Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 8, No. 1, pp. 157–184, January 2012

CEPHOIDS: DUALITY AND REFERENCE VECTORS

Diethard Pallaschke and Joachim Rosenmüller

Abstract: A Cephoid is a Minkowski sum of finitely many ("de Gua") simplices in $\mathbb{R}^n$. Within this paper we introduce the concept of duality for cephoids and prove that the maximal faces of the outer surface of a cephoid respect duality in a suitable way. Next, we show that the reference vector of a maximal face (i.e., the vector listing the dimensions of the various subsimplices of the summands involved in that face) uniquely defines that face. Based on these results, we exhibit two graphs on the outer surface of a cephoid. The first one corresponds to a maximal face and its reference system. The second graph describes the generalized tentacles.

Key words: *Minkowski sum, de Gua simplexes, cephoids, number of facets, bargening*

Mathematics Subject Classification: *52B12, 52B05, 90D12*

1 Introduction

A *cephoid* is an algebraic ("Minkowski") sum of "de Gua" simplexes. More precisely, let $\boldsymbol{a} = (a_1, \ldots, a_n) > 0 \in \mathbb{R}^n$, be a positive vector. Let $\boldsymbol{e}^i$ be the i^{th} unit vector and let $\boldsymbol{a}^i := a_i \boldsymbol{e}^i$ $(i \in \boldsymbol{I} := \{1, \ldots, n\})$. There are two simplexes we associate with $\boldsymbol{a}$.

First of all there is the simplex spanned by the vectors $\boldsymbol{a}^i$ $(i \in \boldsymbol{I})$ which is $\Delta^{\boldsymbol{a}} := \operatorname{conv}\left(\left\{\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}\right)$, the subsimplexes of which are denoted by $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \operatorname{conv}((\boldsymbol{a}^{(i)})_{i \in \boldsymbol{J}})$ for $\emptyset \neq \boldsymbol{J} \subseteq \boldsymbol{I}$ (conv denoting the convex hull).

There is, in addition, the simplex obtained by adding the vector $\mathbf{0}$ to the span, i.e., $\Pi^{\boldsymbol{a}} := \operatorname{conv}\left(\left\{0, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\}\right)$. We call this a ***de Gua simplex*** in view of de Gua's generalization of the Pythagorean Theorem (see [1]). Previously, we have also used the term ***prism*** in order to refer to this kind of simplex. The notation is used in order to distinguish the de Gua simplex (prism) $\Pi^{\boldsymbol{a}}$ from its (outward) surface $\Delta^{\boldsymbol{a}}$.

Next, let $\boldsymbol{a}^{\bullet} = (\boldsymbol{a}^{(k)})_{k=1}^K$ be be a *family* of positive vectors in $\mathbb{R}^n$. We write $\boldsymbol{K} := \{1, \ldots, K\}$. Then the algebraic or Minkowski sum

$$\Pi = \Pi^{\boldsymbol{a}^{\bullet}} := \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} = \sum_{k \in \boldsymbol{K}} \Pi^{(k)} \tag{1.1}$$

Π is called a ***cephoid***. Cephoids have been introduced in [7], see also [5],[6]. Throughout this paper we assume that a cephoid is ***nondegenerate***, see [7] for the details.

The outer surface ("Pareto surface","cephoidal surface") of a cephoid Π is denoted by $\partial\Pi$. In order to describe the maximal faces of $\partial\Pi$, it is appropriate to recall the "Coincidence

ISSN 1348-9151

Theorem" ([7]). It states the following. Given some maximal face $\boldsymbol{F}$ of $\partial\Pi$, there is, for each $k \in \boldsymbol{K}$ an index set $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ and a corresponding subsimplex $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\Delta^{(k)}$ such that

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{1.2}$$

holds true. Furthermore, there is a set of coefficients (unique up to a positive multiple constant) $\boldsymbol{c}^\star = (c^\star_k)_{k\in\boldsymbol{K}}$ such that $\boldsymbol{F}$ has the same normal as the convex hull of the "adjusted" subfaces $c^\star_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$. In this context we prefer to also use the term "maximum" for the convex hull referring to the partial ordering induced by inclusion on convex sets; hence this convex hull is denoted by

$$\bigvee_{k\in\boldsymbol{K}} c^\star_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}.$$

The sets $\boldsymbol{J}^{(k)}$ are called the ***reference sets***, the collection $\mathcal{J} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ is the ***reference system*** of $\boldsymbol{F}$. The reference system defines $\boldsymbol{F}$ uniquely, the ***adjustment coefficients*** $c^\star_k$ are determined uniquely up to a positive multiple. The set

$$\boldsymbol{L} \;:=\; \{l \in \boldsymbol{I} \mid l \text{ appears in at least two of the sets } \boldsymbol{J}^{(k)}\}$$

is called the ***adjustment set***. It serves to determine the normal of $\boldsymbol{F}$ as follows. We write $\boldsymbol{L}^{(k)} \;:=\; \boldsymbol{L}\cap\boldsymbol{J}^{(k)}$ and

$$\mathrm{L} \;:=\; \left\{(k,l) \,\middle|\, l\in\boldsymbol{L},\ \boldsymbol{J}^{(k)} \ni l\right\} \;=\; \left\{(k,l) \,\middle|\, l \in \boldsymbol{L}^{(k)}\right\} \tag{1.3}$$

and obtain the ***linear adjustment system*** which is the homogeneous linear system of equations in variables (c_k, λ_l), $((k,l)\in\mathrm{L})$, given by

$$c_k a^{(k)}_l = \lambda_l \quad ((k,l)\in\mathrm{L}). \tag{1.4}$$

As shown in [7], this system (for fixed $\boldsymbol{F}$) has a unique solution (up to a positive constant) $(c^\star_\bullet, \lambda^\star_\bullet)$, the first ingredients of which yield the adjustment coefficients. Moreover, the normal $\boldsymbol{n}^\star$ of $\boldsymbol{F}$ is obtained by computing

$$a^\star_i \;:=\; \max_{k\in\boldsymbol{K}} c^\star_k a^{(k)}_i \ (i\in\boldsymbol{I}) \tag{1.5}$$

and

$$\boldsymbol{n}^\star \;=\; \left(\frac{1}{a^\star_1},\ldots,\frac{1}{a^\star_n}\right). \tag{1.6}$$

Finally, note that the linear functional $\boldsymbol{x} \mapsto \boldsymbol{n}^\star\boldsymbol{x}$ attains its maximum restricted to $\Pi^{(k)} = \Pi^{\boldsymbol{a}^{(k)}}$ exactly on $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$. "Adjustment" means that the maximal value of this functional relative to $c^\star_k\Pi^{(k)}$ (which is attained exactly on $c^\star_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$) equals some common value $t^\star$ for all $k\in\boldsymbol{K}$ – which is why $\boldsymbol{n}^\star$ is indeed the normal to

$$\bigvee_{k\in\boldsymbol{K}} c^\star_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

as well as to $\boldsymbol{F}$.

Within this paper we start out by introducing a notion of *duality* of cephoids. Based on this we exhibit the bijection between reference vectors and maximal faces. Then we continue to explore the structure of faces as well the one of cephoidal surfaces.

Remark 1.1. Within the framework of *Game Theory* a cephoid may be seen as a certain type of bargaining problem (a lottery between bargaining opportunities about money in various currency domains), see MASCHLER--PERLES [10], PALLASCHKE--ROSENMÜLLER, [6], ROSENMÜLLER [12].

In theoretical economics a three-dimensional cephoid implicitly appears in the context of "Ricardian" production and free trade, see the paper of MCKENZIE [4].

From the viewpoint of *Optimization* there is another, rather obvious interpretation of cephoids. Suppose a hiker wanting to ascend a mountain wishes to limit the weight of his rucksack to a unit (of 20 kg, say). He intends to pack various foods $i = 1, \ldots, n$. The weight per unit of food i is given by $\frac{1}{a_i}$. Now, the hiker wants to obtain maximal nourishment from what he carries and it is known that the nutritive quality of a unit of food i is given by n_i.

Consider any plan $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n_+$ of the hiker, implying that he takes the quantity x_i of food i. Then the weight to be attached to this collection of foods is

$$\sum_{i=1}^{n} \frac{x_i}{a_i}$$

and must not exceed 1. Therefore, the hiker has to solve the LP suggested by

$$\max \left\{ \sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \mathbb{R}^n_+,\ \sum_{i=1}^{n} \frac{x_i}{a_i} \leq 1 \right\} = \max \left\{ \sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \Pi^{\boldsymbol{a}} \right\} .$$

This kind of a simple LP is generally called a "rucksack problem".

Now it may happen that there is a small elevator available at the mountain area. This device is very sturdy, so the weight to be carried is not a restriction, at least as far as foods are concerned. However, the volume to be transported is limited; for convenience assume that the device carries a unit in volume maximally.

If food i yields a volume of $\frac{1}{b_i}$ per unit, then any plan $\boldsymbol{y} \in \mathbb{R}^n$ of transporting a volume of y_i $(i \in \boldsymbol{I})$ by the elevator results in a total volume of

$$\sum_{i=1}^{n} \frac{y_i}{b_i},$$

hence maximal nourishment is obtained by solving the LP suggested by

$$\max \left\{ \sum_{i=1}^{n} n_i y_i \,\middle|\, \boldsymbol{y} \in \mathbb{R}^n_+,\ \sum_{i=1}^{n} \frac{y_i}{b_i} \leq 1 \right\} = \max \left\{ \sum_{i=1}^{n} n_i y_i \,\middle|\, \boldsymbol{y} \in \Pi^{\boldsymbol{b}} \right\} .$$

Finally, a hiker having available both, his rucksack and the elevator, is obviously looking for

$$\max \left\{ \sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \mathbb{R}^n_+,\ \boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}'' \ ,\ \ \boldsymbol{x}' \in \Pi^{\boldsymbol{a}}\ , \boldsymbol{x}'' \in \Pi^{\boldsymbol{b}} \right\}$$

which is

$$\max \left\{ \sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}} \right\} .$$

Therefore, a family $\boldsymbol{a}^{\bullet}$ of positive vectors generating the cephoid $\Pi = \Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k=1}^{K} \Pi^{(k)}$ together with a a linear functional $\boldsymbol{x} \mapsto \boldsymbol{n}\boldsymbol{x}$ is obviously interpreted such that each $\Pi^{(k)}$

is representing a production process called "plant k". All plants produce the same good ("nourishment"). A unit of raw material or production factor i put into activity at plant k requires an amount of $\frac{1}{a_i^{(k)}}$ of the capacity of plant k. The plants can be operated independently and the results can be added. Thus, maximizing the linear functional defined above amounts to determining

$$\max\left\{\sum_{i=1}^{n} n_i x_i \;\middle|\; \boldsymbol{x} \in \Pi^{\boldsymbol{a}^{\bullet}}\right\} \quad .$$

Clearly, we are now motivated to provide a description of the "outward" faces of Π, in particular, the *maximal faces*, that is, those of dimension $n-1$.

Let $\boldsymbol{F}$ be a maximal face with normal $\boldsymbol{n}^{\star}$. Let $\boldsymbol{c}^{\star}$ denote the adjustment coefficients which can be computed by means of the linear adjustment system. Consider the "global" rucksack problem suggested by

$$\max\left\{\boldsymbol{n}^{\star}\boldsymbol{x} \;\middle|\; \boldsymbol{x} \in \bigvee_{k\in\boldsymbol{K}} c_k^{\star}\Pi^{(k)}\right\}.$$

It will turn out (Section 2) that the optimal solutions of this problem are the ones of the original "many plants" problem (suggested by the cephoid). The de Gua simplex

$$\widehat{\Pi} \;:=\; \bigvee_{k\in\boldsymbol{K}} c_k^{\star}\Pi^{(k)}$$

represents the new "global" plant which is obtained via

$$a_i^{\star} = \max_{k\in\boldsymbol{K}} c_k a_k^{(i)} \ ,$$

i.e., $\widehat{\Pi} = \Pi^{\boldsymbol{a}^{\star}}$. Thus, the capacities of this plant are defined by

$$\frac{1}{a_i^{\star}} = \frac{1}{\max_{k\in\boldsymbol{K}} c_k a_k^{(i)}} = \min_{k\in\boldsymbol{K}} \frac{1}{c_k a_k^{(i)}} \ .$$

That is, we adjust the capacities of the various plants appropriately (in order to admit comparison of productivity) and then take the minimal capacity in order to obtain the global production process.

The optimal solutions are of the form $c_k^{\star}\boldsymbol{a}^{(k)i}(\iota \in \boldsymbol{J}^{(k)})$. Finally, we obtain

$$\begin{aligned}
\max\left\{\boldsymbol{n}^{\star}\boldsymbol{x} \;\middle|\; \boldsymbol{x} \in \bigvee_{k\in\boldsymbol{K}} c_k^{\star}\Pi^{(k)}\right\} &= \boldsymbol{n}^{\star}c_k^{\star}\boldsymbol{a}^{(k)i} \\
&= t^{\star} \\
&= n_i^{\star}a_i^{(k)}c_k^{\star} \\
&= n_i^{\star}\bar{a}_k^{(i)}c_k^{\star} \\
&= n_i^{\star}\bar{\boldsymbol{a}}^{(i)k}\boldsymbol{c}^{\star} \\
&= t^{\star} \\
&= \max\left\{\boldsymbol{y}\boldsymbol{c}^{\star} \;\middle|\; \boldsymbol{y} \in \bigvee_{i\in\boldsymbol{I}} n_i^{\star}\bar{\Pi}^{(i)}\right\}
\end{aligned} \tag{1.7}$$

which is the "duality theorem of cephoidal programming".

