Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 8, No. 1, pp. 121–134, January 2012

JOINT PRICING AND PRODUCTION PLANNING OF MULTI-PERIOD MULTI-PRODUCT SYSTEMS WITH UNCERTAINTY IN DEMAND

Louis Caccetta and Elham Mardaneh

Abstract: In this paper, we present a multiperiod model for production planning in a make-to-stock manufacturing system with constant pricing under uncertainty. We consider a multiproduct capacitated setting and introduce a demand-based model where the demand is a function of the price. There is an assumption that the production setup costs are negligible. A key part of the model is that the uncertain price / demand function is chosen from a discrete set of scenarios. As a result of this, the problem becomes a non linear programming problem with the nonlinearities only in the objective function. We develop a robust optimization model for this problem that considers the optimality and feasibility of all scenarios. The robust solution is obtained by solving a series of nonlinear programming problems. We illustrate our methodology with detailed numerical examples.

Key words: *production planning, pricing, coordination of marketing and operations decisions, demand uncertainty*

Mathematics Subject Classification: *90B30, 90B50, 90C30, 91B06*

---

1 Introduction

During the past decade, there has been increasing interest in the integration of pricing and production / inventory decisions. Previously, the price of each product has been determined exogenously. In more recent times however, researchers have focused on the coordination of endogenous pricing and production planning / inventory control problems, by which decision makers estimate the production / inventory amount and the price of each product subject to a set of constraints that satisfy the induced market demand and yield maximum profit over a specified time horizon. A comprehensive survey of the coordinated pricing and inventory control problems can be found in Z.J.Shen, D.Simchi-Levi and S.D.Wu (2004), in which around 160 articles and books are reviewed and classified according to a number of characteristics of the problem or assumptions made by the researchers, like deterministic or uncertain demand.

S.M.Gilbert (2000) deals with the problem of jointly determining prices and production schedules for a set of items that are produced on the same production equipment. Under the assumptions that the production setup costs are negligible and that demand is seasonal but price dependent, S.M.Gilbert (2000) exploits the special structure of the problem to develop a solution procedure based on network optimization and nonlinear programming methodology. However, in his model backorders are not allowed.

ISSN 1348-9151

L.Caccetta and E.Mardaneh (2010) extend the work done by S.M.Gilbert (2000) to the case that backorders are allowed in the model. By designing a search tree structure, the authors find the optimal price and production plan of multiple products over a multi-period horizon for a manufacturing system with deterministic demand/price relationship.

S.Kachani and G.Perakis (2002) study a pricing and inventory problem with multiple products sharing production resources, and they apply fluid methodology to make their pricing, production, and inventory decisions. In their case, they consider the sojourn or delay time of a product in inventory, where the delay is a deterministic function of initial inventory and price (including competitor's prices). For the continuous time formulation, they establish when the general model has a solution, and for the discrete case they provide an algorithm for computing pricing policies. Fluid dynamic models are used also by E.Adida and G.Perakis (2007) to study a make-to-stock manufacturing system by deterministic demand. In that problem, they introduce and study an algorithm that computes the optimal production and pricing policy as a function of the time on a finite time horizon, and discuss some insights. Their results illustrate the role of capacity and the effects of the dynamic nature of demand in the model. S.Biller et al (2005) analyze a pricing and production problem where (in extensions), multiple products may share limited production capacity. When the demand for products is independent and revenue curves are concave, the authors show that an application of the greedy algorithm provides the optimal pricing and production decisions.

Although numerous models have been developed to solve deterministic joint pricing and production planning problems of multiple products, little work has been done on multi-product systems over a multi-period horizon under uncertainty.

G.Gallego and G.Van Ryzin (1997) extend their previous work (1994), which considers a firm producing a single product, to focus on a multi-market problem, with multiple products sharing common resources. They model demand as a stochastic point process function of time and the prices of all products: the vector of demand for n products, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, is determined by time and the vector of prices, $p = (p_1, p_2, \ldots, p_n)$. Revenue is assumed to be concave, and the null price condition (price is set to infinity when inventory is zero) as applied. G.Gallego and G.Van Ryzin (1997) formulate a deterministic problem, which they show gives a bound on the expected revenue. This problem also motivates the creation of a make-to-stock (MTS) and a make-to-order (MTO) heuristic. The MTS heuristic requires that all products be preassembled, and the price path is determined from the deterministic solution. The MTO heuristic also uses the prices from the deterministic solution but produces and sells products as they are requested. An order is rejected if the components are not available to assemble it. The authors show that each of these heuristics is asymptotically optimal as the expected sales increases.

L.M.Chan et al (2006) consider a general stochastic demand function over multiple periods, where production capacity is limited but set-up costs are not incurred. Excess demand is lost, and sales are discretionary, i.e., inventory may be set aside to satisfy future demand even at the expense of lost sales in the current period. The authors develop a dynamic programming model that solves the problem to optimality for discrete possibilities of fixed prices and compares the results to a deterministic approach.

