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STRONG CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS WITH NONLINEAR OPERATORS IN HILBERT SPACES

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Abstract: In this paper, we introduce an iterative sequence for finding a common element of the set of fixed points of a nonspreading mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality problem for a monotone and Lipschitz-continuous mapping in a Hilbert space. We show that the sequence converges strongly to a common element of the above three sets.

Key words: *Hilbert space, equilibrium problem, fixed point, nonspreading mapping, hybrid method, monotone mapping*

Mathematics Subject Classification: *47H05, 47H09, 47H20*

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1 Introduction

Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of solution (1.1) is denoted by $EP(f)$. A mapping A of C into H is called monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in C$. The variational inequality problem is to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. A mapping A of C into H is called α-inverse-strongly monotone if there exists a positive real number α such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. It is obvious that any α-inverse-strongly monotone mapping A is monotone and Lipschitz continuous; see, for example, [18]. A mapping S of C into itself is called nonexpansive if $\|Su - Sv\| \leq \|u - v\|$ for all $u, v \in C$. A mapping S of C into itself is called nonspreading [6, 7] if

$$2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2$$

for all $u, v \in C$. We denote by $F(S)$ the set of fixed points of S. Recently, in the case when S is a nonexpansive mapping, Nadezhkina and Takahashi [8] introduced an iterative process for finding the common element of the set $F(S)$ and the set $VI(C, A)$ for a monotone and

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Lipschitz-continuous mapping by using the extragradient method introduced in Korpelevich [5]. On the other hand, Tada and Takahashi [13, 14] and Takahashi and Takahashi [15] obtained weak and strong convergence theorems for finding a common element of the set $EP(f)$ and the set $F(S)$ in a Hilbert space. Very recently, Shinzato and Takahashi [12] established a strong convergence theorem for finding a common element of the set $EP(f)$, the set $VI(C, A)$ for an inverse-strongly monotone mapping and the set $F(S)$ of a nonexpansive mapping in a Hilbert space by using the shrinking projection method introduced in Takahashi, Takeuchi and Kubota [17]. We know also a strong convergence theorem [2] for finding a common element of the set $EP(f)$ and the set of fixed points of a finite family of nonexpansive mappings in a Hilbert space.

In this paper, motivated by Shinzato and Takahashi [12] and Nadezhkina and Takahashi [8], we prove a strong convergence theorem for finding a common element of the set $EP(f)$, the set $VI(C, A)$ for a monotone, Lipschitz-continuous mapping and the set $F(S)$ of a nonspreading mapping in a Hilbert space by using the shrinking projection method and the extragradient method.

2 Preliminaries

In this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y\rangle \geq 0$,

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.1}$$

for all $x \in H$ and $y \in C$. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, this implies

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u)$$

for all $\lambda > 0$. It is also known that H satisfies the Opial condition [10]. That is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. We also know that H has the Kadec-Klee property, that is, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. In fact, from

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2,$$

we get that a Hilbert space has the Kadec-Klee property. An operator $A : H \to 2^H$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if} \quad v \in C, \\ \emptyset, & \text{if} \quad v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [11]. For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemmas.

Lemma 2.1. *The following equality holds in a Hilbert space H: For all $u, v \in H$,*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle.$$

Lemma 2.2 ([1])**.** *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 2.3 ([3])**.** *Assume that $f : C \times C \to R$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}T\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

We know also the following result.

Lemma 2.4. *Let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself with $F(S) \neq \emptyset$. Then, $F(S)$ is closed and convex.*

Proof. A mapping $S : C \to C$ is nonspreading, i.e.,

$$2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2$$

for all $u, v \in C$. If $v = Sv$, then we have $2\|Su - v\|^2 \leq \|Su - v\|^2 + \|v - u\|^2$ and hence $\|Su - v\|^2 \leq \|v - u\|^2$. This implies that S is quasi-nonexpansive. So, we have from [4] that $F(S)$ is closed and convex. □

$\boxed{3}$ Main Results

In this section, we prove two strong convergence theorems for monotone mappings and nonspreading mappings with equilibrium problems in a Hilbert space. First, we prove a strong convergence theorem by the shrinking projection method [17].