The last paragraphs of this remark will become more obvious in the light of the duality theory which is the topic of the next section.

$\boxed{2}$ Duality

A cephoid is provided by a family of positive vectors or, equivalently, by a positive matrix the rows of which represent the various de Gua simplexes. The dual cephoid, is provided by the transposed matrix. Thus we have

Definition 2.1. Let $\boldsymbol{a}^{\bullet} = \left(\boldsymbol{a}^{(k)}\right)_{k\in\boldsymbol{K}}$ be a family of positive vectors and $\Pi = \Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be the cephoid generated. Put $\bar{a}_k^{(i)} := a_i^{(k)} (i \in \boldsymbol{I},\ k \in \boldsymbol{K})$. We call the family

$$\left(\bar{\boldsymbol{a}}^{(i)}\right)_{i\in\boldsymbol{I}} \tag{2.1}$$

the ***dual family*** and the cephoid

$$\bar{\Pi} \;=\; \Pi^{\bar{\boldsymbol{a}}^{\bullet}} = \sum_{i\in\boldsymbol{I}} \Pi^{\bar{\boldsymbol{a}}^{(i)}} \tag{2.2}$$

the ***dual cephoid***.

More detailed, $\left(\Pi, \bar{\Pi}\right)$ constitutes a dual pair. Yet, it is convenient to speak of the "primal" and "dual" cephoid despite the fact that each is "the dual" of the other one. If the "primal" family $\boldsymbol{a}^{\bullet}$ is regarded as a matrix , then the "dual" family is represented by the transposed matrix $(\bar{a}_k^{(i)})_{i\in\boldsymbol{I},k\in\boldsymbol{K}}$. Throughout this presentation we assume ***nondegeneracy*** for the primal and dual cephoid simultaneously, see Pallaschke--Rosenmüller [7].

Definition 2.2. Let $\boldsymbol{F}$ be a maximal face of Π and let $\mathcal{J} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ be the reference system. Define, for $i \in \boldsymbol{I}$

$$\bar{\boldsymbol{J}}^{(i)} \;:=\; \left\{k \in \boldsymbol{K} \;\middle|\; i \in \boldsymbol{J}^{(k)}\right\}. \tag{2.3}$$

Then we call

$$\bar{\mathcal{J}} \;=\; \left(\bar{\boldsymbol{J}}^{(i)}\right)_{i\in\boldsymbol{I}} \tag{2.4}$$

the ***dual reference system***.

Clearly we have, for any $k \in \boldsymbol{K}$

$$J^{(k)} \;=\; \left\{i \in \boldsymbol{I} \;\middle|\; k \in \bar{\boldsymbol{J}}^{(i)}\right\}, \tag{2.5}$$

so $(\mathcal{J}, \bar{\mathcal{J}})$ again constitute a dual pair. In fact, we may introduce the set

$$\mathrm{J} \;:=\; \{(k,i) \,|\, i \in \boldsymbol{J}^{(k)}\} = \{(i,k) \,|\, k \in \bar{\boldsymbol{J}}^{(i)}\} \tag{2.6}$$

which yields both families simultaneously as cuts in coordinate directions. As a consequence, we have

$$n + K - 1 = \sum_{k\in\boldsymbol{K}} |\boldsymbol{J}^{(k)}| = |\mathrm{J}| = \sum_{i\in\boldsymbol{I}} |\bar{\boldsymbol{J}}^{(i)}| \quad . \tag{2.7}$$

Definition 2.3.

$$\begin{aligned} \bar{\boldsymbol{L}} \;:=\; & \left\{k \in \boldsymbol{K} \;\middle|\; k \text{ is in at least two different } \bar{\boldsymbol{J}}^{(i)}\right\} \\ =\; & \left\{k \in \boldsymbol{K} \;\middle|\; \boldsymbol{J}^{(k)} \text{ contains at least two different indices } i\right\} \\ =\; & \left\{k \in \boldsymbol{K} \;\middle|\; |\boldsymbol{J}^{(k)}| \geq 2\right\} \end{aligned} \tag{2.8}$$

is the ***dual adjustment set***. The analogous property of the primal adjustment set reads now

$$\begin{aligned} \boldsymbol{L} \;:=\; & \left\{ i \in \boldsymbol{I} \,\middle|\, i \text{ is in at least two of the } \boldsymbol{J}^{(k)} \right\} \\ =\; & \left\{ i \in \boldsymbol{I} \,\middle|\, |\bar{\boldsymbol{J}}^{(i)}| \geq 2 \right\}. \end{aligned} \tag{2.9}$$

Recalling the notation

$$\mathrm{L} \;:=\; \left\{ (k,l) \,\middle|\, l \in \boldsymbol{L},\ \boldsymbol{J}^{(k)} \ni l \right\} \;=\; \left\{ (k,l) \,\middle|\, l \in \boldsymbol{L}^{(k)} \right\} \;, \tag{2.10}$$

we obtain the dual version

$$\begin{aligned} \bar{\mathrm{L}} \;:=\; & \left\{ (i,s) \,\middle|\, s \in \bar{\boldsymbol{L}},\ \bar{\boldsymbol{J}}^{(i)} \ni s \right\} \;=\; \left\{ (i,s) \,\middle|\, s \in \bar{\boldsymbol{L}}^{(i)} \right\} \\ =\; & \left\{ (i,s) \,\middle|\, i \in \boldsymbol{J}^{(s)}, \left| \boldsymbol{J}^{(s)} \right| \geq 2 \right\}. \end{aligned} \tag{2.11}$$

Now we are in the position to formulate

Theorem 2.4. *Let $\boldsymbol{a}^{\bullet}$ and its dual be in general position. Let $\boldsymbol{F}$ be a maximal face of Π with reference system $\mathfrak{J}$. Let*

$$(\boldsymbol{c}^{\star}, \boldsymbol{\lambda}^{\star}) \;=\; (c_k^{\star}, \lambda_l^{\star})_{(k,l)\in \mathrm{L}}$$

be a solution of the linear adjustment system corresponding to $\boldsymbol{F}$. Then

$$\bar{\boldsymbol{F}} \;:=\; \sum_{i \in \boldsymbol{I}} \bar{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}. \tag{2.12}$$

is a maximal face of $\bar{\Pi}$ with adjustment set $\bar{\boldsymbol{L}}$ and normal $\boldsymbol{c}^{\star}$.

Proof. **1$^{\text{st}}$STEP** : Let $\boldsymbol{n}^{\star}$ denote the normal of $\boldsymbol{F}$; then we know that the function $\boldsymbol{x} \mapsto \boldsymbol{n}^{\star}\boldsymbol{x}$ attains its maximal value – say t_k – relative to the simplex $\Delta^{(k)}$ exactly on the subsimplex $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$. Moreover, the joint maximal value $t^{\star}$ is attained on every $c_k^{\star}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ (with a suitable choice of $c_k^{\star}$, say $c_k^{\star} = \frac{t^{\star}}{t_k}$).

Consequently, we have

$$\boldsymbol{n}^{\star} c_k^{\star} \boldsymbol{a}^{(k)i} \quad \begin{cases} = & t^{\star} \quad ((k,i) \in \mathrm{J}) \\ < & t^{\star} \quad ((k,i) \notin \mathrm{J}) \end{cases}$$

which can as well be written

$$n_i^{\star} a_i^{(k)} c_k^{\star} \quad \begin{cases} = & t^{\star} \quad ((k,i) \in \mathrm{J}) \\ < & t^{\star} \quad ((k,i) \notin \mathrm{J}) \end{cases} \quad . \tag{2.13}$$

Equivalently we have

$$c_k^{\star} \overline{a}_k^{(i)} n_i^{\star} \quad \begin{cases} = & t^{\star} \quad ((k,i) \in \mathrm{J}) \\ < & t^{\star} \quad ((k,i) \notin \mathrm{J}) \end{cases}$$

which is also

$$c_k^{\star} n_i^{\star} \bar{\boldsymbol{a}}^{(i)k} \quad \begin{cases} = & t^{\star} \quad (k \in \bar{\boldsymbol{J}}^{(i)}) \\ < & t^{\star} \quad (k \notin \bar{\boldsymbol{J}}^{(i)}) \end{cases} \quad . \tag{2.14}$$

Now, equation (2.14) shows that, for each $i \in \boldsymbol{I}$, the function $\boldsymbol{y} \mapsto \boldsymbol{c}^\star \boldsymbol{y}$ attains its maximal value $t^\star$ relative to $n_i^\star \overline{\Delta}^{(i)}$ exactly on $n_i^\star \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}$. Thus, $\boldsymbol{c}^\star$ is normal to

$$\widehat{\widehat{\Delta}} \;:=\; \bigvee_{i \in \boldsymbol{I}} n_i^\star \overline{\Delta}^{(i)} \;. \tag{2.15}$$

2ndSTEP : Since $\boldsymbol{J}^{(k)} \neq \emptyset$ for all $k \in \boldsymbol{K}$, there is, for any $k \in \boldsymbol{K}$, some $i \in \boldsymbol{I}$ such that $i \in \boldsymbol{J}^{(k)}$ holds true. Therefore

$$\bigcup_{i \in \boldsymbol{I}} \bar{\boldsymbol{J}}^{(i)} \;=\; \bigcup_{i \in \boldsymbol{I}} \left\{ k \in \boldsymbol{K} \,\middle|\, i \in \boldsymbol{J}^{(k)} \right\} = \boldsymbol{K}$$

Now, as $\widehat{\widehat{\Delta}}$ is spanned by $n_i^\star \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}$, we conclude that the dimension is $\dim \widehat{\widehat{\Delta}} = K - 1$, that is, the simplex $\widehat{\widehat{\Delta}}$ has maximal dimension. Moreover, we have for the dimension of the spanning subsimplices

$$\sum_{i \in \boldsymbol{I}} \dim \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} = \sum_{i \in \boldsymbol{I}} (|\bar{\boldsymbol{J}}^{(i)}| - 1) = (\sum_{i \in \boldsymbol{I}} \bar{r}_i) - n = (n + K - 1) - n = K - 1 \quad , \tag{2.16}$$

where the second equation follows from $|\boldsymbol{I}| = n$ and the third one from equations (2.7). Also, we have introduced $\bar{r}_i \;:=\; |\bar{\boldsymbol{J}}^{(i)}|$.

3rdSTEP : The function $\boldsymbol{y} \mapsto \boldsymbol{c}^\star \boldsymbol{y}$ takes its maximal value relative to $\overline{\Delta}^{(i)}$ exactly on $\overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}$; this value is $\frac{t^\star}{n_i^\star}$ for $i \in \boldsymbol{I}$. Therefore it is seen that

$$\bar{\boldsymbol{F}} \;=\; \sum_{i \in \boldsymbol{I}} \bar{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}. \tag{2.17}$$

as specified in (2.12) is a face of $\bar{\Pi}$ with normal $\boldsymbol{c}^\star$.

We show that $|\bar{\boldsymbol{J}}^{(i)} \cap \bar{\boldsymbol{J}}^{(j)}| \leq 1$ for all $i \neq j$. Assume that, on the contrary, we have $r, s \in \bar{\boldsymbol{J}}^{(1)} \cap \bar{\boldsymbol{J}}^{(2)}$ for some $r \neq s$. In view of (2.13) we obtain the following equations:

$$\begin{aligned} n_r^\star a_r^{(1)} c_1^\star \;&=\; n_s^\star a_s^{(1)} c_1^\star \\ n_r^\star a_r^{(2)} c_2^\star \;&=\; n_s^\star a_s^{(2)} \;.c_2^\star \end{aligned}$$

Dividing both equations we obtain

$$\frac{a_r^{(1)} c_1^\star}{a_r^{(2)} c_2^\star} \;=\; \frac{a_s^{(1)} c_1^\star}{a_s^{(2)} c_2^\star} \;,$$

that is,

$$\frac{a_r^{(1)}}{a_r^{(2)}} \;=\; \frac{a_s^{(1)}}{a_s^{(2)}} \;,$$

contradicting nondegeneracy.

Consequently, all subsimplices $\overline{\Delta}^{i}_{\bar{\boldsymbol{J}}^{(i)}}$ are located in pairwise orthogonal subspaces. This implies

$$\dim \left(\sum_{i \in \boldsymbol{I}} \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} \right) \;=\; \sum_{i \in \boldsymbol{I}} \dim \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} \;=\; K - 1 \;, \tag{2.18}$$

meaning that $\bar{\boldsymbol{F}}$ is indeed maximal.