E.Adida and G.Perakis (2006) use robust optimization and fluid dynamic models to study a make-to-stock manufacturing system with uncertain demand. They show that the robust formulation is of the same order of complexity as the nominal (deterministic) problem and demonstrate how to adapt the nominal solution algorithm to the robust problem.

Generally in the case of uncertainty, it has been assumed that the demand has some known portion based on price (e.g., linear demand curve), with an additional stochastic element. However, in this paper we deal with problems in which the demand/price relationship is uncertain with some known probabilities. In other words, we consider a discrete set of scenarios for the relationship between the chosen price and induced market demand.

The application that motivated this research is manufacturing pricing, where the products are non-perishable assets and can be stored to fulfill the future demands. We assume that the firm is not flexible to change the price list frequently and usually has long-term contracts with Original Equipment Manufacturers (OEMs). Additionally, in some companies, the price announcement to market is done by publishing the price lists which cannot be adjusted easily. Hence the price change will bring a considerable cost to them. In general, choosing a constant price over a finite horizon facilitates the maintenance of a stable set of loyal customers.

The purpose of this paper is to develop a robust optimization (RO) model to determine the optimal production planning and constant pricing of a manufacturing system with multiple products over a multiple period horizon to maximize the total profit which consists of sales revenue, production and inventory holding costs under demand/price uncertainty. To our knowledge, this is the first time that the robust optimization approach has been used in the case of discrete time production planning and pricing.

Our paper is organized as follow: Section 2 briefly reviews the robust optimization approach and its formulation in the case of a Linear Programming problem. Section 3 presents the deterministic model for the problem of joint pricing and production planning and develops a robust optimization model for the uncertain case. Section 4 discusses solution methods. Section 5 illustrates our model and its solution with two numerical examples.

2 Robust Optimization Approach

Our work is based on the robust optimization tools developed by J.M.Mulvey, R.J.Vanderbei and S.A.Zenios (1995) which incorporates a goal programming structure with a set of scenarios involving stochastic inputs. Their optimization model has the following structure:

$$\text{Minimize } c^T x + d^T y \tag{2.1}$$

Subject to

$$Ax = b \tag{2.2}$$

$$Bx + Cy = e \tag{2.3}$$

$$x \in R^{n_1},\ y \in R^{n_2} \text{ and } x, y \geq 0\ . \tag{2.4}$$

Equation(2.2) denotes the structural constraints whose coefficients are fixed and free of noise. Equation (2.3) denotes the control constraints. The coefficients of this constraint set are subject to noise. Inequality (2.4) expresses the non-negativity restrictions.

To define the robust optimization problem, a discrete set of scenarios $\Omega = \{1, 2, 3, \ldots, S\}$ is introduced. With each scenario $s \in \Omega$ associate the set $\{d_s, B_s, C_s, e_s\}$ of realizations for the coefficients of the control constraints, and the probability of the scenario P_s, $(\sum_{s=1}^{S} P_s = 1)$. The optimal solution of the mathematical program (2.1)-(2.4) will be robust with respect to optimality if it remains "close" to optimal for any realization of the scenario $s \in \Omega$. It is then termed *"solution robust"*. The solution is also robust with respect to feasibility if it remains "almost" feasible for any realization of s. It is then termed *"model robust"*.

Because it is unlikely that any solution to problem (2.1)-(2.4) will remain both feasible and optimal for all scenario indices $S \in \Omega$, a model is needed that will allow us to measure the tradeoff between the solution and model robustness. The robust optimization model proposed by J.M.Mulvey, R.J.Vanderbei and S.A.Zenios (1995) formalises a way to measure this tradeoff.

Let $\{y_1, y_2, \ldots, y_s\}$ be a set of control variables for each scenario $s \in \Omega$ and $\{z_1, z_2, \ldots, z_s\}$ a set of error vectors that measure the infeasibility allowed in the control constraints under scenario s. Consider now the following formulation of the robust optimization model.

$$\text{Minimize } \sigma(x, y_1, \ldots, y_s) + \omega\rho(z_1, z_2, \ldots, z_s) \tag{2.5}$$

Subject to

$$Ax = b \tag{2.6}$$

$$B_s x + C_s y_s + z_s = e_s \text{ for all } s \in \Omega \tag{2.7}$$

$$x \geq 0,\ y_s \geq 0,\ \text{for all } s \in \Omega\ . \tag{2.8}$$

With multiple scenarios, the objective function $\varepsilon = c^T x + d^T y$ becomes a random variable taking the value $\varepsilon_s = c^T x + d_s^T y_s$, with probability P_s. Hence, there is no longer a single choice for an aggregate objective. We could use the mean value

$$\sigma(.) = \sum_{s\in\Omega} P_s \varepsilon_s \tag{2.9}$$

which is the function used in stochastic linear programming formulations. In a worst-case analysis the model minimizes the maximum value, and the objective function is defined by

$$\sigma(.) = \max_{s\in\Omega} \varepsilon_s \tag{2.10}$$

Both of these choices are special cases of Robust Optimization (RO).

