mu + (\nabla_\sigma \phi(x^*,\mu^*,\bar\sigma))^T d\sigma = 0_m,\end{array} \tag{2.4}$$

where

$$\nabla_x\phi_j(x^*,\mu^*,\bar\sigma) = \begin{cases}\nabla_x g_j(x^*,\bar\sigma), & j\in I,\\ 0, & j\in N.\end{cases}$$
$$\nabla_\mu\phi_j(x^*,\mu^*,\bar\sigma) = \begin{cases}0, & j\in I,\\ 1, & j\in N.\end{cases}$$
$$\nabla_\sigma\phi_j(x^*,\mu^*,\bar\sigma) = \begin{cases}\nabla_\sigma g_j(x^*,\bar\sigma), & j\in I,\\ 0, & j\in N.\end{cases}$$

Denote

$$\begin{array}{l}F_x = \nabla_x f(x^*,\bar\sigma), F_\sigma = \nabla_\sigma f(x^*,\bar\sigma),\\ F_{xx} = \nabla_{xx} f(x^*,\bar\sigma) - \sum_{k=1}^{l}\lambda_k^*\nabla_{xx}h_k(x^*,\bar\sigma) - \sum_{j=1}^{m}\mu_j^*\nabla_{xx}g_j(x^*,\bar\sigma),\\ F_{x\sigma} = \nabla_{x\sigma} f(x^*,\bar\sigma) - \sum_{k=1}^{l}\lambda_k^*\nabla_{x\sigma}h_k(x^*,\bar\sigma) - \sum_{j=1}^{m}\mu_j^*\nabla_{x\sigma}g_j(x^*,\bar\sigma),\\ H_x = (\nabla_x h(x^*,\bar\sigma))^T, H_\sigma = (\nabla_\sigma h(x^*,\bar\sigma))^T, G_x = (\nabla_x g(x^*,\bar\sigma))^T,\\ \phi_x = (\nabla_x\phi(x^*,\mu^*,\bar\sigma))^T, \phi_\sigma = (\nabla_\sigma\phi(x^*,\mu^*,\bar\sigma))^T, \phi_\mu = (\nabla_\mu\phi(x^*,\mu^*,\bar\sigma))^T.\end{array}$$

In matrix form, the system (2.4) can be written as

$$\begin{bmatrix} F_x & F_\sigma & \mathbf{0} & \mathbf{0} & -1\\ F_{xx} & F_{x\sigma} & -H_x^T & -G_x^T & \mathbf{0}\\ H_x & H_\sigma & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \phi_x & \phi_\sigma & \mathbf{0} & \phi_\mu & \mathbf{0}\end{bmatrix}\begin{bmatrix} dx\\ d\sigma\\ d\lambda\\ d\mu\\ dz\end{bmatrix} = \mathbf{0}. \tag{2.5}$$

Obviously, (2.5) is a system of linear equation, so we can get all the sensitivities from (2.5) easily. Furthermore, (2.5) can be written as

$$U[dx, d\lambda, d\mu, dz]^T = S d\sigma, \tag{2.6}$$

where U, S are

$$U = \begin{bmatrix} F_x & \mathbf{0} & \mathbf{0} & -1\\ F_{xx} & -H_x^T & -G_x^T & \mathbf{0}\\ H_x & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \phi_x & \mathbf{0} & \phi_\mu & \mathbf{0}\end{bmatrix}, S = -\begin{bmatrix} F_\sigma\\ F_{x\sigma}\\ H_\sigma\\ \phi_\sigma\end{bmatrix}.$$

It is obvious that U is a square matrix and we can get the following conclusion.

Theorem 2.1. *Let $\bar{\sigma}$ be a given parameter, (x^*, λ^*, μ^*) be the corresponding local optimal solution which is regular and nondegenerate, z^* be the optimal objective function value, assume that*

$$\begin{vmatrix} F_{xx} & -H_x^T & -G_{Ix}^T \\ H_x & \mathbf{0} & \mathbf{0} \\ G_{Ix} & \mathbf{0} & \mathbf{0} \end{vmatrix} \neq 0, \tag{2.7}$$

where $G_{Ix} = (\nabla_x g_j(x^*, \bar{\sigma}))^T, j \in I$, then (2.6) has a unique sensitivity solution

$$[\partial x/\partial\sigma, \partial\lambda/\partial\sigma, \partial\mu/\partial\sigma, \partial z/\partial\sigma]^T = U^{-1}S$$

under small perturbations in σ around $\bar{\sigma}$.