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let A be a monotone and k-Lipschitz continuous mapping of C into H and let S be a nonspreading mapping of C into itself such that $F(S) \cap VI(C,A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$, $C_1 = C$ and*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ w_n = \alpha_n S x_n + (1-\alpha_n) P_C(u_n - \lambda_n A y_n), \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \ n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \lambda_n \leq b < \dfrac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ and $0 < r \leq r_n$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)\cap EP(f)}x$, where $P_{F(S)\cap VI(C,A)\cap EP(f)}$ is the metric projection of H onto $F(S) \cap VI(C,A) \cap EP(f)$.

Proof. Put $v_n = P_C(u_n - \lambda_n A y_n)$ for every $n \in \mathbb{N}$ and take

$$x^* \in F(S) \cap VI(C,A) \cap EP(f).$$

Then, we have $x^* = P_C(x^* - \lambda_n A x^*) = T_{r_n} x^*$. We first show by induction that

$$F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$$

for all $n \in \mathbb{N}$. It is obvious that $F(S) \cap VI(C,A) \cap EP(f) \subseteq C_1 = C$. Suppose that

$$F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$$

for some $n \in \mathbb{N}$ and take $x^* \in F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$. Then, we have from (2.1) and Lemma 2.1 that

$$\begin{aligned}
\|v_n - x^*\|^2 &\leq \|u_n - \lambda_n A y_n - x^*\|^2 - \|u_n - \lambda_n A y_n - v_n\|^2 \\
&= \|u_n - x^*\|^2 - \|\lambda_n A y_n\|^2 - 2\langle u_n - \lambda_n A y_n - x^*, \lambda_n A y_n\rangle \\
&\quad - \|u_n - v_n\|^2 + \|\lambda_n A y_n\|^2 + 2\langle u_n - \lambda_n A y_n - v_n, \lambda_n A y_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\langle x^* - v_n, \lambda_n A y_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle A y_n, x^* - v_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 \\
&\quad + 2\lambda_n(\langle A y_n - A x^*, x^* - y_n\rangle + \langle A x^*, x^* - y_n\rangle + \langle A y_n, y_n - v_n\rangle) \\
&\leq \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\
&\quad + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\
&= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + 2\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle.
\end{aligned}$$

From $y_n = P_C(u_n - \lambda_n A u_n)$, we have

$$\begin{aligned}
&\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle \\
&\quad = \langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle + \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\
&\quad \le \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\
&\quad \le \lambda_n k \|u_n - y_n\| \|v_n - y_n\|.
\end{aligned}$$

Since $a^2 + b^2 \ge 2ab$ for all $a, b \in \mathbb{R}$, we have that

$$\begin{aligned}
\|v_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + 2\lambda_n k \|u_n - y_n\| \|v_n - y_n\| \\
&\le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\
&\quad + \lambda_n^2 k^2 \|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\
&= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2.
\end{aligned}$$

So, we have from $u_n = T_{r_n} x_n$, $x^* = T_{r_n} x^*$ and $\lambda_n^2 k^2 < 1$ that

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|\alpha_n(Sx_n - x^*) + (1-\alpha_n)(v_n - x^*)\|^2 \\
&\le \alpha_n \|Sx_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\le \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.1) \\
&= \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|T_{r_n} x_n - T_{r_n} x^*\|^2 \\
&\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.2) \\
&\le \|x_n - x^*\|^2.
\end{aligned}$$

Then $x^* \in C_{n+1}$. This implies that $F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \in \mathbb{N}$. For $z \in C_n$, we know from Nakajo and Takahashi [9] that $\|w_n - z\| \le \|x_n - z\|$ is equivalent to

$$\|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - z\rangle \le 0.$$

So, C_{n+1} is closed and convex. Then for any $n \in \mathbb{N}$, C_n is closed and convex. From $x_n = P_{C_n} x$, we have $\langle x - x_n, x_n - z\rangle \ge 0$ for all $z \in C_n$. Since $x^* \in F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$, we also have $\langle x - x_n, x_n - x^*\rangle \ge 0$. So, we have

$$\begin{aligned}
0 \le \langle x - x_n, x_n - x^*\rangle &= \langle x - x_n, x_n - x + x - x^*\rangle \\
&= -\langle x - x_n, x - x_n\rangle + \langle x - x_n, x - x^*\rangle \\
&\le -\|x - x_n\|^2 + \|x - x_n\| \|x - x^*\|.
\end{aligned}$$