The second term in the objective function $\rho(z_1, \ldots, z_s)$ is a feasibility penalty function. It is used to penalize violations of the control constraints under some of the scenarios. The above model takes a multi-criteria objective form. The first term measures optimality robustness, whereas the penalty term is a measure of model robustness. The goal programming weight ω is used to derive a spectrum of values that tradeoff solution for model robustness.

The specific choice of the penalty function is problem dependent, and also has implications for the accompanying solution algorithm. There are two alternative penalty functions considered

$\rho(z_1, \ldots, z_s) = \sum_{s\in\Omega} P_s z_s^T z_s$. This quadratic penalty function is applicable to equality constrained problems.

$\rho(z_1, \ldots, z_s) = \sum_{s\in\Omega} P_s \max\{0, z_s\}$. This exact penalty function applies to inequality control constraints when only positive violations are of interest.

3 Joint Pricing and Production Planning of Multiple Products Over a Multi-period Horizon

3.1 Notation and Model Variables

Decision variables

p_j : the price of product $j, j = 1, 2, \ldots, n$

P : the price vector

D_j : the induced demand intensity for product j, which is a function of its price

x_{jt} : the amount of product j produced in period t; $j = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, T$

y_{jt} : the amount of product j held in inventory at the end of period t

X : the $n \times T$ production matrix

Y : the $n \times T$ inventory matrix

Parameters and constants

c_j : the production cost of one unit of item $j, j = 1, 2, \ldots, n$

h_j : the holding cost of one unit of item j in inventory for one period

K_t : the total amount of available capacity in period t

β_{jt} : he seasonality parameter of item j in period t

Functions

$D_j(p)$: the relationship between price and induced demand of j, for example: $D_j(p) = a_j - b_j \cdot p_j$

3.2 Single Scenario Optimization Model(Deterministic Case)

The problem of jointly determining the price and production plan of multiple products over a multi-period horizon can be modeled as follows.

3.2.1 Objective

The main components of the objective function in the problem of joint pricing and production planning include sales revenue, production cost and holding inventory cost. So, to consider the multi-product problem stated earlier as an aim of this paper, the objective function is given as follows

$$\max_{P,X,Y \geq 0} \Big\{ \pi(P, X, Y) = \underbrace{\sum_{j=1}^{n} \sum_{t=1}^{T} p_j \cdot D_j(p) \cdot \beta_{jt}}_{\text{Revenue}} - \underbrace{\sum_{j=1}^{n} \sum_{t=1}^{T} c_j \cdot x_{jt}}_{\text{Production cost}} - \underbrace{\sum_{j=1}^{n} \sum_{t=1}^{T} h_j \cdot y_{jt}}_{\text{Inventory cost}} \Big\}. \quad (3.1)$$

The first term in (3.1) is the sales revenue as a product of the chosen price and the induced demand brought by the chosen price. The second and third terms are production and inventory holding costs respectively.

(3.1) can be written as

$$\min_{P,X,Y\geq 0}\Big\{-\pi(P,X,Y)=-\sum_{j=1}^{n}\sum_{t=1}^{T}p_j\cdot D_j(p)\cdot\beta_{jt}+\sum_{j=1}^{n}\sum_{t=1}^{T}c_j\cdot x_{jt}+\sum_{j=1}^{n}\sum_{t=1}^{T}h_j\cdot y_{jt}\Big\}. \quad (3.2)$$

3.2.2 Constraints

$$\sum_{j=1}^{n}\sum_{t=1}^{T}\beta_{jt}\cdot D_j(p)\leq\sum_{t=1}^{T}K_t \quad (3.3)$$

$$x_{jt}+y_{jt-1}-y_{jt}=D_j(p)\cdot\beta_{jt},\ \text{for } j=1,\dots,n \text{ and } t=1,\dots,T \quad (3.4)$$

$$\sum_{j=1}^{n}x_{jt}\leq K_t,\ \text{for } t=1,\dots,T \quad (3.5)$$

$$x_{jt},y_{jt},p_j\geq 0,\ \text{for } j=1,\dots,n \text{ and } t=1,\dots,T \quad (3.6)$$

Constraint (3.3) ensures that only demand intensity vectors which result in a feasible solution have been considered. Constraint (3.4) is a set of flow balance equations that ensure that all of the induced demand is satisfied. Constraint (3.5) ensures that there is an adequate amount of capacity in period t to produce all n items based on the plan. Inequality (3.6) expresses the non-negativity restrictions. We have a mathematical programming problem with a nonlinear objective and linear constraints.

3.3 Robust Optimization Model for Multiple Scenarios

We assume that in the case of uncertainty, the price / induced demand function, $D_j(p)$, is not known in advance. Instead there is a discrete set of scenarios by which the relationship between price and induced demand is defined with a known probability. To find out the effect of uncertainty on the joint pricing and production planning, we need to redefine the objective function and some constraints, which are subject to the uncertainty, of the above single scenario model.