Proof. From the definition of ϕ_x, ϕ_μ in (2.4), U can be written as

$$U = \begin{bmatrix} F_x & \mathbf{0} & \mathbf{0} & \mathbf{0} & -1 \\ F_{xx} & -H_x^T & -G_{Ix}^T & \mathbf{0} & \mathbf{0} \\ H_x & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_{Ix} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_N & \mathbf{0} \end{bmatrix},$$

where I_N is the unit matrix whose cardinality is equal to the number of all inactive inequalities (i.e. $g_j(x^*, \sigma) > 0, j \in N$). Let $q = (q^1, q^2, q^3, q^4, q^5) \in R^n \times R^l \times R^{m_I} \times R^{m_N} \times R$ be an appropriately partitioned vector with $Uq = 0$, where m_I is the the number of active inequalities and m_N is the the number of inactive inequalities. Then we can get that

$$\begin{aligned} &F_x q^1 - q^5 = 0, \\ &F_{xx} q^1 - H_x^T q^2 - G_{Ix}^T q^3 = 0_n, \\ &H_x q^1 = 0_l, \\ &G_{Ix} q^1 = 0_{m_I}, \\ &q^4 = 0_{m_N}. \end{aligned}$$

This together with the condition (2.7) implies $q = (q^1, q^2, q^3, q^4, q^5) = \mathbf{0}$. So we can get that $det(U) \neq 0$ and (2.6) has a unique solution

$$[\partial x/\partial\sigma, \partial\lambda/\partial\sigma, \partial\mu/\partial\sigma, \partial z/\partial\sigma]^T = U^{-1}S.$$

□

Theorem 2.1 allows one diriving all the sensitivities with respect to all the parameters simultaneously. It is easy to see that there are two particular cases in which the nonsingularity of (2.7) is guaranteed.

Case 1: $l + m_I = n$, and the matrix

$$M = \begin{pmatrix} H \\ G \end{pmatrix}$$

is invertible.

Case 2: F_{xx} is positive and the matrix

$$M = \begin{pmatrix} H \\ G \end{pmatrix}$$

is full row rank.

Moreover, if the matrices involved in Theorem 2.1 become singular, we can get the sensitivity of the solution by the linear system (2.6) directly. But the corresponding solution has the structure of a cone in this case.

$\boxed{2.3}$ Local Sensitivity Formulas for the Case of Regular Degenerate Solution

In this case, $I_0 \neq \emptyset$, i.e. there exists the case of $\mu_j^* = g_j(x^*, \bar{\sigma}) = 0, j \in I_0$. So $\phi_j(x^*, \mu^*, \sigma)$ is not differentiable but semismooth at this time. From the semismooth properties of

$$\varphi(\mu_j^*, g_j(x^*, \bar{\sigma})) = \mu_j^* + g_j(x^*, \bar{\sigma}) - \sqrt{(\mu_j^*)^2 + (g_j(x^*, \bar{\sigma}))^2},$$

we know that if the changes of μ_j^* or $g_j(x^*, \bar{\sigma})$ (denote $d\mu_j^*$ or dg_j) is not equal to zero, we have

$$\begin{aligned} &\phi_j(x^* + dx, \mu^* + d\mu, \bar{\sigma} + d\sigma) - \phi_j(x^*, \mu^*, \bar{\sigma}) \\ &= \varphi(\mu^* + d\mu, g_j(x^* + dx, \bar{\sigma} + d\sigma)) - \varphi(\mu^*, g_j(x^*, \bar{\sigma})) \\ &= V_{jx} dx + V_{ja} d\sigma + V_{j\mu} d\mu + o(\|(dx, d\sigma, d\mu)\|), \end{aligned}$$

where $V_{jx} = \nabla_x g_j(x^*, \bar{\sigma})(1 - \frac{dg_j}{\sqrt{(dg_j)^2 + (d\mu_j)^2}}),\ V_{j\sigma} = \nabla_\sigma g_j(x^*, \bar{\sigma})(1 - \frac{dg_j}{\sqrt{(dg_j)^2 + (d\mu_j)^2}}),\ V_{j\mu} = 1 - \frac{d\mu_j}{\sqrt{(dg_j)^2 + (d\mu_j)^2}}$.