This implies that $\|x - x_n\| \le \|x - x^*\|$. From $x_n = P_{C_n} x$ and $x_{n+1} = P_{C_{n+1}} x \in C_{n+1} \subset C_n$, we also have

$$\langle x - x_n, x_n - x_{n+1}\rangle \ge 0. \qquad (3.3)$$

From (3.3), we have that for $n \in \mathbb{N}$,

$$\begin{aligned}0 \le \langle x - x_n, x_n - x_{n+1}\rangle &= \langle x - x_n, x_n - x + x - x_{n+1}\rangle \\ &= -\langle x - x_n, x - x_n\rangle + \langle x - x_n, x - x_{n+1}\rangle \\ &\le -\|x - x_n\|^2 + \|x - x_n\|\|x - x_{n+1}\|\end{aligned}$$

and hence $\|x - x_n\| \le \|x - x_{n+1}\|$. Thus $\{\|x_n - x\|\}$ is bounded and monotone and increasing. So, $\lim_{n\to\infty} \|x_n - x\|$ exists. Next, we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. In fact, from (3.3) we have

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x + x - x_{n+1}\|^2 \\ &= \|x_n - x\|^2 + 2\langle x_n - x, x - x_{n+1}\rangle + \|x - x_{n+1}\|^2 \\ &= -\|x_n - x\|^2 + 2\langle x_n - x, x_n - x + x - x_{n+1}\rangle + \|x - x_{n+1}\|^2 \\ &= -\|x_n - x\|^2 + 2\langle x_n - x, x_n - x_{n+1}\rangle + \|x - x_{n+1}\|^2 \\ &\le -\|x_n - x\|^2 + \|x - x_{n+1}\|^2.\end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x\|$ exists, we have that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\|w_n - x_n\| \le \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\|.$$

Then $\lim_{n\to\infty} \|w_n - x_n\| = 0$. Further, from

$$\begin{aligned}0 &\le \|x_n - x^*\|^2 - \|w_n - x^*\|^2 \\ &= (\|x_n - x^*\| + \|w_n - x^*\|)(\|x_n - x^*\| - \|w_n - x^*\|) \\ &\le (\|x_n - x^*\| + \|w_n - x^*\|)\|x_n - w_n\| \to 0,\end{aligned}$$

we obtain

$$\|x_n - x^*\|^2 - \|w_n - x^*\|^2 \to 0.$$

From (3.2), we also obtain

$$\|u_n - y_n\|^2 \le \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|w_n - x^*\|^2)$$

and

$$\begin{aligned}\|y_n - v_n\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(u_n - \lambda_n A y_n)\|^2 \\ &\le \|u_n - \lambda_n A u_n - (u_n - \lambda_n A y_n)\|^2 \\ &= \|\lambda_n A y_n - \lambda_n A u_n\|^2 \\ &\le \lambda_n^2 k^2 \|y_n - u_n\|^2.\end{aligned}$$

Then we have $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and $\lim_{n\to\infty} \|y_n - v_n\| = 0$. Consider

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \\ &\le \langle T_{r_n} x_n - T_{r_n} x^*, x_n - x^*\rangle \\ &= -\langle u_n - x^*, x^* - x_n\rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2).\end{aligned}$$