First, we redefine the objective function in (3.2) under each scenario, which consists of revenue, production cost and inventory holding cost.

$$RV^s(\text{revenue})=\sum_{j=1}^{n}\sum_{t=1}^{T}p_j\cdot D_j^s(p)\cdot\beta_{jt} \quad (3.7)$$

$$PC\ (\text{production cost})=\sum_{j=1}^{n}\sum_{t=1}^{T}c_j\cdot x_{jt},\ \text{and} \quad (3.8)$$

$$IC^s(\text{inventory cost})=\sum_{j=1}^{n}\sum_{t=1}^{T}h_j\cdot y_{jt}^s \quad (3.9)$$

In the above $D_j^s(p)$ is the price/induced demand function of product j and y_{jt}^s is the inventory of product j at the end of period t under scenario $s\in\Omega=\{1,2,3,\dots,S\}$ which happens with a probability of $Pr(s)$. The objective function for the joint pricing and production planning problem with noisy data comes to a random variable with the probability of each scenario, which is formulated as follows

$$\min_{P,X,Y^s\geq 0} \sum_{j=1}^{n}\sum_{t=1}^{T} c_j \cdot x_{jt} + \sum_{j=1}^{n}\sum_{t=1}^{T} h_j \cdot y_{jt}^s - \sum_{j=1}^{n}\sum_{t=1}^{T} p_j \cdot D_j^s(p) \cdot \beta_{jt}. \tag{3.10}$$

There may be an idea that the expected objective function can be considered to cover all possible scenarios. But the optimal solution of the expected objective function is not likely to be optimal for all scenarios.

$$\begin{aligned}\min_{P,X,Y^s\geq 0} \quad & \sum_{s\in\Omega} Pr(s)\cdot\left(\sum_{j=1}^{n}\sum_{t=1}^{T} c_j \cdot x_{jt} + \sum_{j=1}^{n}\sum_{t=1}^{T} h_j \cdot y_{jt}^s - \sum_{j=1}^{n}\sum_{t=1}^{T} p_j \cdot D_j^s(p)\cdot\beta_{jt}\right)\\ = \quad & \sum_{s\in\Omega} Pr(s)\cdot(PC + IC^s - RV^s)\end{aligned} \tag{3.11}$$

Hence, the idea of adding a weight of the variance of the expected solution to the objective function causes choosing such solutions which are close to optimal for all scenarios. However, if we put a zero weight for the variance of the expected solution in the objective function, then we would have the general form of stochastic programming problem. On the other hand, in the case of uncertainty the matter of feasibility should be taken into account as well as the optimality. To find a solution which remains almost feasible under all scenarios, we penalize the violation of feasibility of each constraint subject to uncertainty.

It is worthwhile to mention that mostly the robust optimization approach has been used in linear programming problems as developed by J.M.Mulvey, R.J.Vanderbei and S.A.Zenios (1995), but here, we develop a nonlinear programming model with the use of the RO approach. Now, the robust optimization model for our targeted problem can be formally expressed as

$$\begin{aligned}\min_{P,X,Y^s\geq 0} \quad & \sum_{s\in\Omega} Pr(s)\cdot(PC + IC^s - RV^s)\\ + \quad & \lambda\sum_{s\in\Omega} Pr(s)\cdot\left[(PC + IC^s - RV^s) - \sum_{s^0\in\Omega} Pr(s^0)\cdot(PC + IC^{s^0} - RV^{s^0})\right]^2\\ + \quad & \omega\sum_{s\in\Omega} Pr(s)\left[z^s + \sum_{j=1}^{n}\sum_{t=1}^{T} z_{jt}^s\right]\end{aligned} \tag{3.12}$$

Subject to

$$\sum_{j=1}^{n} x_{jt} \leq K_t, \text{ for } t = 1,\dots,T \tag{3.13}$$

$$\sum_{j=1}^{n}\sum_{t=1}^{T} \beta_{jt}\cdot D_j^s(p) - z^s \leq \sum_{t=1}^{T} K_t, \text{ for all } s\in\Omega \tag{3.14}$$

$$x_{jt} + y_{jt-1}^s - y_{jt}^s + z_{jt}^s = D_j^s(p)\cdot\beta_{jt},\ j = 1,\dots,n;\ t = 1,\dots,T.\ \ s\in\Omega \tag{3.15}$$

$$x_{jt},\ y_{jt}^s,\ p_j,\ z^s,\ z_{jt}^s \geq 0.\ j = 1,\dots,n;\ t = 1,\dots,T.\ \ s\in\Omega \tag{3.16}$$

Note that z_{jt}^s is the under-fulfillment of demand of product j in period t under scenario s. Also z^s is the total under-fulfillment of demand of all products over the total planning horizon under scenario s.

The first and second terms in the objective function (3.12) are mean and variance of the objective function respectively, which measure the solution robustness. The third term in (3.12) is set to measure the model's robustness with respect to infeasibility associated with control constraints (3.14) and (3.15) under scenario s.