$\square$

Remark 2.5. The set $\bar{\boldsymbol{L}}$ is the adjustment set for $\bar{\boldsymbol{F}}$. We write $\bar{L} := |\bar{\boldsymbol{L}}|$. That is, whenever $s \in \bar{\boldsymbol{L}}$, say $s \in \bar{\boldsymbol{J}}^{(i^{'})} \cap \bar{\boldsymbol{J}}^{(i^{''})}$ for suitable $i^{'}, i^{''} \in \boldsymbol{I}$, then the vertex

$$n^{\star}_{i'} \bar{\boldsymbol{a}}^{(i^{'})s} = n^{\star}_{i''} \bar{\boldsymbol{a}}^{(i^{''})s} \tag{2.19}$$

is common to the ("de Gua") subsimplices $n^{\star}_{i'} \bar{\Pi}^{\bar{\boldsymbol{a}}^{(i')}}_{\bar{\boldsymbol{J}}^{(i')}}$ and $n^{\star}_{i''} \bar{\Pi}^{\bar{\boldsymbol{a}}^{(i'')}}_{\bar{\boldsymbol{J}}^{(i'')}}$. Now recall that, for $i \in \boldsymbol{I} \setminus \boldsymbol{L}$, the set $\bar{\boldsymbol{J}}^{(i)}$ consists of just one element. Therefore, using (2.7) and writing $\bar{r}_i := |\bar{\boldsymbol{J}}^{(i)}|$, we obtain

$$\begin{aligned} \sum_{i \in \boldsymbol{L}} \bar{r}_i = & \sum_{i \in \boldsymbol{I}} \bar{r}_i - \sum_{i \in \boldsymbol{I} \setminus \boldsymbol{L}} \bar{r}_i = \sum_{i \in \boldsymbol{I}} \bar{r}_i - \sum_{i \in \boldsymbol{I}, \bar{r}_i = 1} \bar{r}_i \\ = & \ (n + K - 1) - (n - L) = K + L - 1 \end{aligned} \tag{2.20}$$

or

$$\sum_{i \in \boldsymbol{L}} |\bar{\boldsymbol{J}}^{(i)}| = K + L - 1 = \sum_{k \in \boldsymbol{K}} |\boldsymbol{L}^{(k)}|, \tag{2.21}$$

the last equation is derived from [7], Section 3, (3.11).

The analogue equation connecting the primal reference sets with the dual adjustment sets in size is based in the definition $\bar{\boldsymbol{L}}^{(i)} := \bar{\boldsymbol{L}} \cap \bar{\boldsymbol{J}}^{(i)}$ $(i \in \boldsymbol{I})$ and reads

$$\sum_{k \in \bar{\boldsymbol{L}}} |\boldsymbol{J}^{(k)}| = n + \bar{L} - 1 = \sum_{i \in \boldsymbol{I}} |\bar{\boldsymbol{L}}^{(i)}|. \tag{2.22}$$

Remark 2.6. Recall that the *linear adjustment system* with respect to the face $\boldsymbol{F}$ is given by

$$c_k a^{(k)}_l = \lambda_l \quad ((k, l) \in \mathrm{L}). \tag{2.23}$$

Clearly, the ***dual linear adjustment system*** is the linear system of equations in variables $(n_{\bullet}, \mu_{\bullet})$

$$\overline{a}^{(i)}_s n_i \ = \ \mu_s \quad ((i, s) \in \overline{\mathrm{L}}). \tag{2.24}$$

As in the primal case, every solution $\boldsymbol{n}^{\star}$ induces the normal of the dual face. Indeed, define

$$\overline{a}^{\star}_k \ := \ \max_{i \in \boldsymbol{I}} \boldsymbol{n}^{\star} \bar{\boldsymbol{a}}^{(i)k} \ = \ \max_{i \in \boldsymbol{I}} n^{\star}_k \overline{a}^{(i)}_k, \tag{2.25}$$

then the vector $\bar{\boldsymbol{a}}^{\star}$ defines the simplex $\widehat{\overline{\Delta}}$, i.e., $\widehat{\overline{\Delta}} = \Delta^{\bar{\boldsymbol{a}}^{\star}}$ and hence

$$c^{\star}_k = \frac{1}{\overline{a}^{\star}_k} \quad (k \in \boldsymbol{K}) \tag{2.26}$$

defines the normal $\boldsymbol{c}^{\star}$ to the dual face $\bar{\boldsymbol{F}}$. This way we see that *the adjustment coefficients of the primal face constitute the normal of the dual face* and vice versa. In particular, the system (2.24) directly serves to compute the normal of the face. Using only primal terms, we write this system

$$a^{(s)}_i n_i \ = \ \mu_s \quad \Big((i, s) \in \boldsymbol{I} \times \boldsymbol{K}, i \in \boldsymbol{J}^{(s)}, \mid \boldsymbol{J}^{(s)} \mid \ \geq 2\Big) \tag{2.27}$$

Corollary 2.7. *Let* $(\Pi, \bar{\Pi})$ *be a dual pair. Let* $\boldsymbol{F}$ *and* $\widetilde{\boldsymbol{F}}$ *be adjacent maximal faces of* Π. *Then the dual faces* $\bar{\boldsymbol{F}}$ *and* $\bar{\widetilde{\boldsymbol{F}}}$ *are adjacent.*

Proof. Let $\mathcal{J} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ and $\widetilde{\mathcal{J}} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ be the reference systems to $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ respectively. By the Neighborhood Theorem (Theorem 3.3. of [7]) there are indices $k_0, l_0 \in \boldsymbol{K}$ as well as $p, q \in \boldsymbol{I}$ such that $p \notin \boldsymbol{J}^{(k_0)}, q \in \widetilde{\boldsymbol{J}}^{(k_0)}$

$$\widetilde{\boldsymbol{J}}^{(k_0)} = \boldsymbol{J}^{(k_0)} \cup \{p\}\ ,\quad \widetilde{\boldsymbol{J}}^{(l_0)} = \boldsymbol{J}^{(l_0)} \setminus \{q\} \tag{2.28}$$

while for all indices $k \in \boldsymbol{K}, k \neq k_0, l_0$ the reference sets $\boldsymbol{J}^{(k)}$ and $\widetilde{\boldsymbol{J}}^{(k)}$ coincide. Inspection of Definition 2.2 shows that

$$\widetilde{\bar{\boldsymbol{J}}}^{(p)} = \bar{\boldsymbol{J}}^{(p)} \cup \{k_0\}\ ,\quad \widetilde{\bar{\boldsymbol{J}}}^{(q)} = \bar{\boldsymbol{J}}^{(q)} \setminus \{l_0\} \tag{2.29}$$

while for all $i \neq p, q$ the reference sets $\bar{\boldsymbol{J}}^{(i)}$ and $\widetilde{\bar{\boldsymbol{J}}}^{(i)}$ coincide. From this it follows that $\bar{\boldsymbol{F}}$ and $\widetilde{\bar{\boldsymbol{F}}}$ are adjacent. □

Note, however, that the partial ordering of *all faces* of $\partial\Pi$ is *not* preserved during the transition to the dual. The following example enlightens vividly the relevant aspects.

Example 2.8. Consider in particular the case that $K = 2$, i.e., $\boldsymbol{K} = \{1, 2\}$, thus we have $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$. It is known (Theorem 4.1. of [7]) that the cephoidal surface $\partial\Pi$ is completely described by a permutation or ordering $\prec$ such that all maximal faces are of the shape

$$\boldsymbol{F}^{\prec i_0} \;=\; \Delta^{\boldsymbol{a}}_{\{i|i\preceq i_0\}} + \Delta^{\boldsymbol{b}}_{\{i|i_0\preceq i\}}.$$

The reference system for $\boldsymbol{F}^{\prec i_0}$ is thus

$$\left\{\boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)}\right\} = \{\{i|i \preceq i_0\}, \{i|i_0 \preceq i\}\}\,.$$

Consequently, we find for the corresponding dual reference system

$$\bar{\boldsymbol{J}}^{(i)} = \{k \in \boldsymbol{K} \,\big|\, i \in \boldsymbol{J}^{(k)}\} \tag{2.30}$$

the following: whenever $i \prec i_0$, then obviously

$$\bar{\boldsymbol{J}}^{(i)} = \{1\}, \tag{2.31}$$

and whenever $i_0 \prec i$, then

$$\bar{\boldsymbol{J}}^{(i)} = \{2\} \tag{2.32}$$

holds true. For $i = i_0$ we obtain

$$\bar{\boldsymbol{J}}^{(i)} = \{1, 2\} \tag{2.33}$$

so that the dual face to $\boldsymbol{F}^{\prec i_0}$ is

$$\bar{\boldsymbol{F}}^{\prec i_0} = \sum_{i\prec i_0} \bar{\boldsymbol{a}}^{(1)i} + \Delta^{(i_0)}_{\{1,2\}} + \sum_{i_0\prec i} \bar{\boldsymbol{a}}^{(2)i}\ . \tag{2.34}$$

Thus, the Pareto or cephoidal surface of $\bar{\Pi}$ is a linear curve with line segments being the translates of the various $\Delta^{(i_0)}_{\{1,2\}}$. If i_0 is the first w.r.t. $\prec$, then the face

$$\bar{\boldsymbol{F}}^{\prec i_0} = \Delta^{(i_0)}_{\{1,2\}} + \sum_{i\neq i_0} \bar{\boldsymbol{a}}^{(2)i}\ .$$

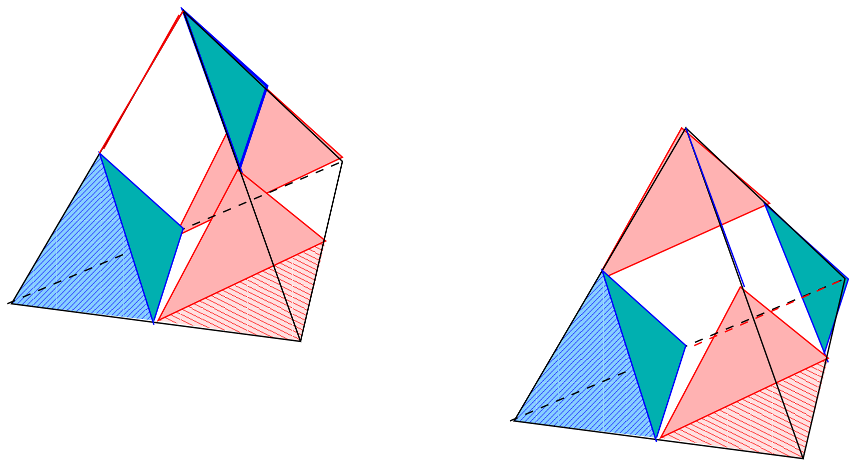

Figure 1: The sum of two de Gua simplexes for $n = 4$

is the "uppermost" line segment, i.e., the first in the ordering induced by the slope when we begin with the smallest slope (in absolute value). Thus, it is seen that $\prec$ represents as well the ordering of the line segments within the cephoidal surface of the dual cephoid.

In particular, consider Example 4.4. of [7] which treats a case with $n = 4$. A sketch of the canonical representation is provided by Figure 1. Assuming that the translate of $\Delta^{\boldsymbol{a}}$ occupies the first vertex of the sum (i.e., $2\boldsymbol{e}^1$), and the translate of $\Delta^{\boldsymbol{b}}$ the second one, the left hand version of Figure 1 corresponds to the ordering $\prec\ = (2341)$. The 3-dimensional faces are given by

$$\begin{aligned}
\boldsymbol{F}^{\prec 2} &= \Delta^{\boldsymbol{a}}_{2} + \Delta^{\boldsymbol{b}}_{2341} \\
\boldsymbol{F}^{\prec 3} &= \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{341} \\
\boldsymbol{F}^{\prec 4} &= \Delta^{\boldsymbol{a}}_{234} + \Delta^{\boldsymbol{b}}_{41} \\
\boldsymbol{F}^{\prec 1} &= \Delta^{\boldsymbol{a}}_{2341} + \Delta^{\boldsymbol{b}}_{1}\ .
\end{aligned} \tag{2.35}$$

The ordering $\prec$ represents the neighborhood structure of the four faces simultaneously indicating the unique extremal vector $\boldsymbol{c}^i = \boldsymbol{a}^i + \boldsymbol{b}^i$ assigned to a face. If we start with $\boldsymbol{F}^{\prec 2}$ containing $\boldsymbol{c}^2$, then the unique neighbor is $\boldsymbol{F}^{\prec 3}$ containing $\boldsymbol{c}^3$ etc.. Thus, while running through the extremals $\boldsymbol{c}^i$ according to $\prec$ one also passes from one face to its neighbor.

The same situation occurs with respect to the dual cephoid $\bar{\Pi}$. The dual face to $\boldsymbol{F}^{\prec 2}$ i.e., generated by the reference system $\mathfrak{J} = \{\{2\},\{2341\}\}$ is $\bar{\boldsymbol{F}}^{\prec 2}$ which is given by

$$\bar{\mathfrak{J}}\ =\ \{\{2\},\{12\},\{2\},\{2\}\}$$

i.e.,

$$\bar{\boldsymbol{F}}^{\prec 2} = \Delta^{(1)}_{\{2\}} + \Delta^{(2)}_{\{12\}} + \Delta^{(3)}_{\{2\}} + \Delta^{(4)}_{\{2\}}$$

this is a translate of $\bar{\Delta}^{(2)}_{\{12\}}$ by means of $\boldsymbol{a}^{(1)2} + \boldsymbol{a}^{(3)2} + \boldsymbol{a}^{(4)2}$. Similarly,

$$\bar{\boldsymbol{F}}^{\prec 3} = \Delta^{(1)}_{\{2\}} + \Delta^{(2)}_{\{1\}} + \Delta^{(3)}_{\{12\}} + \Delta^{(4)}_{\{2\}}$$

is a translate of $+\Delta^{(3)}_{\{12\}}$. The further two dual faces are.

$$\bar{\boldsymbol{F}}^{\prec 4} = \Delta^{(1)}_{\{2\}} + \Delta^{(2)}_{\{1\}} + \Delta^{(3)}_{\{1\}} + \Delta^{(4)}_{\{12\}}$$

and

$$\bar{\boldsymbol{F}}^{\prec 1} = \Delta^{(1)}_{\{12\}} + \Delta^{(2)}_{\{1\}} + \Delta^{(3)}_{\{1\}} + \Delta^{(4)}_{\{1\}} \ .$$

The cephoidal surface $\partial\bar{\Pi}$ is sketched together with its canonical representation in Figure 2. When we start in the uppermost face and run through the faces according to $\prec$, then we

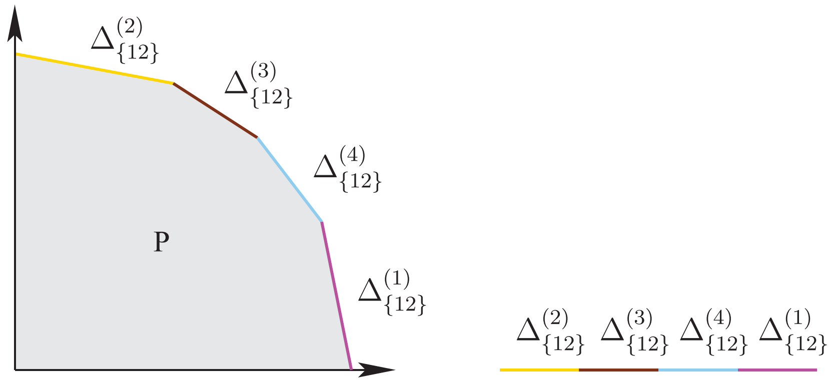

Figure 2: The dual surface and its canonical representation

pass all faces in downwards direction.