So in the case of regular degenerate solution, if $d\mu_j^*$ or dg_j is not equal to zero and we aim to perturb $x^*, \sigma, \lambda^*, \mu^*, z^*$ in such a way that the KKT systems still hold, we have

$$\begin{array}{l} (\nabla_x f(x^*, \bar{\sigma}))^T dx + (\nabla_\sigma f(x^*, \bar{\sigma}))^T d\sigma - dz = 0, \\ (\nabla_{xx} f(x^*, \bar{\sigma}) - \sum_{k=1}^{l} \lambda_k^* \nabla_{xx} h_k(x^*, \bar{\sigma}) - \sum_{j=1}^{m} \mu_j^* \nabla_{xx} g_j(x^*, \bar{\sigma}))dx \\ +(\nabla_{x\sigma} f(x^*, \bar{\sigma}) - \sum_{k=1}^{l} \lambda_k^* \nabla_{x\sigma} h_k(x^*, \bar{\sigma}) - \sum_{j=1}^{m} \mu_j^* \nabla_{x\sigma} g_j(x^*, \bar{\sigma}))d\sigma \\ -\nabla_x h(x^*, \bar{\sigma}) d\lambda - \nabla_x g(x^*, \bar{\sigma}) d\mu = 0_n, \\ (\nabla_x h(x^*, \bar{\sigma}))^T dx + (\nabla_\sigma h(x^*, \bar{\sigma}))^T d\sigma = 0_l, \\ (\nabla_x \phi(x^*, \mu^*, \bar{\sigma}))^T dx + (\nabla_\mu \phi(x^*, \mu^*, \bar{\sigma}))^T d\mu + (\nabla_\sigma \phi(x^*, \mu^*, \bar{\sigma}))^T d\sigma = 0, \end{array} \tag{2.8}$$

where

$$\nabla_x \phi_j(x^*, \mu^*, \bar{\sigma}) = \begin{cases} \nabla_x g_j(x^*, \bar{\sigma}), & j \in I_+, \\ V_{jx}, & j \in I_0, \\ 0, & j \in N. \end{cases}$$
$$\nabla_\mu \phi_j(x^*, \mu^*, \bar{\sigma}) = \begin{cases} 0, & j \in I_+, \\ V_{j\mu}, & j \in I_0, \\ 1, & j \in N. \end{cases}$$
$$\nabla_\sigma \phi_j(x^*, \mu^*, \bar{\sigma}) = \begin{cases} \nabla_\sigma g_j(x^*, \bar{\sigma}), & j \in I_+, \\ V_{j\sigma}, & j \in I_0, \\ 0, & j \in N. \end{cases}$$

Though (2.8) is not a system of linear equations, we can get the corresponding values of $V_{jx}, V_{j\sigma}, V_{j\mu}$ according to the adjustment of the active inequality with $(\mu_j^*, g_j(x^*, \sigma)) = (0, 0)$. We consider two cases as follows:

(i) If we want this inequality remain active, we take $dg_j = 0, d\mu_j > 0$, so we get $V_{jx} = \nabla_x g_j(x^*, \bar{\sigma}), V_{j\mu} = 0, V_{j\sigma} = \nabla_\sigma g_j(x^*, \bar{\sigma})$.

(ii) If we want this inequality become inactive, we can take $d\mu_j = 0, dg_j > 0$, and get $V_{jx} = 0, V_{j\mu} = 1, V_{j\sigma} = 0$.

In a regular degenerate case, there are two possible changes for the constraint $g_j(x^*, \sigma)$, we can decide $g_j(x^*, \sigma)$ remain active or inactive according the need of an actual problem.