4 Solution Methods

Given the capacity limitations and uncertainty in the problems parameters, the firm must decide upon production quantities, inventory levels for each item as well as a constant price at which it commits to sell the products over the total planning horizon.

As already been noted, corresponding to each specific λ and ω, which define the optimality and feasibility preferences, the problem (3.12)-(3.16) comes to an optimization problem with nonlinear objective function and linear constraints. There is a vast literature on such problems. The book by D.G.Luenberger (2003) presents different methods designed to solve a Nonlinear Programming problem which has n variables and m constraints. Methods devised for solving this problem that work in spaces of dimension $n-m$, n, m, or $n+m$ are Primal Methods, Penalty and Barrier Methods, Dual and Cutting Plane Methods and Lagrange Methods, respectively.

Primal methods work on the original problem directly by searching through the feasible region, which has dimension $n-m$, for the optimal solution. Each point in the process is feasible and the value of the objective function constantly improves. Penalty and Barrier methods approximate constrained optimization problems by unconstrained problems with adding a term to the objective function. In the case of penalty methods the term prescribes a high cost for violation of the constraints, and in the case of barrier methods the term favors points interior to the feasible region over those near the boundary. Dual methods are based on the viewpoint that it is the Lagrange multipliers which are the fundamental unknowns associated with a constrained problem. Once these multipliers are known, the determination of the solution point is simple. Dual methods do not attack the original constrained problem directly but instead attack an alternate problem, the dual problem, whose unknowns are the Lagrange multipliers of the first problem. Cutting plane algorithms develop a series of ever-improving approximating linear programs, whose solutions converge to the solution of the original problem. Lagrange methods directly solve the Lagrange first-order necessary conditions. The set of necessary conditions is a system of $n+m$ equations in the $n+m$ unknowns.

In the computation part of this research we have utilized the existing optimization packages with the capability of dealing with the nonlinear objective functions. We use the library subroutine 'NLPSolve' of Maple in which there is a method option to select the proper one for solving the specific problem, such as Quadratic Interpolation, Branch-and Bound, Modified Newton, Nonlinear Simplex, Preconditioned Conjugate Gradient and Sequential Quadratic Programming (SQP). According to the described criteria for each method, we have selected SQP to optimize each NLP problem.

Our strategy to find the optimal pricing and production planning under uncertainty is to solve a sequence of the above nonlinear programming problems for a range of λ and ω. We first choose a fixed small value (e.g. 0.01) of λ and a considerably vast range of ω (e.g. 0-300, with suitable increments) and find the optimal solution of each specific problem within the selected range. We draw three different plots of all specific problems as follows, the expected profit, the solutions standard deviation (as a measurement of the optimality) and the demand under fulfillment (as a measurement of the feasibility). Next, if the expected profit doesn't level out within the chosen range of λ, we extend the range to observe a leveled out expected profit. By increasing the value of λ, we solve the sequence of nonlinear problems again within the same modified range of ω. Comparing the resulted plots of the new λ and the previous one, and bearing in the mind the decision makers preferences, the next value of λ may be selected to continue the computation. As would be expected, the robust zone of any specific case of joint pricing and production planning is highly dependent on the resulting plots to compare the expected profit with the optimality and feasibility measures.

In the next section we illustrate this RO approach and our strategy to find the robust solution for a detailed small example. We further demonstrate the ability to handle a large, more realistic case.

5 Numerical Example

In order to make the model more clear, two different cases are presented in this section. First, we start modeling and solving a smaller example with $n = 2$ products, $T = 6$ periods and $\Omega = \{1, 2\}$. The second example consists of $n = 10$ products, $T = 12$ periods and $\Omega = \{1, 2, 3\}$.

5.1 Two Products, Six Periods and Two Scenarios

The parameters for the example are as shown in Table 1.

Table 1: Parameters of the example

Product	Scenario	$D_j^s(p)$	h_j	c_j	β_{j1}	β_{j2}	β_{j3}	β_{j4}	β_{j5}	β_{j6}
j=1	$Pr(s=1)=0.8$	$150-5p_1$	6	16	0.6	0.5	0.2	3	1.5	0.2
	$Pr(s=1)=0.2$	$150-4p_1$								
j=2	$Pr(s=1)=0.8$	$150-5p_2$	2.5	11	1	1	1	1	1	1
	$Pr(s=1)=0.2$	$150-4p_2$								

We assume a fixed production capacity, $K_t = I40$, for all periods in the planning horizon.