Example 2.9. Next we discuss an example with $n = 4, K = 3$ named "The Marriage of a Windmill and a Circle". The canonical representation is given by Figure 3. We use $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ for the primal family assuming that $\Delta^{(\boldsymbol{a})}$ corresponds to "blue", $\Delta^{(\boldsymbol{b})}$ corresponds to "red", and $\Delta^{(\boldsymbol{c})}$ corresponds to "green".

The maximal faces as indicated are defined by the following reference sets:

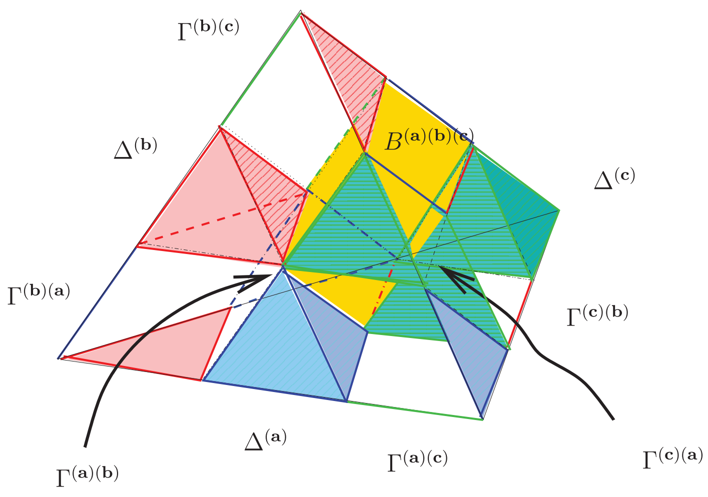

Figure 3: The Marriage of a windmill and a circle

$$
\begin{array}{rlrl}
\textit{Name} & \boldsymbol{J}^{(\boldsymbol{a})} & \boldsymbol{J}^{(\boldsymbol{b})} & \boldsymbol{J}^{(\boldsymbol{c})} \\
\Delta^{(\boldsymbol{a})} & \{1234\} & \{2\} & \{1\} \\
\Gamma^{(\boldsymbol{a})(\boldsymbol{c})} & \{234\} & \{2\} & \{12\} \\
\Gamma^{(\boldsymbol{a})(\boldsymbol{b})} & \{134\} & \{23\} & \{1\} \\
 & & & \\
\Delta^{(\boldsymbol{b})} & \{4\} & \{1234\} & \{1\} \\
\Gamma^{(\boldsymbol{b})(\boldsymbol{c})} & \{4\} & \{234\} & \{14\} \\
\Gamma^{(\boldsymbol{b})(\boldsymbol{a})} & \{14\} & \{123\} & \{1\} \\
 & & & \\
\Delta^{(\boldsymbol{c})} & \{3\} & \{3\} & \{1234\} \\
\Gamma^{(\boldsymbol{c})(\boldsymbol{b})} & \{3\} & \{23\} & \{124\} \\
\Gamma^{(\boldsymbol{c})(\boldsymbol{a})} & \{34\} & \{2\} & \{124\} \\
 & & & \\
B^{(\boldsymbol{a})(\boldsymbol{b})(\boldsymbol{c})} & \{34\} & \{23\} & \{14\}
\end{array}
\tag{2.36}
$$

The dual cephoid is the sum of 4 de Gua simplexes in 3 dimensions, we denote the dual family by $\bar{\boldsymbol{a}}^{(1)}$. The canonical representation is given by the following sketch.

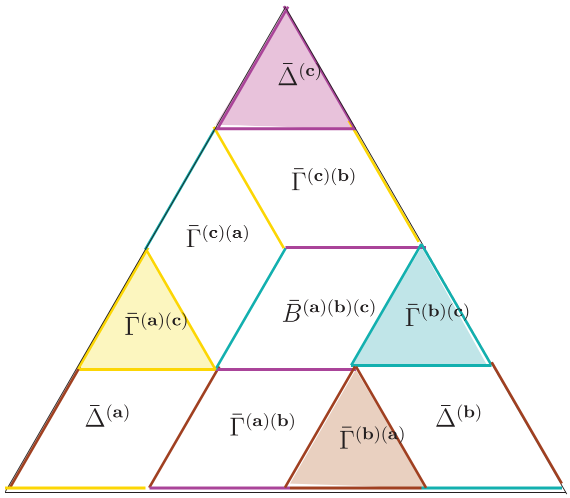

Figure 4: The Dual Marriage

The maximal faces are listed in the same order as its primal counterparts and indicated accordingly. We obtain the following list.

$$\begin{array}{rllll}
Name & \overline{\boldsymbol{J}}^{(1)} & \overline{\boldsymbol{J}}^{(2)} & \overline{\boldsymbol{J}}^{(3)} & \overline{\boldsymbol{J}}^{(4)} \\
\bar{\Delta}^{(\boldsymbol{a})} & \{13\} & \{12\} & \{1\} & \{1\} \\
\bar{\Gamma}^{(\boldsymbol{a})(\boldsymbol{c})} & \{3\} & \{123\} & \{1\} & \{1\} \\
\bar{\Gamma}^{(\boldsymbol{a})(\boldsymbol{b})} & \{13\} & \{2\} & \{12\} & \{1\} \\
& & & & \\
\bar{\Delta}^{(\boldsymbol{b})} & \{23\} & \{2\} & \{2\} & \{12\} \\
\bar{\Gamma}^{(\boldsymbol{b})(\boldsymbol{c})} & \{3\} & \{2\} & \{2\} & \{123\} \\
\bar{\Gamma}^{(\boldsymbol{b})(\boldsymbol{a})} & \{123\} & \{2\} & \{2\} & \{1\} \\
& & & & \\
\bar{\Delta}^{(\boldsymbol{c})} & \{3\} & \{3\} & \{123\} & \{3\} \\
\bar{\Gamma}^{(\boldsymbol{c})(\boldsymbol{b})} & \{3\} & \{23\} & \{12\} & \{3\} \\
\bar{\Gamma}^{(\boldsymbol{c})(\boldsymbol{a})} & \{3\} & \{23\} & \{1\} & \{13\} \\
& & & & \\
\bar{B}^{(\boldsymbol{a})(\boldsymbol{b})(\boldsymbol{c})} & \{3\} & \{2\} & \{12\} & \{13\}
\end{array} \tag{2.37}$$

$\boxed{3}$ The Reference Vector

Given a maximal face

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

of a cephoid Π, we write $r_k \;:=\; |\boldsymbol{J}^{(k)}| \;\; (k \in \boldsymbol{K})$ and call $\boldsymbol{r} = (r_1, \ldots, r_K)$ the ***reference vector*** of $\boldsymbol{F}$. We are going to show that the reference vector uniquely defines the face. To this end, we list the main properties of such a vector.

Definition 3.1. Let $n, K \in \mathrm{N}$. A vector of positive integers $\boldsymbol{r} \;=\; (r_1, \ldots, r_K) \in \mathrm{N}_0^K$ is said to be a (K, n)–***reference code*** if

$$r_k \leq n \;\; (k \in \boldsymbol{K}) \tag{3.1}$$

and

$$\sum_{\kappa=1}^{K} r_k \;\leq\; K + n - 1 \;\; . \tag{3.2}$$

A reference code $\boldsymbol{r}$ is ***maximal*** if an equation holds true in (3.2).

Clearly, a reference vector of a maximal face is a maximal reference code.

Theorem 3.2. *Let $\boldsymbol{a}^{\bullet} \;=\; \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ be a nondegenerate family of positive vectors in $\mathbb{R}^n$. Then, for every maximal (K, n)–reference code $\boldsymbol{r}$ there exists a unique maximal face $\boldsymbol{F}$ of $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ with reference system*

$$\left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}} \tag{3.3}$$

with

$$|\boldsymbol{J}^{(k)}| = r_k \quad (k \in \boldsymbol{K}) \ , \tag{3.4}$$

i.e., $\boldsymbol{r}$ *is the reference vector of* $\boldsymbol{F}$*.*

Proof. **1$^{\mathbf{st}}$STEP** : For $n = 2$ the Theorem is obvious. For $K = 2$ the Theorem follows from Theorem 4.1 of [7]. Of course, the case $K = 2$ follows as well by duality as explained in the previous section.

Now we proceed by induction.

2$^{\mathbf{nd}}$STEP : Within this step we show that any two maximal faces necessarily have different reference vectors.

First of all assume $K \leq n-1$. Let $\boldsymbol{F}$ be a maximal face and let

$$\boldsymbol{F} \ = \ \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

be the representation via the reference system. Also, let $\boldsymbol{L}$ be the set of adjustment indices. As $|\boldsymbol{L}| = L \leq K-1 \leq n-2$ (Theorem 3.1, (ii),(a) of [7]) there are at least two indices, say 1 and n, that do not belong to $\boldsymbol{L}$. As a consequence $\boldsymbol{F}|_{\mathbb{R}_{\boldsymbol{I}-1}}$ and $\boldsymbol{F}|_{\mathbb{R}_{\boldsymbol{I}-n}}$ are maximal faces of the families $\boldsymbol{a}^{\bullet}|_{\mathbb{R}_{\boldsymbol{I}-1}}$ and $\boldsymbol{a}^{\bullet}|_{\mathbb{R}_{\boldsymbol{I}-n}}$ respectively. The reference vectors of these faces are reference codes of the form $(r_1, \ldots, r_\kappa - 1, \ldots, r_K)$ and $(r_1, \ldots, r_l - 1, \ldots, r_K)$ with suitable $\kappa, l \in \boldsymbol{K}$. By induction, these reference codes uniquely determine the reference systems

$$\left\{\boldsymbol{J}^{(k)} \setminus \{1\}\right\}_{k\in\boldsymbol{K}} \quad , \quad \left\{\boldsymbol{J}^{(k)} \setminus \{n\}\right\}_{k\in\boldsymbol{K}}$$

of two faces of the restrictions of Π to $\mathbb{R}^{\boldsymbol{I}-1}$ and $\mathbb{R}^{\boldsymbol{I}-n}$ respectively. Hence the reference system

$$\left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$$

of $\boldsymbol{F}$ is uniquely determined by $\boldsymbol{r}$. This shows, that there is at most one face corresponding to a reference code, provided $K \leq n-1$ holds true.

But for $K \geq n$ we know that every maximal face is the sum of at most $n-1$ subfaces of the $\Delta^{(k)}$ plus a number of vertices from the remaining ones. In other words, every face is r–full for some $r \leq n-1$ (see [7]) By the above argument, with respect to the $n-1$ faces that yield reference sets of size at least 2, these reference sets are uniquely defined. The remaining vertices, however, are uniquely defined as well.

Thus, a reference code defines a face uniquely, if at all.

3$^{\mathbf{rd}}$STEP :

On the other hand, given a family $\boldsymbol{a}^{\bullet}$ and the cephoid Π generated, let $\mathcal{F}(K, n)$ be the set of maximal faces of Π. Let $\Pi|_{-n}$ be the cephoid generated by the family $\boldsymbol{a}|_{\mathbb{R}_{\boldsymbol{I}-n}}$ of vectors projected onto $\mathbb{R}_{\boldsymbol{I}-n}$ and let $\mathcal{F}(K, n-1)$ denote the family of its maximal faces.

Similarly, let $\Pi^{(-K)} \ = \ \sum_{k=1}^{K-1} \Pi^{(k)}$ be the sum of the first $K-1$ de Gua simplexes and let $\mathcal{F}(K-1, n)$ denote the system of maximal faces of this cephoid. The induction hypothesis applies to both cephoids constructed.

Now, let $\boldsymbol{F} \in \mathcal{F}(K, n)$ and let $\boldsymbol{r}$ be its reference vector. First of all, assume that $r_K = 1$ is the case, that is, $\boldsymbol{F}$ consists of a face $\boldsymbol{F}^{(-K)}$ of $\Pi^{(-K)}$ plus a vertex of $\Delta^{(K)}$. By induction, the face $\boldsymbol{F}^{(-K)}$ is uniquely defined by $(r_1, \ldots, r_{K-1})$ By nondegeneracy the remaining vertex of $\Delta^{(K)}$ is uniquely defined as well. On the other hand, every face $\Pi^{(-K)}$ together with a suitable and unique vertex of $\Delta^{(K)}$ yields a face in $\mathcal{F}(K-1, n)$. Thus, $\mathcal{F}(K-1, n)$ and the set $\left\{ \boldsymbol{F} \in \mathcal{F}(K,n) \,\middle|\, r_K = |\boldsymbol{J}^{(K)}| = 1 \right\}$ are bijectively mapped into each other in a canonical way.