Then
$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2 \le \|x_n - x^*\|^2. \tag{3.4}$$
From this equality and (3.1), we have
$$\begin{aligned}\|w_n - x^*\|^2 &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2\\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n)\|x_n - u_n\|^2.\end{aligned}$$
So, we have
$$\|x_n - u_n\|^2 \le \frac{1}{1-\alpha_n}(\|x_n - x^*\|^2 - \|w_n - x^*\|^2),$$
which implies that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Since $\alpha_n(Sx_n - x_n) = w_n - (1-\alpha_n)v_n - \alpha_n x_n$, we have
$$\begin{aligned}\alpha_n\|Sx_n - x_n\| &\le \|w_n - v_n\| + \|v_n - x_n\|\\ &\le \|w_n - x_n\| + 2\|v_n - x_n\|\\ &\le \|w_n - x_n\| + 2(\|v_n - y_n\| + \|y_n - u_n\| + \|u_n - x_n\|).\end{aligned}$$
So we obtain $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ for some $p \in C$. From $\|x_n - Sx_n\| \to 0$, we have $Sx_{n_i} \rightharpoonup p$. Next, let us show $p \in F(S)$. Since S is nonspreading, we have
$$\begin{aligned}2\|Sx_{n_i} - Sp\|^2 &\le \|Sx_{n_i} - p\|^2 + \|x_{n_i} - Sp\|^2\\ &= \|Sx_{n_i} - p\|^2 + \|x_{n_i} - Sx_{n_i}\|^2 + 2\langle x_{n_i} - Sx_{n_i}, Sx_{n_i} - Sp\rangle\\ &\quad + \|Sx_{n_i} - Sp\|^2.\end{aligned}$$
Then
$$\|Sx_{n_i} - Sp\|^2 \le \|Sx_{n_i} - p\|^2 + \|x_{n_i} - Sx_{n_i}\|^2 + 2\langle x_{n_i} - Sx_{n_i}, Sx_{n_i} - Sp\rangle.$$
Suppose $Sp \ne p$. From Opial's theorem [10] and $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$, we obtain
$$\begin{aligned}\liminf_{i\to\infty}\|Sx_{n_i} - p\|^2 &< \liminf_{i\to\infty}\|Sx_{n_i} - Sp\|^2\\ &\le \liminf_{i\to\infty}(\|Sx_{n_i} - p\|^2 + \|x_{n_i} - Sx_{n_i}\|^2 + 2\langle x_{n_i} - Sx_{n_i}, Sx_{n_i} - Sp\rangle)\\ &= \liminf_{i\to\infty}\|Sx_{n_i} - p\|^2.\end{aligned}$$
This is a contradiction. Hence $Sp = p$. Next, let us show $p \in VI(C, A)$. Let
$$Tv = \begin{cases} Av + N_C v, & \text{if} \quad v \in C,\\ \emptyset, & \text{if} \quad v \notin C.\end{cases}$$
Then, from [11] T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence, $w - Av \in N_C v$. So, we have $\langle v - u, w - Av\rangle \ge 0$ for all $u \in C$. On the other hand, from $v_n = P_C(u_n - \lambda_n Ay_n)$ and $v \in C$, we have $\langle u_n - \lambda_n Ay_n - v_n, v_n - v\rangle \ge 0$ and hence $\langle v - v_n, \dfrac{v_n - u_n}{\lambda_n} + Ay_n\rangle \ge 0$.

Therefore, from $w - Av \in N_C v$ and $v_n \in C$, we have

$$\begin{aligned}
\langle v - v_{n_i}, w\rangle &\geq \langle v - v_{n_i}, Av\rangle \\
&\geq \langle v - v_{n_i}, Av\rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i}\rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle \\
&\quad - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\
&\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle.
\end{aligned}$$

Since $\lim_{n\to\infty} \|v_n - x_n\| = 0$, $\lim_{n\to\infty} \|v_n - u_n\| = 0$, $\lim_{n\to\infty} \|y_n - v_n\| = 0$ and A is Lipschitz continuous, we obtain $\langle v - p, w\rangle \geq 0$. Since T is maximal monotone, we have $p \in T^{-1}0$ and hence $p \in VI(C, A)$. Let us show $p \in EP(f)$. Since $f(u_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq 0$ for all $y \in C$. From (A2), we also have

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq f(y, u_{n_i})$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}).$$