Here we bring the problem formulation based on the proposed RO approach as follow

$RV^1 = 6p_1(150 - 5p_1) + 6p_2(150 - 5p_2) = 900p_1 + 900p_2 - 30p_1^2 - 30p_2^2$
$RV^2 = 6p_1(150 - 4p_1) + 6p_2(150 - 4p_2) = 900p_1 + 900p_2 - 24p_1^2 - 24p_2^2$
$PC = 16(x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16}) + 11(x_{21} + x_{22} + x_{23} + x_{24} + x_{25} + x_{26})$
$IC^1 = 6(y_{11}^1 + y_{12}^1 + y_{13}^1 + y_{14}^1 + y_{15}^1) + 2.5(y_{21}^1 + y_{22}^1 + y_{23}^1 + y_{24}^1 + y_{25}^1)$
$IC^2 = 6(y_{11}^2 + y_{12}^2 + y_{13}^2 + y_{14}^2 + y_{15}^2) + 2.5(y_{21}^2 + y_{22}^2 + y_{23}^2 + y_{24}^2 + y_{25}^2)$

$$\begin{aligned}\min \quad & PC + 0.8(IC^1 - RV^1) + 0.2(IC^2 - RV^2) \\ & +0.8\lambda[0.2(IC^1 - RV^1) - 0.2(IC^2 - RV^2)]^2 \\ & +0.2\lambda[0.8(IC^{12} - RV^2) - 0.8(IC^1 - RV^1)]^2 \\ & +0.8\omega[z^1 + z_{11}^1 + z_{12}^1 + z_{13}^1 + z_{14}^1 + z_{15}^1 + z_{16}^1 + z_{21}^1 + z_{22}^1 + z_{23}^1 + z_{24}^1 + z_{25}^1 + z_{26}^1] \\ & +0.2\omega[z^2 + z_{11}^2 + z_{12}^2 + z_{13}^2 + z_{14}^2 + z_{15}^2 + z_{16}^2 + z_{21}^2 + z_{22}^2 + z_{23}^2 + z_{24}^2 + z_{25}^2 + z_{26}^2]\end{aligned}$$

Subject to

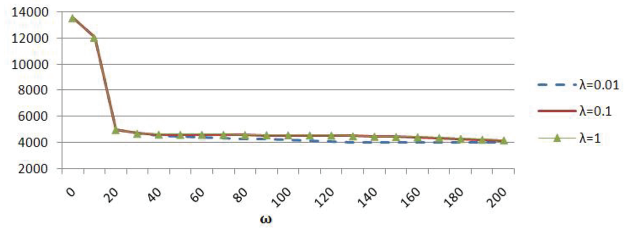
$$x_{1t} + x_{2t} \leq 140 \quad \text{for } t = 1, \ldots, 6$$
$$6(150 - 5p_1) + 6(150 - 5p_2) - z^1 \leq 840$$
$$6(150 - 4p_1) + 6(150 - 4p_2) - z^2 \leq 840$$
$$x_{jt} + y^s_{jt-1} - y^s_{jt} + z^s_{jt} = D^s_j(p) \cdot \beta_{jt}, \text{ for } j = 1, 2;\ t = 1, \ldots, 6 \text{ and all } s \in \{1, 2\}$$
$$x_{jt}, y^s_{jt}, p_j, z^s, z^s_{jt} \geq 0, \text{ for } j = 1, 2;\ t = 1, \ldots, 6 \text{ and all } s \in \{1, 2\}.$$

The Nonlinear Programming problem has been solved for a range of λ and ω. The optimized value of the decision variables can be obtained for each specific problem with a particular λ and ω.

Based on the decision maker's preferences the violation of the optimality and feasibility can be penalized by choosing the appropriate value for λ and ω respectively. The higher value of results in a solution with less standard deviation from the expected one and a big ω brings more feasibility to all control constraints under each scenario. Hence, a single decision cannot be made instantly for this type of problem with uncertainty; instead there should be a reasonable discussion revealing the importance of the optimality and feasibility for each case.

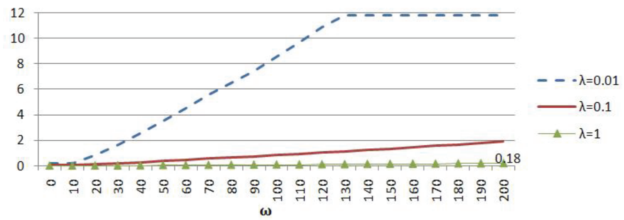
Now we bring the result of the specific problem for a chosen range of $\omega(0 - 200)$ and some fixed value of $\lambda(0.01, 0.1, 1)$.

Figure 1: The expected total profit



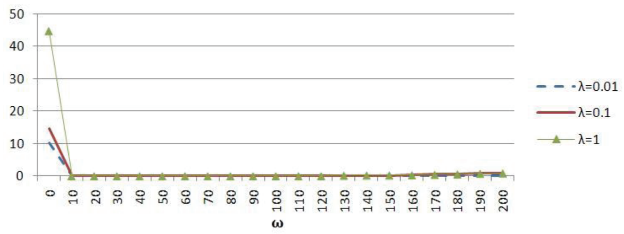
As can be seen, the expected profit drops dramatically when the value of ω increases by 20. Since then, by increasing the value of ω, the change of expected profit is not considerable. When the value of λ rises from 0.1 to 1, we can say that the total profit remains very similar. As a result, choosing the preferred λ depends on the other aspects of the decision making, which are optimality and infeasibility measurements illustrated in the following figures. Figure 2. illustrates the solution's standard deviation as a percentage of the expected total profit, which is actually an indicator of the optimality. Figure 3. shows the total demand under-fulfillment as a percentage of the total expected demand, which measures the feasibility.