So we can also get a linear equations to calculate all the sensitivities for the case of regular degenerate solution according to the adjustment of the active inequality with

$(\mu_j^*, g_j(x^*, \sigma)) = (0,0)$ as follows,

$$\begin{bmatrix} F_x & F_\sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} & -1 \\ F_{xx} & F_{x\sigma} & -H_x^T & -G_{I'x}^T & \mathbf{0} & \mathbf{0} \\ H_x & H_\sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_{I'x} & G_{I'\sigma} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_N' & \mathbf{0} \end{bmatrix} \begin{bmatrix} dx \\ d\sigma \\ d\lambda \\ d\mu \\ dz \end{bmatrix} = \mathbf{0}, \tag{2.9}$$

where $G_{I'x}$ and $G_{I'\sigma}$ represent the partial derivatives with respect to x and a of all the active inequalities which include the corresponding inequalities with$(\mu_j, g_j(x^*, \bar{\sigma})) = (0,0)$ but are required to remain active. I_N' is the unit matrix whose cardinality is equal to the number of all inactive inequalities which include the corresponding inequalities with $(\mu_j, g_j(x^*, \bar{\sigma})) = (0,0)$ but are required to become inactive. Similarly to the proof of Theorem 2.1, we get the following conclusion.

Theorem 2.2. *Let $\bar{\sigma}$ be a given parameter, (x^*, λ^*, μ^*) be the corresponding local optimal solution which is regular and degenerate, z^* be the optimal objective function value, assume that*

$$\begin{vmatrix} F_{xx} & -H_x^T & -G_{I'x}^T \\ H_x & \mathbf{0} & \mathbf{0} \\ G_{I'x} & \mathbf{0} & \mathbf{0} \end{vmatrix} \neq 0,$$

then (2.9) has a unique solution

$$[\partial x/\partial\sigma, \partial\lambda/\partial\sigma, \partial\mu/\partial\sigma, \partial z/\partial\sigma]^T = U_1^{-1} S_1 \tag{2.10}$$

under small perturbations in σ around $\bar{\sigma}$, where

$$U_1 = \begin{bmatrix} F_x & \mathbf{0} & \mathbf{0} & \mathbf{0} & -1 \\ F_{xx} & -H_x^T & -G_{I'x}^T & \mathbf{0} & \mathbf{0} \\ H_x & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ G_{I'x} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_N' & \mathbf{0} \end{bmatrix}, S_1 = -\begin{bmatrix} F_\sigma \\ F_{x\sigma} \\ H_\sigma \\ G_{I'\sigma} \\ \mathbf{0} \end{bmatrix}.$$

3 Illustrative Examples

In this section, we illustrate the method developed in section 2 by its application to a regular nondegenerate example and a regular degenerate example.

3.1 Regular Nondegenerate Example

Considering the following optimization problem:

$$\min_{a,b} z = (\frac{a+b}{2} - \bar{y})^2 + (\frac{1}{12}(b-a)^2 - \sigma^2)^2$$

$$s.t.\ \ g_1(y,a) = a - y_{min} \leq 0, \tag{3.1}$$

$$g_2(y,a) = y_{max} - b \leq 0. \tag{3.2}$$

where $\bar{y}$ and σ^2 are the sample mean and variance respectively. Then we take $y_{max} = y_n, y_{min} = y_1, n = 5, y_1 = 0.2, y_2 = 0.3, y_3 = 0.4, y_5 = 0.95$ and get the local optimal solution $\mu_1^* = 0, \mu_2^* = 0.0053, \hat{a} = -0.00468, \hat{b} = 0.95, y_{max} = y_5 = 0.95, y_{min} = y_1 = 0.2$. So we can see that the inequality (3.1) is the inactive constraint, the inequality (3.2) is the active constraint, this is a case of regular nondegenerate example. The KKT system of this problem is

$$\begin{cases} (\frac{(a+b)}{2} - \bar{y})(\frac{(b-a)^2}{12} - \sigma^2)\frac{(a-b)}{3} + \mu_1 = 0, \\ (\frac{(a+b)}{2} - \bar{y})(\frac{(b-a)^2}{12} - \sigma^2)\frac{(b-a)}{3} - \mu_2 = 0, \\ y_{min} - a + \mu_1 - \sqrt{\mu_1^2 + (y_{min} - a)^2} = 0, \\ b - y_{max} + \mu_2 - \sqrt{\mu_2^2 + (b - y_{max})^2} = 0. \end{cases}$$