4^{th}STEP :

Next, let $\boldsymbol{F} \in \mathcal{F}(K,n)$ be such that $r_K = |\boldsymbol{J}^{(K)}| \geq 2$ is true. By induction, there is a unique maximal face, say $\boldsymbol{F}^\star \in \mathcal{F}(K, n-1)$ of $\Pi|_{\mathbb{R}^{\boldsymbol{I}-n}}$ that corresponds to the reference vector $(r_1, \ldots, r_{K-1})$. By the first step, we conclude that

$$\left| \left\{ \mathcal{F}(K,n) \,\middle|\, |\boldsymbol{J}^{(K)}| \geq 2 \right\} \right| \leq |\mathcal{F}(K, n-1)| \tag{3.5}$$

holds true. But by Proposition 6.10 of [7] we know that

$$|\, \mathcal{F}(K,n)\,| \;=\; |\, \mathcal{F}(K-1,n)\,| \,+\, |\, \mathcal{F}(K,n-1).\,|$$

Then, necessarily, equation prevails in formula (3.5) and hence there is indeed for every maximal code $\boldsymbol{r}$ a maximal face that has $\boldsymbol{r}$ as its reference vector.

□

Corollary 3.3. *Let $\boldsymbol{a}^\bullet = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ be a nondegenerate family of positive vectors in $\mathbb{R}^n$. Then, for every $k \in \boldsymbol{K}$ and every $i \in \boldsymbol{I}$ there is a bijection $\mathcal{P}^{(i)}$ which maps*

$$\left\{ \boldsymbol{F} \in \mathcal{F}(K,n) \,\middle|\, |\boldsymbol{J}^{(k)}| = 2 \right\}$$

on

$$\mathcal{F}^{(-i)}(K, n-1) \;:=\; \left\{ \boldsymbol{F} \,\middle|\, \boldsymbol{F} \text{ is a maximal face of } \boldsymbol{a}^\bullet|_{\mathbb{R}_{\boldsymbol{I}\setminus\{-i\}}} \right\}$$

This bijection is obtained by associating with any maximal face $\boldsymbol{F}$ with reference code $\boldsymbol{r}$, $r_k \geq 2$, the maximal face on $\partial\Pi^{(-i)} \;:=\; \partial\Pi_{\,|\,\mathbb{R}^{\boldsymbol{I}\setminus\{i\}}}$ defined via $\boldsymbol{r} - \boldsymbol{e}^k$.

Corollary 3.4. *Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ be a nondegenerate cephoid. There is a bijection of $\mathcal{F} \;:=\; \{\boldsymbol{F}|\boldsymbol{F}$ is a maximal face of $\Pi\}$ onto the set of maximal (K,n)–reference codes.*

4 The Reference Graph

Within the previous sections we discussed necessary conditions for a maximal face, i.e., properties of the reference system, the adjustment set, and the normal determined by the corresponding linear adjustment system.

Now we want to consider sufficient conditions. Let

$$\Pi \;=\; \Pi^{\boldsymbol{a}^\bullet} = \sum_{k=1}^{K} \Pi^{(k)}$$

be a cephoid generated by a family $\boldsymbol{a}^\bullet = \left(\boldsymbol{a}^{(k)}\right)_{k\in\boldsymbol{K}}$. We use $\Delta^{(k)} \;=\; \Delta^{a^{(k)}}, \Pi^{(k)} \;=\; \Pi^{a^{(k)}}$ etc. $(k \in \boldsymbol{K})$. Given a family of subsets of $\boldsymbol{I}$, say

$$\mathcal{J} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}},$$

we may assign to every index set $\boldsymbol{J}^{(k)}$ the simplex

$$\Delta^{(k)}_{\boldsymbol{J}^{(k)}} \;=\; \operatorname{conv}\left(\left\{\boldsymbol{a}^{(k),l}\right\}_{l\in\boldsymbol{J}^{(k)}}\right) \subseteq \Delta^{(k)}.$$

We will exhibit conditions for $\mathfrak{J}$ such

$$\boldsymbol{F} = \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{4.1}$$

is a maximal face of Π. To this end we will discuss some obvious requirements a system of index sets $\mathfrak{J}$ has to satisfy in order to be a candidate for a reference system to a maximal face.

Definition 4.1. Let

$$\mathfrak{J} = \left(\boldsymbol{J}^{(k)}\right)_{k \in \boldsymbol{K}}$$

be a family of subsets of $\boldsymbol{I}$. $\mathfrak{J}$ is called an ***admissible system*** if the following conditions are satisfied:

1. $\bigcup_{k \in \boldsymbol{K}} \boldsymbol{J}^{(k)} \ = \ \boldsymbol{I}$
2. $\sum_{k \in \boldsymbol{K}} |\boldsymbol{J}^{(k)}| \ = \ \sum_{k \in \boldsymbol{K}} j_k \ = \ K + n - 1 \ .$
3. For any two different indices $k, l \in \boldsymbol{K}$ the sets $\boldsymbol{J}^{(k)}$ and $\boldsymbol{J}^{(l)}$ contain at most one common index.
4. For every index $k \in \boldsymbol{K}$ there exists an index $k' \in \boldsymbol{K}$ with $k \neq k'$ and $\left| \left(\boldsymbol{J}^{(k)} \cap \boldsymbol{J}^{(k')} \right) \right| = 1$.

Thus, the reference system of a face of a cephoid is clearly admissible.

For every admissible system $\mathfrak{J}$ we denote by $\boldsymbol{L} \subseteq \boldsymbol{I}$ the set of indices that appear in at least two of the members $\boldsymbol{J}^{(k)}$ of the family. $\boldsymbol{L}$ is called the set of ***critical indices*** (corresponding to $\mathfrak{J}$). Accordingly,

$$\boldsymbol{L}_k \ := \ \boldsymbol{L} \cap \boldsymbol{J}^{(k)} \tag{4.2}$$

defines the ***critical system***

$$\mathfrak{L} = \left(\boldsymbol{L}^{(k)}\right)_{k \in \boldsymbol{K}} \quad . \tag{4.3}$$

The critical system obviously inherits the defining properties from its parental admissible system, i.e., we have:

1. $\bigcup_{k \in \boldsymbol{K}} \boldsymbol{L}^{(k)} \ = \ \boldsymbol{L}$
2. For any two different indices $k, k' \in \boldsymbol{K}$ the sets $\boldsymbol{L}^{(k)}$ and $\boldsymbol{L}^{(k')}$ contain at most one common index.
3. For every index $k \in \boldsymbol{K}$ there exists an index $k' \in \boldsymbol{K}$ with $k \neq k'$ and $\left| \left(\boldsymbol{L}^{(k)} \cap \boldsymbol{L}^{(k')} \right) \right| = 1$.

We use this as a motivation for the following

Definition 4.2. Let $\boldsymbol{L} \subseteq \boldsymbol{I}$ and let

$$\mathcal{L} = \left(\boldsymbol{L}^{(k)}\right)_{k \in \boldsymbol{K}} \quad .$$

be a system of subsets of $\boldsymbol{L}$. We say that $\mathcal{L}$ is ***$\boldsymbol{L}$–admissible*** if the conditions $1., 2.$, and 3. are satisfied.

Thus, the critical system of an admissible set is $\boldsymbol{L}$–admissible with respect to the set $\boldsymbol{L}$ of critical indices.

We wish to associate a graph to an admissible $\boldsymbol{L}$–system as follows.

Definition 4.3. The (undirected) ***associated graph*** generated by an admissible $\boldsymbol{L}$–system $\mathcal{L}$ is the pair

$$(\mathcal{L}, \mathcal{E}) \tag{4.4}$$

given as follows. The ***nodes*** of the graph are the elements of the family $\mathcal{L}$. An ***edge*** or arc of the graph is a pair $\boldsymbol{E} = (\boldsymbol{L}_k, \boldsymbol{L}_{k'})$ such that $\boldsymbol{L}_k \cap \boldsymbol{L}_{k'} \neq \emptyset$ holds true. Colloquially we say that $\boldsymbol{L}_k$ and $\boldsymbol{L}_{k'}$ are connected if $\boldsymbol{E} = (\boldsymbol{L}_k, \boldsymbol{L}_{k'})$ is an edge.

As graph as defined above may have cycles, i.e., in our case a sequence of nodes $\boldsymbol{L}^{(k_1)}$, $\boldsymbol{L}^{(k_2)}, \ldots, \boldsymbol{L}^{(k_T)}$ such that , for any $t \in \{1, \ldots, T-1\}$ the nodes $\boldsymbol{L}^{(k_t)}$ and $\boldsymbol{L}^{(k_{t+1})}$ are connected and $\boldsymbol{L}^{(k_1)} = \boldsymbol{L}^{(k_T)}$ is the case. We call a cycle ***proper*** if the same index $l \in \boldsymbol{L}$ is involved in each edge, i.e., if

$$\boldsymbol{L}^{(k_t)} \cap \boldsymbol{L}^{(k_{t+1})} = \{l\} \tag{4.5}$$

holds true for some $l \in \boldsymbol{L}$ and all $t \in \{1, \ldots, T-1\}$. Otherwise we call the cycle ***improper***.

Definition 4.4. An $\boldsymbol{L}$–admissible family of index sets

$$\mathcal{L} = \left(\boldsymbol{L}^{(k)}\right)_{k \in \boldsymbol{K}}$$

is called a ***pre–adjustment system*** if the following conditions are satisfied:

1. $L \ := \ |\boldsymbol{L}| \leq K - 1$ holds true.
2. $\displaystyle\sum_{k \in \boldsymbol{K}} |\boldsymbol{L}^{(k)}| \ =: \ \sum_{k \in \boldsymbol{K}} L_k \ = \ K + L - 1 \ .$
3. There are at least two indices k^*, k° such that $|\boldsymbol{L}^{k^*}| = |\boldsymbol{L}^{k^\circ}| = 1$ holds true. That is, the associated graph has at least two ***boundary nodes***.
4. The associated graph $(\mathcal{L}, \mathcal{E})$ is connected.
5. The associated graph $(\mathcal{L}, \mathcal{E})$ has no improper cycles.

An admissible family of index sets

$$\mathcal{J} = \left(\boldsymbol{J}^{(k)}\right)_{k \in \boldsymbol{K}}$$

is called a ***pre–reference system*** if the critical set $\boldsymbol{L}$ induces a critical system $\mathcal{L}$ that is a pre–adjustment system. The corresponding ***linear pre–adjustment system*** is the linear system of equations formed in analogy to (1.4) of section 1.

A reference system resulting from a maximal face has the properties listed above. Indeed, a reference system induces a set $\boldsymbol{L}$ of adjustment indices as well as an adjustment system which is $\boldsymbol{L}$–admissible. The associated graph is called the ***adjustment graph***. Now we have

Lemma 4.5. *Let $\boldsymbol{F}$ be a maximal face of a cephoid $\Pi = \Pi^{\boldsymbol{a}^\bullet}$. Then the adjustment graph has no improper cycles.*

Proof. If the graph has an improper cycle, then the linear adjustment system admits of the trivial solution only. More precisely, let (w.l.o.g)

$$\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}, \ldots \boldsymbol{L}^{(\kappa)}, \boldsymbol{L}^{(1)}$$

constitute an improper cycle. Then we find indices $l_1, l_2, \ldots, l_\kappa$ such that

$$l_1 \in \boldsymbol{L}^{(1)} \cap \boldsymbol{L}^{(2)},\ l_2 \in \boldsymbol{L}^{(2)} \cap \boldsymbol{L}^{(3)},\ \ \ldots,\ l_\kappa \in \boldsymbol{L}^{(\kappa)} \cap \boldsymbol{L}^{(1)}$$

holds true. Consider the following subsystem of the linear adjustment system, given by

$$\begin{array}{rcl} c_1 \boldsymbol{a}^{(1)}_{l_1} & = & \lambda_{l_1} \\ c_2 \boldsymbol{a}^{(2)}_{l_1} & = & \lambda_{l_1} \\ c_2 \boldsymbol{a}^{(2)}_{l_2} & = & \lambda_{l_2} \\ \ldots & \ldots & \ldots \\ c_\kappa \boldsymbol{a}^{(\kappa)}_{l_{\kappa-1}} & = & \lambda_{l_{\kappa-1}} \\ c_\kappa \boldsymbol{a}^{(\kappa)}_{l_\kappa} & = & \lambda_{l_\kappa} \\ c_\kappa \boldsymbol{a}^{(1)}_{l_\kappa} & = & \lambda_{l_\kappa}. \end{array} \tag{4.6}$$

This is a system with 2κ variables and 2κ equations. If we write $a_{ik} := a^{(k)}_{l_i}$ just for the moment, the coefficient matrix is

$$\begin{array}{c} \begin{array}{ccccccccc} 1 & & & & \kappa & \kappa+1 & & & 2\kappa \end{array} \\ \begin{pmatrix} a_{11} & & & & & \vdots & 1 & & & & \\ & a_{21} & & & & \vdots & 1 & & & & \\ & a_{22} & & & & \vdots & & 1 & & & \\ & & a_{23} & & & \vdots & & 1 & & & \\ & & & \cdots & & \vdots & & & \cdots & & \\ & & & \cdots & & \vdots & & & \cdots & & \\ & & & & a_{\kappa\kappa-1} & \vdots & & & & 1 & \\ & & & & a_{\kappa\kappa} & \vdots & & & & & 1 \\ a_{1\kappa} & & & & & \vdots & & & & & 1 \end{pmatrix} \end{array}. \tag{4.7}$$

We claim that the matrix (4.7) has full rank. To see this, subtract an $a_{1\kappa}$–multiple of the last column from the first column and, thereafter, omit the last column and the last row. Next, add an $\frac{a_{1\kappa}}{a_{\kappa\kappa}}$–multiple of column κ to column 1. Then, the last row contains the entry

$a_{\kappa\kappa}$ only. Hence (4.7) has full rank if and only if the following matrix (4.8)

$$\begin{array}{c}
\begin{array}{ccccccccccc}
1 & & & & \kappa-1 & & \kappa & & & & 2(\kappa-1)
\end{array} \\
\left(\begin{array}{ccccccccccc}
a_{11} & & & & & : & 1 & & & & \\
 & a_{21} & & & & : & 1 & & & & \\
 & a_{22} & & & & : & & 1 & & & \\
 & & a_{23} & & & : & & 1 & & & \\
 & & & \cdots & & : & & & \cdots & & \\
 & & & \cdots & & : & & & \cdots & & \\
 & & & & a_{\kappa-1\kappa-2} & : & & & & 1 & \\
 & & & & a_{\kappa-1\kappa-1} & : & & & & & 1 \\
\frac{a_{1\kappa}a_{\kappa\kappa-1}}{a_{\kappa\kappa}} & & & & & : & & & & & 1
\end{array}\right)
\end{array} \tag{4.8}$$

has full rank. By induction, we see that (4.7) has full rank indeed.