From $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we get $u_{n_i} \rightharpoonup p$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, it follows by (A4) that $0 \geq f(y, p)$ for all $y \in C$. For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)p$. From $y, p \in C$, we have $y_t \in C$ and hence $f(y_t, p) \leq 0$. So, from (A1) and (A4) we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, p) \leq tf(y_t, y)$$

and hence $0 \leq f(y_t, y)$. From (A3), we have $0 \leq f(p, y)$ for all $y \in C$ and hence $p \in EP(f)$. Thus $p \in F(S) \cap VI(C, A) \cap EP(f)$. Let

$$p^* = P_{F(S)\cap VI(C,A)\cap EP(f)}x \subseteq C_{n+1}.$$

From $x_{n+1} = P_{C_{n+1}}x$, we have $\|x - x_{n+1}\| \leq \|x - p^*\|$. Hence, we have

$$\|x - p^*\| \leq \|x - p\| \leq \liminf_{i\to\infty} \|x - x_{n_i}\| \leq \limsup_{i\to\infty} \|x - x_{n_i}\| \leq \|x - p^*\|.$$

So, we obtain $\lim_{i\to\infty} \|x - x_{n_i}\| = \|x - p\| = \|x - p^*\|$ and $p = p^*$. We can conclude that $x_n \to p^* = P_{F(S)\cap VI(C,A)\cap EP(f)}x$. This completes the proof. □

Next, we prove another strong convergence theorem which is different from Theorem 3.1.

Theorem 3.2. *Let C be a closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let A be a monotone and k-Lipschitz continuous mapping of C into H and let S be a nonspreading mapping of C into itself such that $F(S) \cap VI(C, A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$, $C_1 = C$ and*

$$\begin{cases}
f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \lambda_n Au_n), \\
w_n = \alpha_n x_n + (1 - \alpha_n)SP_C(u_n - \lambda_n Ay_n), \\
C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\
x_{n+1} = P_{C_{n+1}}x, \ n \in \mathbb{N},
\end{cases}$$

where $0 < a \le \lambda_n \le b < \dfrac{1}{k}$, $0 < c \le \alpha_n \le d < 1$ *and* $0 < r \le r_n$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C,A)\cap EP(f)}x$.

Proof. Put $v_n = P_C(u_n - \lambda_n Ay_n)$ for every $n \in \mathbb{N}$ and take

$$x^* \in F(S) \cap VI(C,A) \cap EP(f).$$

Then, we have $x^* = P_C(x^* - \lambda_n Ax^*) = T_{r_n}x^*$. We first show by induction that

$$F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$$

for all $n \in \mathbb{N}$. It is obvious that $F(S) \cap VI(C,A) \cap EP(f) \subseteq C_1 = C$. Suppose that

$$F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$$

for some $n \in \mathbb{N}$. Let $x^* \in F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$. Following the proof of Theorem 3.1, we have

$$\|v_n - x^*\|^2 \le \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2.$$

Hence

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(Sv_n - x^*)\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|Sv_n - x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.5) \\
&\le \|x_n - x^*\|^2.
\end{aligned}$$

Then $x^* \in C_{n+1}$. This implies that $F(S) \cap VI(C,A) \cap EP(f) \subseteq C_n$ for all $n \in \mathbb{N}$. Next, we can follow the proof of Theorem 3.1 to show

1. C_n is closed and convex for all $n \in \mathbb{N}$.
2. $\{\|x_n - x\|\}$ is bounded, monotone and increasing and $\lim_{n\to\infty} \|x_n - x\|$ exists.
3. $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.
4. $\lim_{n\to\infty} \|w_n - x_n\| = 0$.
5. $\lim_{n\to\infty} \|u_n - y_n\| = 0$, $\lim_{n\to\infty} \|y_n - v_n\| = 0$ and $\lim_{n\to\infty} \|x_n - u_n\| = 0$.