Figure 2: The solution's standard deviation



As should be expected, we are interested in the smaller values on this figure, because they give a better solution regarding the optimality. Like the previous figure, by increasing the value of λ from 0.1 to 1, the solution's standard deviation does not change significantly. So, to choose the preferred λ we need the result of the feasibility measurement to be able to summarize the robust decision making.

Figure 3: The total demand under-fulfillment



By increasing the value of ω from zero to 10, the percentage of demand under-fulfillment drops sharply. Also by raising λ, the amount of unsatisfied demand will be increased too. But, because at any nonzero rate of ω the under-fulfillment is less than five percent, we do not have too much concern about choosing larger values for λ which brings more optimality. In other words, for finding the robust answer to this specific example, we just consider first and second figures as the mean and standard deviation of the solution respectively.

By finding the optimal solution for $\lambda = 10$ over the same range of ω and comparison to the above plots, it can be seen that almost all the results remain similar. As a result, $\lambda = 1$ and $\omega = 20$ to 30 provides a reasonable zone to find the robust solution of the problem. As an example we present the output within the robust zone for $\lambda = 1$ and $\omega = 20$.

The total expected profit: 4971.1

The standard deviation: 0.0084%

The total under-fulfillment: 0

$x_{11} = 23.0844$ $x_{12} = 19.2370$ $x_{13} = 7.6948$ $x_{14} = 7.6948$ $x_{15} = 57.7110$ $x_{16} = 63.6037$

$x_{21} = x_{22} = x_{23} = x_{24} = x_{25} = x_{26} = 51.7045$

$p_1 = 22.3051$ $p_2 = 19.6590$

5.2 Ten Products, Twelve Periods and Three Scenarios

We now discuss a realistic size problem in manufacturing pricing, where the firm is not flexible to change the price list frequently and usually has long-term contracts with Original Equipment Manufacturers (OEMs). It is reasonable to expect that in a real situation the number of product classes, which do not have cross price dependency in their demand function is not too large. So the assumption of $n = 10$ can cover a large number of applications. Besides, by considering $T = 12$ we are catering for a whole year of planning on a monthly basis, which is logical for constant pricing. Our three scenarios present good, moderate and weak market situations. For such a realistic size example, Maple is efficient.

The parameters for this example are as shown in Table 2.

We assume a fixed production capacity, $K_t = 100$, for all periods in the planning horizon. Similar to the previous example, the results for this specific problem include the expected profit, the standard deviation of the solution and the total demand under-fulfillment shown as follow

Table 2: Parameters of the example

Product	h_j	c_j	β_{j1}	β_{j2}	β_{j3}	β_{j4}	β_{j5}	β_{j6}	β_{j7}	β_{j8}	β_{j9}	β_{j10}	β_{j11}	β_{j12}
j=1	2	6	0.6	0.5	0.2	0.2	1.5	3	2	1	0.8	0.5	0.7	1
j=2	1	4	0.8	0.6	1	2	1.5	1.1	1	0.9	0.7	1.4	0.3	0.7
j=3	3	8	0.7	0.4	0.3	0.3	1.5	2.8	2	1	0.7	0.6	1	0.7
j=4	2	5	0.9	0.3	0.6	1.2	1.5	1.5	3	0.5	0.6	0.4	0.7	0.8
j=5	4	10	1	1	1	1	1	1	1	1	1	1	1	1
j=6	1	7	2	1	0.8	0.5	0.7	1	0.6	0.5	0.2	0.2	1.5	3
j=7	4	9	1	1	1	1	1	1	1	1	1	1	1	1
j=8	3	4	1.5	1.5	1.2	0.6	0.3	0.9	1	2	0.6	0.7	1	0.7
j=9	2	6	1	1	1	1	1	1	1	1	1	1	1	1
j=10	1	8	1	1	1	1	1	1	1	1	1	1	1	1

Scenario	$D_j(p_j)$
Pr(s=1)=0.7	$150 - 5p_j$
Pr(s=2)=0.2	$150 - 4.5p_j$
Pr(s=3)=0.1	$150 - 4p_j$

Figure 4: The expected total profit

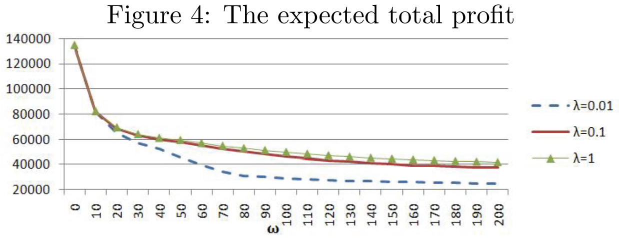


Figure 5: The solution's standard deviation as a percentage of the expected total profit