□

Lemma 4.6. *Let $\boldsymbol{F}$ be a maximal face of a cephoid $\Pi = \Pi^{\boldsymbol{a}^\bullet}$. Then the adjustment graph is connected.*

Proof. The proof runs quite analogously to the one of the previous Lemma 4.5. If the adjustment graph can be decomposed into two disjoint graphs, the each part admits of an independent solution of the linear adjustment system. Hence the solutions span a linear space of dimension at least two – in which case the normal is not uniquely defined up to a constant. A precise version is found in Theorem 2.4. of [5]. □

Lemma 4.7. *Let $\boldsymbol{F}$ be a maximal face of a cephoid $\Pi = \Pi^{\boldsymbol{a}^\bullet}$. Then the adjustment graph has at least two boundary nodes.*

The proof is obvious because the adjustment graph has no improper cycles.

Corollary 4.8. *Let $\boldsymbol{a}^\bullet$ be a family of positive vectors. Let $\boldsymbol{F}$ be a maximal face of the corresponding cephoid Π. Then the reference system defining $\boldsymbol{F}$ is a pre–reference system. The adjustment system is a pre–adjustment system.*

Clearly, to any pre–adjustment system that arises from a pre–reference system we may associate the polyhedron

$$\boldsymbol{F}_{\boldsymbol{L}} := \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{L}^{(k)}}. \tag{4.9}$$

Now we have

Theorem 4.9. *Let $\mathfrak{J}_F = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ be a family of subsets of $\boldsymbol{I}$. Then*

$$\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{4.10}$$

is a maximal face of Π if and only if the following holds true.

1. *$\mathfrak{J}$ is a pre–adjustment system.*
2. *The solution $(\boldsymbol{c}^\star, \lambda^\star)$ to the linear pre–adjustment system of equations satisfies*

$$\begin{array}{rcl} c^\star_k a^{(k)}_l & = & \lambda^\star_l \quad ((k,l) \in \mathrm{L}) \\ & \geq & c^\star_{k'} a^{(k')}_l ((k,l) \notin \mathrm{L}). \end{array} \tag{4.11}$$

Proof. The inequalities in *item 2* ensure that the vector $\boldsymbol{n}^\star = \left(\frac{1}{a_1^\star}, \ldots, \frac{1}{a_n^\star}\right)$ constructed via

$$a_i^\star := \max_{k \in \boldsymbol{K}} c_k^\star a_i^{(k)} \quad (i \in \boldsymbol{I}) \tag{4.12}$$

constitutes a linear function that achieves its maximum relative to $\Delta^{(k)}$ exactly on $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$, thus is a normal to Π and, clearly, the normal to $\boldsymbol{F}$. Because of *item 2* of Definition 4.1, the dimension of $\boldsymbol{F}$ is at most $n-1$. Because of *items 4* and *5* of Definition 4.4 the normal to $\boldsymbol{F}$ is uniquely defined, hence $\boldsymbol{F}$ has exactly dimension $n-1$.

On the other hand, if $\boldsymbol{F}$ is a maximal face, then the representation by (4.10) is unique and $\mathfrak{J}$ is the reference system of $\boldsymbol{F}$. Then $\mathfrak{J}$ is *a fortiory* a pre–adjustment system and the inequalities in *item 2* result from (1.4) and (1.5).

□

$\boxed{5}$ Adjacent Faces: The Normal Cone

The Neighborhood Theorem (Theorem 3.3. of [7]) describes the shape of the $(n-2)$–dimensional subface that is the intersection of two adjacent faces. Based on the duality theory we formulate a generalization. Thereafter we discuss some properties of the normal cone of the $(n-2)$–dimensional intersection of two adjacent maximal faces.

Theorem 5.1. *Let Π be a cephoid and let $\boldsymbol{F}, \widetilde{\boldsymbol{F}}$ be adjacent maximal faces (i.e., with $(n-2)$–dimensional intersection) with reference systems $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$. Then there exist indices $p, q \in \boldsymbol{K}$, $p \neq q$, and $i_0, i_1 \in \boldsymbol{I}$, $i_0 \neq i_1$, with $i_0 \in \boldsymbol{L}$, $i_1 \in \widetilde{\boldsymbol{L}}$, such that the following holds:*

$$\begin{aligned} \boldsymbol{J}^{(k)} &= \widetilde{\boldsymbol{J}}^{(k)} \quad (k \neq p, q) \\ \boldsymbol{J}^{(p)} &= \widetilde{\boldsymbol{J}}^{(p)} \cup \{i_0\} \\ \widetilde{\boldsymbol{J}}^{(q)} &= \boldsymbol{J}^{(q)} \cup \{i_1\} \end{aligned} \tag{5.1}$$

Proof. **1stSTEP** : As $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ are neighbors, the intersection is a face

$$\boldsymbol{F} \cap \widetilde{\boldsymbol{F}} = \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)} \cap \widetilde{\boldsymbol{J}}^{(k)}} \tag{5.2}$$

with dimension $n-2$, hence

$$\sum_{k \in \boldsymbol{K}} \left| \boldsymbol{J}^{(k)} \cap \widetilde{\boldsymbol{J}}^{(k)} \right| = n - 2 + K \; .$$

As the corresponding sums for the two faces yield $n-1+K$ we must necessarily find p, q such that

$$\begin{aligned} \boldsymbol{J}^{(p)} &= \left(\boldsymbol{J}^{(p)} \cap \widetilde{\boldsymbol{J}}^{(p)}\right) \cup \{i_0\} &= \widetilde{\boldsymbol{J}}^{(p)} \cup \{i_0\} \\ \widetilde{\boldsymbol{J}}^{(q)} &= \left(\boldsymbol{J}^{(q)} \cap \widetilde{\boldsymbol{J}}^{(q)}\right) \cup \{i_1\} &= \boldsymbol{J}^{(q)} \cup \{i_1\} \end{aligned} \tag{5.3}$$

say,

$$\begin{array}{cc} \boldsymbol{J}^{(p)} & \widetilde{\boldsymbol{J}}^{(q)} \\ 1\,2\,\ldots\,t \quad i_0 & s\,\ldots\,r \\ 1\,2\,\ldots\,t & s\,\ldots\,r \quad i_1 . \end{array} \tag{5.4}$$

2ndSTEP : Now $p = q$ is not possible as we would have $|\boldsymbol{J}^{(p)}| = |\widetilde{\boldsymbol{J}}^{(p)}|$. This would imply equal reference vectors for both faces, hence they would coincide. So we know $p \neq q$.

3rdSTEP : Now assume that $i_0 \notin \boldsymbol{L}$ and $i_1 \notin \widetilde{\boldsymbol{L}}$ is the case. Then we have $\mathrm{L} = \widetilde{\mathrm{L}}$. As the system L determines $\boldsymbol{F}$ uniquely, it would follow that $\boldsymbol{F} = \widetilde{\boldsymbol{F}}$ holds true. On the other hand, assume e.g. $i_0 \in \mathrm{L}, i_1 \notin \widetilde{\mathrm{L}}$. Then we have $\widetilde{\mathrm{L}} \subseteq \mathrm{L}$. Now the linear adjustment system of equations is again uniquely attached to L. It follows that all equations corresponding to $\widetilde{\mathrm{L}}$ appear in the system attached to L as well. But both system must have maximal rank, i.e., generate a solution space of dimension 1. Evidently, the two systems have the same solution space, in which case the normals coincide. Hence again we would find $\boldsymbol{F} = \widetilde{\boldsymbol{F}}$, which cannot happen. Hence $i_0 \in \boldsymbol{L},\ i_1 \in \widetilde{\boldsymbol{L}}$ is true indeed.

Finally we check $i_0 \neq i_1$. Consider the reference system of the dual faces, i.e.,

$$\bar{\boldsymbol{J}}^{(i)} = \{k \in \boldsymbol{K} | i \in \boldsymbol{J}^{(k)}\} .$$

Clearly we have

$$\begin{aligned} \bar{\boldsymbol{J}}^{(i)} &= \overline{\widetilde{\boldsymbol{J}}}^{(i)} \quad (i \neq i_0, i_1) \\ \bar{\boldsymbol{J}}^{(i_0)} &= \overline{\widetilde{\boldsymbol{J}}}^{(i_1)} \cup \{p\} \\ \overline{\widetilde{\boldsymbol{J}}}^{(i_1)} &= \overline{\boldsymbol{J}}^{(i_0)} \cup \{q\}. \end{aligned} \tag{5.5}$$

Therefore we may repeat the argument of the $2^{nd}STEP$: if $i_0 = \widetilde{\iota}_0$, then the reference vector of the dual faces $\overline{\boldsymbol{F}}$ and $\overline{\widetilde{\boldsymbol{F}}}$ would coincide, which is not possible.

□

Obviously formula (5.5) is similar (and in some sense "dual") to formula (5.1). For completeness it is useful to state the consequences.

Theorem 5.2. *Let Π be a cephoid and let $\boldsymbol{F}, \widetilde{\boldsymbol{F}}$ be adjacent maximal faces with reference systems $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$. Then the dual faces are adjacent as well. With the notation in Theorem 5.1, the reference system of the dual faces is given by* (5.5).

Corollary 5.3. *Let Π be a cephoid and let $\boldsymbol{F}, \widetilde{\boldsymbol{F}}$ be adjacent maximal faces (i.e., with $(n-2)$–dimensional intersection) with reference systems $\mathfrak{J}$ and $\widetilde{\mathfrak{J}}$. Let i_0, i_1 and p, q be given by Theorem 5.1. Let $\bar{\mathrm{L}} = \{(i, s) \in \boldsymbol{I} \times \boldsymbol{K},\ i \in \boldsymbol{K}_s,\ |\boldsymbol{K}_s| \geq 2\}$. Then the normal cone to the intersection*

$$\boldsymbol{F}^\star = \boldsymbol{F} \cap \widetilde{\boldsymbol{F}}$$

spans the two–dimensional subspace of $\mathbb{R}^n$ obtained by the projection of the solutions of

$$\bar{a}_s^{(i)} n_i = \mu_s \quad ((i, s) \in \bar{\mathrm{L}},\ \ (i, s) \neq (i_0, p)) \quad \textit{if} \ \ |\boldsymbol{J}^{(p)}| \geq 3, \tag{5.6}$$

or

$$\bar{a}_s^{(i)} n_i = \mu_s \quad ((i, s) \in \bar{\mathrm{L}}\ ,\ (i, s) \neq (i_0, p), (i_5, p)) \quad \textit{if} \ \ |\boldsymbol{J}^{(p)}| = \{i_0, i_5\}. \tag{5.7}$$

The proof is obvious in view of Remark 2.6. Note that with respect to the equations (2.24), for $|\boldsymbol{J}^{(p)}| = 2$ one has to cancel two equations and the variable μ_p, while otherwise one has just to cancel one equation.

Corollary 5.4. *If $|\boldsymbol{J}^{(p)}| = 2$, and $\boldsymbol{J}^{(p)} = \{i_0, i_9\}$, then either $i_9 \in \boldsymbol{L}$ or $i_9 = i_1$.*

For, i_9 has to appear in some $\widetilde{\boldsymbol{J}}^{(s)}$ with $|\widetilde{\boldsymbol{J}}^{(s)}| \geq 2$.

Let $\Pi^{(-i_0)}$ denote the intersection of Π with the subspace $\mathbb{R}^{\boldsymbol{I}\setminus\{i_0\}}$ of $\mathbb{R}^n$. Also, write $\partial\Pi^{(-i)} = \partial\Pi_{\,|\,\mathbb{R}^{\boldsymbol{I}\setminus\{i\}}}$. Now $\Pi^{(-i_0)}$ is not a maximal face in the outer surface of Π but – viewed in $\mathbb{R}^n$ – it can be seen as a maximal face of Π. Thus, the following corollary can be seen as the appropriate version of Corollary 5.3 for faces $\boldsymbol{F}$ that are adjacent to $\Pi^{(-i_0)}$.