From equation (2.4), we have

$$\begin{aligned} 0 &= \{(a+b)/2 - \bar{y} - [(b-a)^2/12 - \sigma^2](b-a)/3\}da \\ &\quad \{(a+b)/2 - \bar{y} + [(b-a)^2/12 - \sigma^2](b-a)/3\}db \\ &\quad -(1/5)\sum_{i=1}^{5}\{a+b-2\bar{y} - 4(y_i - \bar{y})[1/12(b-a)^2/12 - \sigma^2]\}dy_i - dz, \\ 0 &= \{1/2 + (b-a)^2/18 + 1/3[(b-a)^2/12 - \sigma^2]\}da \\ &\quad \{1/2 - [(b-a)^2/18] - 1/3[(b-a)^2/12 - \sigma^2]\}db \\ &\quad -(1/5)\sum_{i=1}^{5}[1 - 2(b-a)(y_i - \bar{y})/3]dy_i + d\mu_1, \\ 0 &= \{1/2 - (b-a)^2/18 - 1/3[(b-a)^2/12 - \sigma^2]\}da \\ &\quad \{1/2 + [(b-a)^2/18] + 1/3[(b-a)^2/12 - \sigma^2]\}db \\ &\quad -(1/5)\sum_{i=1}^{5}[1 + 2(b-a)(y_i - \bar{y})/3]dy_i - d\mu_2, \end{aligned}$$

$$0 = d\mu_1, \quad 0 = -dy_5 + db.$$

So we have

$$U = \begin{bmatrix} 0.0000 & 0.0053 & 0 & 0 & -1 \\ 0.5534 & 0.4465 & 1 & 0 & 0 \\ 0.4465 & 0.5534 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, S = \begin{bmatrix} -0.0007 & 0.0000 & 0.0006 & 0.0013 & 0.0043 \\ 0.2344 & 0.2217 & 0.2090 & 0.1962 & 0.1390 \\ 0.1657 & 0.1784 & 0.1911 & 0.2039 & 0.2611 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000 \end{bmatrix}.$$

Since U is nonsingular, we can get all the partial derivatives

$$\begin{bmatrix} \partial a/\partial y_1 & \dots & \partial a/\partial y_5 \\ \partial b/\partial y_1 & \dots & \partial b/\partial y_5 \\ \partial \mu_1/\partial y_1 & \dots & \partial \mu_1/\partial y_5 \\ \partial \mu_2/\partial y_1 & \dots & \partial \mu_2/\partial y_5 \\ \partial z/\partial y_1 & \dots & \partial z/\partial y_5 \end{bmatrix} = U^{-1}S = \begin{bmatrix} 0.4235 & 0.4005 & 0.3775 & 0.3545 & -0.5560 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \\ 0.0235 & 0.0005 & -0.0225 & -0.0455 & 0.0440 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0007 & 0.0001 & -0.0006 & -0.0013 & 0.0010 \end{bmatrix}.$$

From the numerical results, we can see that the sensitivities of μ_1 with respect to all the parameters are zero, this is because the corresponding inequality with respect to (3.1) is inactive, the local changes in parameters can not influence it.

$\boxed{3.2}$ Regular Degenerate Example

Considering the following nonlinear programming problem:

$$\begin{aligned} \min_{x_1,x_2} \quad & f(x) = a_1 x_1^2 + x_2^2, \\ s.t. \quad & h(x) = x_1 x_2^2 - a_2 = 0, \\ & g(x) = -x_1 + a_3 \leq 0. \end{aligned}$$

Let λ and μ be the dual variables with respect to the equality constraint and the inequality constraints respectively. When $a_1 = a_3 = 1, a_2 = 2$, the local optimal solution is

$$x_1^* = 1, x_2^* = \sqrt{2}, \lambda^* = -1, \mu^* = 0, z^* = 3.$$

We can see that $\mu^* = g(x^*, a) = -x_1^* + a_3 = 0$, so it is a case of regular degenerate example.