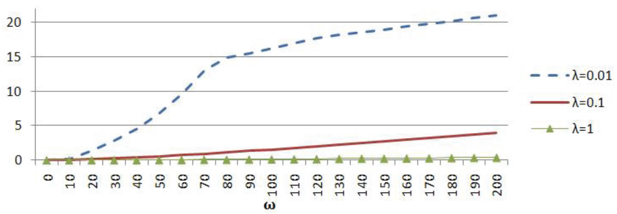
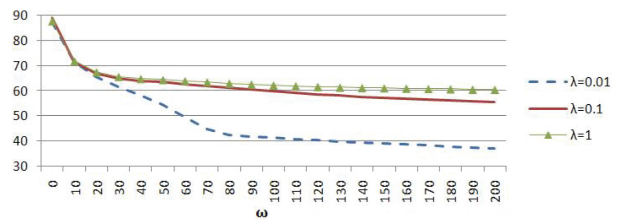


Figure 6: The total demand under-fulfillment as a percentage of the expected total demand



As can be seen, the expected profit in the case of $\lambda = 0.1$ and $\lambda = 1$ remains very similar. We need to consider the optimality and infeasibility measurements simultaneously to choose the robust zone preferred by the decision maker. If it is desired to have the demand satisfied as much as possible, we might choose $\lambda = 0.01$ and $\omega = 80$ to 90 which results in the smallest possible standard deviation and the biggest possible expected profit within the chosen range of λ and ω.On the other hand, if the decision maker is concerned more about the profit regardless of the unsatisfied demand, we might select a bigger value of λ (like 1) and a range of ω(like $20 - 30$) after which the expected profit and infeasibility amount remain much similar. Thus, our methodology presents tools which assist the decision maker explore the range of possibilities.

6 Conclusion

In this paper we have presented a mathematical programming model for determining the optimal production and constant pricing policy for a finite time horizon multiproduct production system with capacity constraints and demand uncertainty. The production set up cost is negligible, and demand for each product is dependent on its price, but the price / demand function is uncertain. Our methodology makes use of Robust Optimization ideas and our model can be effectively implemented utilizing existing computational packages (we use Maple). We illustrate with detailed numerical examples.

References

[1] E. Adida and G. Perakis, A robust optimization approach to dynamic pricing and inventory control with no backorders, *Mathematical Programming* Ser. B 107 (2006) 97–129.

[2] E. Adida and G. Perakis, *A Nonlinear Continuous Time Optimal Control Model of Dynamic Pricing and Inventory Control with No Backorders*, Published online in Wiley Inter Science, 2007.

[3] S. Biller, L.M.Chan, D. Simchi-Levi and J. Swann, Dynamic pricing and the direct-to-customer model in the automotive industry, *Electronic Commerce Research* 5 (2005) 309–334.

[4] L. Caccetta and E. Mardaneh, Joint Pricing and Production Planning for Fixed Priced Multiple Products with Backorders, to be appeared in the Journal of Industrial and Management Optimization, Feb 2010.

[5] L.M. Chan, D. Simchi-Levi and J. Swann, Dynamic pricing models for manufactu ring with stochastic demand and discretionary sales, *Manufacturing and Service Operations Management* 8 (2006) 149–168.

[6] G. Gallego and G. Van Ryzin, Optimal dynamic pricing of inventories with stochastic demand over finite horizons, *Management Science* 40 (1994) 999–1020.

[7] G. Gallego and G. Van Ryzin, A multiproduct dynamic pricing problem and its applications to network yield management, *Operations Research* 45 (1997) 24–41.

[8] S.M. Gilbert, Coordination of pricing and multiple-period production across multiple constant priced goods, *Management Science* 46 (2000) 1602–1616.

[9] S. Kachani and G. Perakis, A fluid model of dynamic pricing and inventory management for make-to-stock manufacturing systems, Sloan School of Management, MIT, pre-print, 2002.

[10] D.G. *Luenberger, Linear and Nonlinear Programming*, Kluwer Academic Publishers, Boston, MA, 2003, pp. 322–454.

[11] J.M. Mulvey, R.J. Vanderbei and S.A. Zenios, Robust optimization of large-scale systems, *Operations Research* 43 (1995) 264–281.

[12] Z.J. Shen, D. Simchi-Levi and S.D. Wu, *Handbook of Qualitative Supply Chain Analysis: Modeling in the E-business Era*, 2004, pp. 335–392.

Manuscript received 20 November 2009
revised 22 February 2010
accepted for publication 1 March 2010

Louis Caccetta
Western Australian Centre of Excellence in Industrial Optimization
Department of Mathematics and Statistics, Curtin University of Technology
GPO Box U1987, WA, 6845, Australia
E-mail address: `L.Caccetta@exchange.curtin.edu.au`

Elham Mardaneh
Western Australian Centre of Excellence in Industrial Optimization
Department of Mathematics and Statistics, Curtin University of Technology
GPO Box U1987, WA, 6845, Australia
E-mail address: `elham.mardaneh@postgrad.curtin.edu.au`