Corollary 5.5. *Let $\boldsymbol{F}$ be a maximal face of a cephoid Π with reference system $\mathfrak{J}$ and normal $\boldsymbol{n}$. For some $p \in \boldsymbol{K}$, let $|\boldsymbol{J}^{(p)}| \geq 2$ and let $i_0 \in \boldsymbol{J}^{(p)}, i_0 \notin \boldsymbol{L}$. Define*

$$\boldsymbol{F}^\star \;:=\; \sum_{k\in\boldsymbol{K}-p} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(p)}_{\boldsymbol{J}^{(p)}\setminus\{i_0\}}. \tag{5.8}$$

Then $\boldsymbol{F}^\star = \boldsymbol{F}\cap\Pi^{(-i_0)} \subseteq \partial\Pi^{(-i_0)}$ is an $(n-2)$ dimensional subface of $\partial\Pi$ and the second extremal to the normal cone of $\boldsymbol{F}^\star$ is $\boldsymbol{n}^\star = \boldsymbol{n} - n_{i_0}\boldsymbol{e}^{i_0}$.

Proof. The dimension of $\boldsymbol{F}$ is obviously $(n-1)$ and vectors on $\boldsymbol{F}$ have zeros at coordinate i_0. $\boldsymbol{F}^\star \subseteq \boldsymbol{F}$ is also obvious. Thus we have to specify the normal cone. However, the normal cone of $\Pi^{(-i_0)}$ (viewed in $\mathbb{R}^n$) is spanned by $\boldsymbol{e}^{i_0}$ hence the one of $\boldsymbol{F}^\star$ is spanned by $\boldsymbol{n}$ and $\boldsymbol{e}^{i_0}$ or equivalently by $\boldsymbol{n}$ and $\boldsymbol{n}^\star$.

Note that $\boldsymbol{n}^\star$ obeys the equations of Corollary 5.3 suitably modified for $\boldsymbol{F}^\star$ if one is willing to see $\Pi^{(-i_0)}$ as a face adjacent to $\boldsymbol{F}$ and $\partial\Pi^{(-i_0)}$ as the intersection of both faces. □

Theorem 5.6. *Let $\boldsymbol{F}$ be a maximal face of a cephoid Π with reference system $\mathfrak{J}$. For some $p \in \boldsymbol{K}$, let $|\boldsymbol{J}^{(p)}| \geq 2$ and let $i_0 \in \boldsymbol{J}^{(p)} \cap \boldsymbol{L}$. Let*

$$\boldsymbol{F}^\star \;:=\; \sum_{k\in\boldsymbol{K}-p} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(p)}_{\boldsymbol{J}^{(p)}\setminus\{i_0\}}. \tag{5.9}$$

Then there is some maximal face $\widetilde{\boldsymbol{F}}$ of Π such that

$$\boldsymbol{F}^\star \;=\; \boldsymbol{F}\cap\widetilde{\boldsymbol{F}}\ .$$

Proof. The normal cone to $\boldsymbol{F}^\star$ is described by Corollary 5.3. Let $\widetilde{\boldsymbol{n}}$ be a further extremal of this cone. By the nondegeneracy assumption, this extremal can either have exactly one zero coordinate $\widetilde{n}_i$ (in which case $\boldsymbol{F}^\star$ is located in the corresponding $\partial\Pi^{(-i)}$), or else it is positive. In the latter case of positvity the theorem is verified.

Now let us assume that one of the components of $\widetilde{\boldsymbol{n}}$ is zero. In this case, whenever $s \neq p$ and $|\boldsymbol{J}^{(s)}| \geq 2$, then $\widetilde{\mu}_s > 0$, for otherwise the equation

$$\bar{a}^{(i)}_s \widetilde{n}_i = \widetilde{\mu}_s \quad (i \in \boldsymbol{J}^{(s)})\ , \tag{5.10}$$

would result in two zero coordinates of $\widetilde{\boldsymbol{n}}$ at least. If $|\boldsymbol{J}^{(p)}| \geq 3$, then the same argument holds true for p: we would have at least *two* equations of the type (5.10), namely for $i \in \widetilde{\boldsymbol{J}}^{(p)} = \{i \in \boldsymbol{J}^{(p)}, i \neq i_0\}$. Therefore, it remains to study the case suggested by equations (5.7), where i_5 is the second index in $\boldsymbol{J}^{(p)}$. Now, if for the first index we have $\widetilde{n}_{i_0} = 0$, then $\boldsymbol{F}^\star$ would be located in $\partial\Pi^{(-i_0)}$, contradicting the fact that $i_0 \in \boldsymbol{L}$ and nondegeneracy. If, on the other hand $\widetilde{n}_{i_5} = 0$ holds true, then it is seen at once that $\widetilde{\boldsymbol{n}}$ satisfies the equations of Corollary 5.3 with respect to

$$\boldsymbol{F}^{\star\star} \;:=\; \sum_{k\in\boldsymbol{K}-p} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(p)}_{\boldsymbol{J}^{(p)}\setminus\{i_5\}},$$

which again contradicts nondegeneracy. This implies that $\widetilde{\boldsymbol{n}}$ is indeed positive and hence the normal of an adjacent maximal face $\widetilde{\boldsymbol{F}}$. Since $\boldsymbol{J}^{(p)} \cap \boldsymbol{L} \neq \emptyset$ and $\widetilde{\boldsymbol{n}}$ belongs to the normal cone to $\boldsymbol{F}^\star$ this implies that $\boldsymbol{F}^\star = \boldsymbol{F} \cap \widetilde{\boldsymbol{F}}$, which completes the proof. □

Theorem 5.7. *Let $\boldsymbol{r}$ be a reference vector and let $\boldsymbol{F}$ be the unique corresponding face. Then, for every $k \in \boldsymbol{K}$ with $r_k \geq 2$, there exists uniquely some $\widetilde{k} \in \boldsymbol{K}$, $\widetilde{k} \neq k$, such that the face $\widetilde{\boldsymbol{F}}$ corresponding to $\widetilde{\boldsymbol{r}} := \boldsymbol{r} + \boldsymbol{e}^{\widetilde{k}} - \boldsymbol{e}^k$ is adjacent to $\boldsymbol{F}$.*

Proof. We know that $\boldsymbol{J}^{(k)} \cap \boldsymbol{L} \neq \emptyset$ for every $k \in \boldsymbol{K}$. Hence, whenever $|\boldsymbol{J}^{(k)}| \geq 2$, we can pick some $i_0 \in \boldsymbol{J}^{(k)}$ that satisfies the conditions of Theorem 5.6. Applying Theorem 5.6 we find at once that the reference vector of the adjacent face $\widetilde{\boldsymbol{F}}$ has the required property.

□

6 Generalized Tentacles

Within this section we describe the generalized version of the tentacle system (see Corollary 4.5 of [7]). The tentacle system exhibited in [7] refers to the de Gua simplexes involved in the construction of a cephoid. For each such de Gua simplex a translate of its outer face appears on the Pareto surface $\partial\Pi$. This translate is the center of a system of "arms" or "tentacles" connecting it with each boundary $\partial\Pi^{(-i)} = \partial\Pi_{\mid \mathbb{R}^{\boldsymbol{I}\setminus\{i\}}}$ via a system of cylinders. This structure gave rise to the name "Cephoid" as explained in the introductory remarks of it Section 1 of [7].

Now we shall show that *every* maximal face $\boldsymbol{F}$ generates a tentacle system. By this we mean a well defined system of maximal faces connected by the adjacency relation. Again we observe that each of the $(n-1)$ subfaces of $\boldsymbol{F}$ gives rise to a "tentacle", i.e., a sequence of faces connecting $\boldsymbol{F}$ to a well defined boundary $\partial\Pi^{(-i)}$.

We fix some maximal face $\boldsymbol{F}$ identified by its reference vector $\boldsymbol{r}$. Also, we fix some element of $\boldsymbol{K} := \{1, \ldots, K\}$, which, w.o.l.g. is the final element $K \in \boldsymbol{K}$ and assume that $r_K \geq 2$. Accordingly we fix the corresponding member of the family $\boldsymbol{a}^\bullet$, say $\boldsymbol{a}^{(K)}$. Of course any other element of $\boldsymbol{K}$ could play the role of K.

Definition 6.1. Let $\boldsymbol{a}^\bullet$ be a nondegenerate family of positive vectors, let $\boldsymbol{F}$ be a maximal face of Π and $\boldsymbol{r}$ the corresponding reference vector. Let $r_K \geq 2$.

1. The reference vectors
$$\boldsymbol{r} - \boldsymbol{e}^K + \boldsymbol{e}^k \quad (k \in \boldsymbol{K}) \tag{6.1}$$
are called the ***tentacle–codes*** or ***T–codes*** of $\boldsymbol{F}$.

2. The corresponding maximal faces $\boldsymbol{F}^{-K,+k}$ are called the (tentacle–faces or) ***T–faces*** of $\boldsymbol{F}$.

3. The $n-2$ dimensional face of $\partial\Pi^{(-i)} := \partial\Pi_{\mid \mathbb{R}^{\boldsymbol{I}\setminus\{i\}}}$ corresponding to the reference vector $\boldsymbol{r} - \boldsymbol{e}^K$ is denoted by $\mathcal{P}^i(\boldsymbol{F})$.

4. An $n-2$–dimensional subface $\boldsymbol{F}^0$ of a $\boldsymbol{T}$–face with reference vector $\boldsymbol{r} - \boldsymbol{e}^K$ is called a ***cylinder basis***.

Our aim is to show that the family of $\boldsymbol{T}$–faces constitute a simple connected graph without circles, that is a ***tree***.The root is $\boldsymbol{F}$. The terminal nodes of this tree are located exactly on the $(n-2)$–dimensional boundary faces of $\partial\Pi$.

Lemma 6.2. *There are K different $\boldsymbol{T}$–faces including $\boldsymbol{F} = \boldsymbol{F}^{-K,+K}$. For any $i \in \boldsymbol{I}$, $\partial\Pi^{(-i)}$ is the outer surface of the cephoid generated by $\boldsymbol{a}^{\bullet} \big|_{\boldsymbol{I}-i}$ and there exists a $\boldsymbol{T}$–face $\widehat{\boldsymbol{F}}$ of Π such that $\widehat{\boldsymbol{F}} \cap \mathbb{R}^{(\boldsymbol{I}\setminus\{i\})} \subseteq \partial\Pi^{(-i)}$ is a cylinderbasis.*

Proof. The first statement is obvious by Theorem 3.2. The second one follows from the Coincidence Theorem (Theorem 3.1. of [7]), *item 2 (d)* and by Corollary 3.3 as $\mathcal{P}^i(\boldsymbol{F})$ is the desired cylinderbasis on the boundary $\partial\Pi^{(-i)}$.

□

Lemma 6.3. *Let $\boldsymbol{F}^{-K,+l}$ be a T–face. Then the number of $(n-2)$–dimensional subfaces of $\boldsymbol{F}^{-K,+l}$ that are cylinder bases is at least 2. More precisely, the number of such subfaces is*

$$\begin{array}{rcll} r_K & \geq & 2 \ , & \textit{for } l = K, \textit{ i.e., for } \boldsymbol{F} = \boldsymbol{F}^{-K,+K} \\ r_l + 1 & \geq & 2 \ , & \textit{for } l \neq K \end{array} \tag{6.2}$$

Proof. Consider the case $l \neq K$. Let $\left\{\widehat{\boldsymbol{J}}^{(k)}\right\}_{k\in\boldsymbol{K}}$ denote the reference system of $\boldsymbol{F}^{-K,+l}$. Then $|\widehat{\boldsymbol{J}}^{(l)}| = r_l + 1 \geq 2$ holds true. For every $i_0 \in \widehat{\boldsymbol{J}}^{(l)}$ the system

$$\left\{\widehat{\boldsymbol{J}}^{(k)}\right\}_{k\in\boldsymbol{K}-l}, \quad \widehat{\boldsymbol{J}}^{(l)} - i_0 \tag{6.3}$$

constitutes a reference system defining an $(n-2)$–dimensional subface of $\boldsymbol{F}^{-K,+l}$ which obviously is a cylinder basis.

□

Lemma 6.4. *Let $\boldsymbol{F}^{-K,+l}$ be a $\boldsymbol{T}$–face and let $\widehat{\boldsymbol{F}}^0$ be an $(n-2)$–dimensional subface that is a cylinder basis. If $\iota_0 \in \boldsymbol{I}$ is* not *an adjustment index in the reference system of $\boldsymbol{F}^{-K,+l}$ and ι_0 does not appear in the reference system of $\widehat{\boldsymbol{F}}^0$ (cf. formula* (6.3)*), then $\widehat{\boldsymbol{F}}^0 \subseteq \partial\Pi^{(-i_0)}$, i.e., $\widehat{\boldsymbol{F}}^0 = \mathcal{P}^{i_0}(\boldsymbol{F})$ is the image of $\boldsymbol{F}$ in $\partial\Pi^{(-i_0)}$ under the bijective mapping established in Corollary 3.3.*

Proof. If an index i_0 does not appear in the reference system of an $(n-2)$–dimensional subface, then there is no summand of any $\Pi^{(k)}$ contributing to the i_0–coordinate, hence this coordinate vanishes for all elements of $\widehat{\boldsymbol{F}}$. □

Lemma 6.5. *Let $\bar{\boldsymbol{F}} = \boldsymbol{F}^{-K,+k}$ and $\widehat{\boldsymbol{F}} = \boldsymbol{F}^{-K,+l}$ be different $\boldsymbol{T}$–faces. Let $\bar{\boldsymbol{J}}^{(K)}$, $\widehat{\boldsymbol{J}}^{(K)}$ denote the reference system corresponding to $\Pi^{(K)}$. If $|\bar{\boldsymbol{J}}^{(K)} \setminus \widehat{\boldsymbol{J}}^{(K)}| \geq 2$ or $|\widehat{\boldsymbol{J}}^{(K)} \setminus \bar{\boldsymbol{J}}^{(K)}| \geq 2$ holds true, the $\bar{\boldsymbol{F}}$ and $\widehat{\boldsymbol{F}}$ are not neighbors.*

Proof. Obvious by the Neighborhood Theorem (Theorem 3.3 of [7]).