If we want the inequality constraint remain active, from (2.10) we get

$$U_1 = \begin{pmatrix} 2 & 2\sqrt{2} & 0 & 0 & -1 \\ 2 & -2\sqrt{2} & 2 & -1 & 0 \\ -2\sqrt{2} & 0 & 2\sqrt{2} & 0 & 0 \\ 2 & \sqrt{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so we can get the corresponding sensitivities as follows:

$$\begin{bmatrix} \partial x_1/\partial a_1 & \partial x_1/\partial a_2 & \partial x_1/\partial a_3 \\ \partial x_2/\partial a_1 & \partial x_2/\partial a_2 & \partial x_2/\partial a_3 \\ \partial \lambda/\partial a_1 & \partial \lambda/\partial a_2 & \partial \lambda/\partial a_3 \\ \partial \mu/\partial a_1 & \partial \mu/\partial a_2 & \partial \mu/\partial a_3 \\ \partial z/\partial a_1 & \partial z/\partial a_2 & \partial z/\partial a_3 \end{bmatrix} = U^{-1}S = \begin{bmatrix} 0 & 0 & 1.0000 \\ 0 & 0.3536 & -0.7071 \\ 0 & 0 & 1.0000 \\ 2.0000 & -1.0000 & 6.0000 \\ 1.0000 & 1.0000 & 0 \end{bmatrix}.$$

From the results, we can see that when the parameter a_1 increases a unit, the variables x_1, x_2, λ do not change, but the varibles μ, z will change 2.0000 and 1. When the parameter a_2 increases a unit, the variables x_1, λ do not change, but the variables x_2, μ, z will change $0.3536, 1, -1$. When the parameter a_3 increases a unit, the variables x_1, x_2, λ, μ will change $1, -0.7071, 1, 6$, but the optimal value z will not change.

If we want the inequality constraint to become inactive, then

$$U_1 = \begin{bmatrix} 2 & 2\sqrt{2} & 0 & 0 & -1 \\ 2 & -2\sqrt{2} & 2 & -1 & 0 \\ -2\sqrt{2} & 0 & 2\sqrt{2} & 0 & 0 \\ 2 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, S_1 = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and we can get the corresponding sensitivities as follows.

$$\begin{bmatrix} \partial x_1/\partial a_1 & \partial x_1/\partial a_2 & \partial x_1/\partial a_3 \\ \partial x_2/\partial a_1 & \partial x_2/\partial a_2 & \partial x_2/\partial a_3 \\ \partial \lambda/\partial a_1 & \partial \lambda/\partial a_2 & \partial \lambda/\partial a_3 \\ \partial \mu/\partial a_1 & \partial \mu/\partial a_2 & \partial \mu/\partial a_3 \\ \partial z/\partial a_1 & \partial z/\partial a_2 & \partial z/\partial a_3 \end{bmatrix} = U_1^{-1}S_1 = \begin{bmatrix} -0.3333 & 0.1667 & 0 \\ 0.2357 & 0.2357 & 0 \\ -0.3333 & 0.1667 & 0 \\ 0 & 0 & 0 \\ 1.0000 & 1.0000 & 0 \end{bmatrix}.$$

From the above results, we can see that when the parameter a_1 increases a unit, the variables $x_1, x_2, \lambda, \mu, z$ will change $-0.3333, 0.2357, -0.3333, 0, 1$; When the parameter a_2 increases a unit, the variables $x_1, x_2, \lambda, \mu, z$ will change $0.1667, 0.2357, 0.1667, 0, 1$; When the parameter a_3 increases a unit, the variables $x_1, x_2, \lambda, \mu, z$ do not change.

$\boxed{4}$ Conclusion

Based on a semismooth NCP function, we transfer the KKT systems of a parametric optimization problem which satisfies the linear independent constraint qualification(LICQ) into a system of semismooth equation. Then we get a system of linear equation to calculate all the sensitivities with respect to all the parameters simultaneously by using the semismooth properties of the semismooth equations. Special attention is given to the case of regular degenerate solution in which we can also get a linear equation to solve the local sensitivity according the adjustment of the corresponding active inequalities. As a by-product of our analysis, we obtain a sufficient condition for the existence of the sensitivity solution to the problem we discussed. This new method is a development of the method presented in ref.[9].

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