Since $(1-\alpha_n)(Sv_n - v_n) = w_n - \alpha_n x_n - (1-\alpha_n)v_n$, we have

$$\begin{aligned}
(1-\alpha_n)\|Sv_n - v_n\| &\le \|w_n - v_n\| + \|v_n - x_n\| \\
&\le \|w_n - x_n\| + 2\|v_n - x_n\| \\
&\le \|w_n - x_n\| + 2(\|v_n - y_n\| + \|y_n - u_n\| + \|u_n - x_n\|).
\end{aligned}$$

Therefore, we also obtain $\lim_{n\to\infty} \|Sv_n - v_n\| = 0$. As $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} \rightharpoonup p$ for some $p \in C$. From $\lim_{n\to\infty} \|Sv_n - v_n\| = 0$, we obtain $Sv_{n_i} \rightharpoonup p$. Next, let us show $p \in F(S)$.

$$\begin{aligned}2\|Sv_{n_i} - Sp\|^2 &\le \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sp\|^2 \\ &= \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sv_{n_i}\|^2 + 2\langle v_{n_i} - Sv_{n_i}, Sv_{n_i} - Sp\rangle \\ &\quad + \|Sv_{n_i} - Sp\|^2.\end{aligned}$$

Then we have

$$\|Sv_{n_i} - Sp\|^2 \le \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sv_{n_i}\|^2 + 2\langle v_{n_i} - Sv_{n_i}, Sv_{n_i} - Sp\rangle.$$

Suppose $Sp \neq p$, From Opial's theorem [10] and $\lim_{n\to\infty} \|Sv_n - v_n\| = 0$, we obtain

$$\begin{aligned}\liminf_{i\to\infty} \|Sv_{n_i} - p\|^2 &< \liminf_{i\to\infty} \|Sv_{n_i} - Sp\|^2 \\ &\le \liminf_{i\to\infty} (\|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sv_{n_i}\|^2 + 2\langle v_{n_i} - Sv_{n_i}, Sv_{n_i} - Sp\rangle) \\ &= \liminf_{i\to\infty} \|Sv_{n_i} - p\|^2.\end{aligned}$$

This is a contradiction. Hence $Sp = p$, i.e., $p \in F(S)$. Following the proof of Theorem 3.1, we get $p \in VI(C, A)$, $p \in EP(f)$ and $x_n \to p^* = P_{F(S)\cap VI(C,A)\cap EP(f)}x$. □

4 Applications

Using Theorems 3.1 and 3.2, we prove four theorems in a real Hilbert space.

Theorem 4.1. *Let C be a closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let S be a nonspreading mapping of C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$, $C_1 = C$ and*

$$\begin{cases} w_n = \alpha_n Sx_n + (1-\alpha_n)T_{r_n}x_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \ n \in \mathbb{N}, \end{cases}$$

where $0 < a \le \lambda_n \le b < \frac{1}{k}$, $0 < c \le \alpha_n \le d < 1$ and $0 < r \le r_n$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)}x$.

Proof. Putting $A = 0$ in Theorem 3.1, we obtain the desired result. □

Theorem 4.2. *Let C be a closed convex subset of a Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let S be a nonspreading mapping of C into itself such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$, $C_1 = C$ and*

$$\begin{cases} w_n = \alpha_n x_n + (1-\alpha_n)ST_{r_n}x_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \ n \in \mathbb{N}, \end{cases}$$

where $0 < a \le \lambda_n \le b < \frac{1}{k}$, $0 < c \le \alpha_n \le d < 1$ and $0 < r \le r_n$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)}x$.

Proof. Putting $A = 0$ in Theorem 3.2, we obtain the desired result. □

Theorem 4.3. *Let C be a closed convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H and let S be a nonspreading mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C,\ C_1 = C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ w_n = \alpha_n S x_n + (1 - \alpha_n) P_C(x_n - \lambda_n A y_n), \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x,\ n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \lambda_n \leq b < \dfrac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ *and* $0 < r \leq r_n$. *Then $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(C,A)} x$.*

Proof. Putting $f(x, y) = 0$ for all $x, y \in C$ in Theorem 3.1, we obtain the desired result. □

Theorem 4.4. *Let C be a closed convex subset of a Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H and let S be a nonspreading mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C,\ C_1 = C$ and*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ w_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_{n+1} = \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x,\ n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \lambda_n \leq b < \dfrac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ *and* $0 < r \leq r_n$. *Then $\{x_n\}$ converges strongly to $P_{F(S) \cap VI(C,A)} x$.*

Proof. Putting $f(x, y) = 0$ for all $x, y \in C$ in Theorem 3.2, we obtain the desired result. □

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