□

Definition 6.6. The ***T–graph*** of $\boldsymbol{F}$ (with respect to K) is the graph $\mathsf{T} = (\mathsf{V}, \mathsf{E})$ with nodes $\mathsf{V} = \{\boldsymbol{T}\text{–faces}\}$ and edges $\mathsf{E} = \{$cylinder bases common to two $\boldsymbol{T}$–faces$\}$.

Note that two nodes are connected by an edge if the cylinder basis is a joint $(n-2)$–dimensional subface of both faces. A $\boldsymbol{T}$–face is called ***terminal*** if, for some $i \in \boldsymbol{I}$, the intersection $\boldsymbol{F} \cap \partial\Pi^{(-i)}$ is a cylinder basis. $\boldsymbol{F} \cap \partial\Pi^{(-i)}$ is called the $i-th$ ***boundary cylinderbasis***.

Theorem 6.7. *The $\boldsymbol{T}$–graph T of $\boldsymbol{F}$ (w.r.t. K) is a tree. For every $i_0 \in \boldsymbol{I}$ there is a unique path from $\boldsymbol{F}$ to a terminal node which contains the boundary cylinder basis $\mathcal{P}^{i_0}(\boldsymbol{F})$.*

Proof. **1^{st}STEP** :

Let

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

with $r_k = |\boldsymbol{J}^{(k)}|$ $(k \in \boldsymbol{K})$ and $r_K \geq 2$. First of all we consider the case that $i_0 \in \boldsymbol{J}^{(K)}$ holds true. Then the $(n-1)$–dimensional subface

$$\overset{\circ}{\boldsymbol{F}}{}^{i_0} \;=\; \sum_{k=1}^{K-1} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(K)}_{\boldsymbol{J}^{(K)}-i_0}$$

is a cylinder basis. This basis is a boundary cylinder basis if $i_0 \notin \boldsymbol{L}$. If so, we are done.

2^{nd}STEP :

If $\overset{\circ}{\boldsymbol{F}}{}^{i_0}$ is not a boundary cylinder basis, then $i_0 \in \boldsymbol{L}$. According to Theorem 5.6 and Theorem 5.7, there is an adjacent face, say $\widetilde{\boldsymbol{F}} \;=\; \boldsymbol{F}^{-K,+\kappa}$. The reference system at position K consists of the index set $\boldsymbol{J}^{(K)} \setminus \{i_0\}$ and at position κ of, say, $\boldsymbol{J}^{(\kappa)} \cup \{i_1\}$. In view of the Neighborhood Theorem, we have $i_1 \in \boldsymbol{L}$.

3^{rd}STEP :

If it so happens that $i_0 \in \boldsymbol{J}^{(\kappa)}$, then we proceed as in the first step. We remove i_0 from $\boldsymbol{J}^{(\kappa)}$ an add a suitable index to some $\boldsymbol{J}^{(\gamma)}$. We have again reached a $\boldsymbol{T}$–face $\widehat{\boldsymbol{F}}$ with the number of appearances of i_0 reduced by 1.

4^{th}STEP :

Suppose that $i_0 \in \boldsymbol{J}^{(\kappa)}$ does not hold. Then we focus on $i_1 \in \boldsymbol{J}^{(\kappa)} \cap \boldsymbol{L}$ as specified in the $3^{rd}STEP$. We can remove i_1 (and possibly further indices) from $\boldsymbol{J}^{(\kappa)}$ and add another index to some $\boldsymbol{J}^{(\lambda)}$ and so on. If there are more indices feasible besides i_1, then two or more paths will branch off at the node reflected by $\widetilde{\boldsymbol{F}}$.

The path can never return to $\boldsymbol{F}$ because any other path leaving $\boldsymbol{F}$ is characterized by a missing index $i_9 \neq i_0$ instead of i_0. No face with missing index i_9 is a neighbor to a face with missing index i_0 by Lemma 6.5. By a similar argument, the path cannot return to another face met during the construction. Of course, when our path splits into several branches, then we can trace all these branches.

As there are only finitely many $\boldsymbol{T}$–vectors, we must eventually reach some face $\widetilde{\widehat{\boldsymbol{F}}}$ with a reference vector $\widetilde{\widehat{\boldsymbol{r}}}$ that is a $\boldsymbol{T}$–code and yields $\widetilde{\widehat{\boldsymbol{r}}}_\nu = \boldsymbol{r}_\nu + 1$ at some coordinate ν with $i_0 \in \boldsymbol{J}^\nu$. Then we can again reduce the number of appearances of i_0 by one.

5^{th}STEP :

Obviously we can proceed in this way, reducing the appearances of i_0 step by step until $i_0 \notin \boldsymbol{L}$ is the case. Then we have found a face with cylinder basis $\mathcal{P}^{i_0}$. The path constructed in the $\boldsymbol{T}$–graph connects $\boldsymbol{F}$ and $\mathcal{P}^{i_0}$.

6^{th}STEP :

On the other hand, we may consider some $\mathcal{P}^i(\boldsymbol{F})$ which corresponds to a unique face of $\Pi_{\mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}}}$. This face can be uniquely extended to a face $\overset{\circ}{\boldsymbol{F}}{}^i$ of Π (the set $\boldsymbol{L}$ is the same for both and characterizes both faces). Starting the procedure explained above, we can connect $\mathcal{P}^i(\boldsymbol{F})$ with every $\mathcal{P}^{i\prime}(\boldsymbol{F})$ for $i\prime \in \boldsymbol{I}, i\prime \neq i$. Thus the $\boldsymbol{T}$–graph has boundary nodes at each $\partial\Pi^{(-i)}$, all of them being connected without loops and circles. $\square$

Example 6.8. The cephoid "FourFour" is a sum of four de Gua simplexes in four dimensions. It is given by the matrix

$$\boldsymbol{A} \;=\; \begin{pmatrix} 701 & 502 & 303 & 104 \\ 205 & 116 & 1007 & 128 \\ 139 & 110 & 611 & 512 \\ 67 & 230 & 444 & 777 \end{pmatrix} \tag{6.4}$$

Figure 5 shows the canonical representation within the simplex $4\Delta^{\boldsymbol{e}}$.

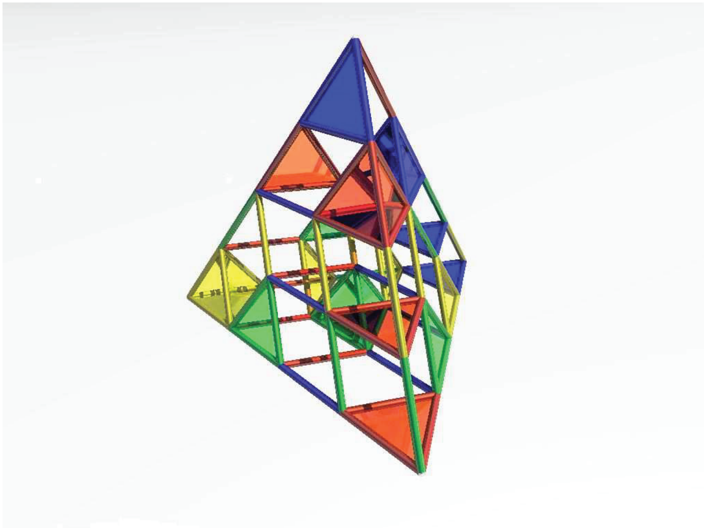

Figure 5: The canonical representation of FourFour

In the following we describe maximal faces by their reference systems. Consider the block without yellow edges

$$\boldsymbol{B}^{y} : \begin{array}{cccc} \mathit{blue} & \mathit{red} & \mathit{green} & \mathit{yellow} \\ 12 & 13 & 24 & 2 \end{array}, \tag{6.5}$$

the reference vector of which is $(2,2,2,1)$. We choose the cylinder bases suggested by the reference vector $(2,1,2,1)$. The block $\boldsymbol{B}^{y}$ has the square $\mathcal{P}^{3}$ as the cylinder basis at $\partial\Pi^{(-3)}$ (the left front side of the tetrahedron). The adjacent face is the cylinder that consists of a green triangle and a blue line segment; it is given by

$$\boldsymbol{Z}^{gr,b} : \begin{array}{cccc} \mathit{blue} & \mathit{red} & \mathit{green} & \mathit{yellow} \\ 12 & 3 & 234 & 2 \end{array}. \tag{6.6}$$

Note that the intersection $\boldsymbol{B}^{y} \cap \boldsymbol{Z}^{gr,b}$ has the correct reference code $(2,1,2,1)$.

At $\boldsymbol{Z}^{gr,b}$ the path has two branches, the boundary cylinder basis $\mathcal{P}^{4}$ at the lower subsimplex is part of the cylinder. If we follow the second path, we reach a block without red. This one is difficult to recognize, it consists of three line segments of blue, green, and yellow color and is described by

$$\boldsymbol{B}^{r} : \begin{array}{cccc} \mathit{blue} & \mathit{red} & \mathit{green} & \mathit{yellow} \\ 12 & 3 & 34 & 24 \end{array}. \tag{6.7}$$

The final face is the cylinder consisting of a blue triangle and a green line segment. It is described by

$$\boldsymbol{Z}^{b,gr} : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 123 & 3 & 34 & 4 \end{array} . \tag{6.8}$$

This cylinder has subfaces $\mathcal{P}^1$ (at the right front side) and $\mathcal{P}^2$ (at the rear side of the tetrahedron), thus we have found all $\mathcal{P}^i,\ i \in \boldsymbol{K}$.

Figure 6 shows the "tentacle" described by the above sequence. The four maximal faces have been isolated from Figure 5. The basis in each face consists of a square with blue and green parallel sides.

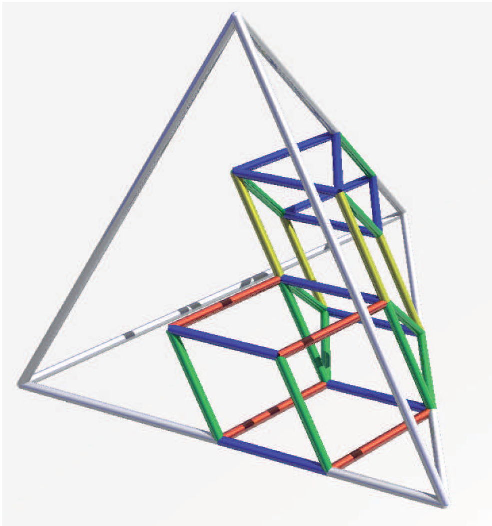

Figure 6: The Tentacle of Example 6.8

References

[1] J.P. de Gua de Malves, The n-dimensional Pythagorean theorem, in *Histoire de l'Academie des Sciences pour l'ann'ee 1783.*

[2] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Mathematics, vol. 168, Springer Verlag, New York, Berlin, Heidelberg, 1996.

[3] M. Maschler and M.A. Perles, The present status of the superadditive solution, in *Essays in Game Theory and Mathematical Economics*, Bibliographisches Institut, Mannheim, 1981.

[4] L. McKenzie, Specialization and efficiency in world production, *Review of Economic Studies* 21 (1954) 165–180.

[5] D. Pallaschke and J. Rosenmüller, Computing The Minkowski sum of prisms, Journal of Global Optimization, 35 (2006) 321–341.

[6] D. Pallaschke and J. Rosenmüller, A superadditive solution for cephoidal bargaining problems, *IJGT* 35 (2007) 569–590.

[7] D. Pallaschke and J. Rosenmüller, Cephoids: Minkowski sums of prisms, *Optimization* 59 (2010) 515–540.

[8] D. Pallaschke and R. Urbański, Pairs of compact convex sets–Fractional arithmetic with convex sets, in *Mathematics and its Applications*, vol.548, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[9] M.A. Perles Non–Existence of superadditive solutions for 3–person games, *IJGT* 11 (1982) 151–161.

[10] M.A. Perles and M. Maschler, The superadditive solution for the Nash bargaining game, *IJGT* 10 (1981) 163–193.

[11] J. Rosenmüller, Game Theory: Stochastics, information, strategies and Cooperation, in *Theory and Decision Library*, C, vol. 25, Kluwer Academic Publishers Boston, Dordrecht, London, 2000.

[12] J. Rosenmüller, *The Theory of Games and Markets*, North Holland Publishing Company, Amsterdam, New York, Oxford, 1981.

[13] R. Schneider, Convex bodies: The Brunn–Minkowski Theory, in *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, UK, vol. 44, 1993.

[14] G.M. Ziegler, Lectures on polytopes, in *Graduate Texts in Mathematics*, Springer Verlag, Heidelberg, Berlin, New York, vol. 152, 1995.

Manuscript received 30 March 2010
revised 30 November 2010
accepted for publication 2 December 2010

Diethard Pallaschke
Institute of Statistics and Mathematical Economics
University of Karlsruhe
D–76128 Karlsruhe
Germany
E-mail address: diethard.pallaschke@kit.edu

Joachim Rosenmüller
Institute of Mathematical Economics
University of Bielefeld
D–33615 Bielefeld
Germany
E-mail address: jr@wiwi.uni-bielefeld.de