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ROBUST OPTIMIZATION EQUILIBRIA FOR BIMATRIX GAME*

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Abstract: In this paper, we investigate robust optimization equilibria with two players, in which each player can not evaluate his own cost matrix accurately while may estimate a bounded set of the cost matrix. We obtain a result that solving this equilibria can be formulated as solving a second-order cone complementarity problem under l_2-norm or a mixed complementarity problem under $l_1 \cap l_\infty$-norm. We present some numerical results to illustrate the relation between robustness and optimality.

Key words: *robust optimization equilibria, bimatrix game, cost matrix uncertainty, second-order cone complementarity problem, mixed complementarity problem*

Mathematics Subject Classification: *74P05, 65K10*

1 Introduction

In [18, 19], Nash first studied non-cooperative games with complete information and proved that each game had an equilibria in mixed strategies. However, in real-world, game-theoretic situations, players are often uncertain of the structure of the game. When payoff functions are uncertain, Harsanyi [12] modeled these incomplete information games as what he called Bayesian game. In that model, the uncertain payoffs were treated as expectation. A shortcoming of this model is that it is not obvious how players can estimate the prior distribution. Holmström and Myerson [15] refined Bayesian games and considered the case where players need not know the distribution. Indeed, in an ex post equilibria no player has an incentive to deviate from the strategy she selects regardless of the realization of the uncertainty. A few recent papers have explored the application of robust optimization to game theory. Hayashi et al. [14] characterized the robust Nash equilibria in simple games as solutions to a second-order cone complementary problem. Aghassi and Bertsimas [1] also considered robust games and proved that robust Nash equilibria always exists. Ordönez and Moses-Stier [20] showed that robust Wardrop equilibria of net-work games always exists with a finite number of players. The technique adopted in [1, 14, 20] could be considered as robust optimization approach. Robust optimization is emerging as a leading methodology to address optimization problems under uncertainty. It converts problems with uncertainty into computationally tractable optimization problems. Ben-Tal and Nemirovski [2, 3, 5], Bertsimas et al. [6] etc.

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derived robust counterparts of uncertain programming problems under the condition that the uncertain set $\mathcal{U}$ was given as an ellipsoid or an intersection of finitely many ellipsoids. Such a robust counterpart exhibits a lateral increase in complexity. Bertsimas and Sim [7, 8] studied a general conic optimization problem from the cardinality of an uncertain set. Using this approach, the robust counterpart is tractable preserving the computational complexity of the nominal problem, i.e., robust LPs (linear programming problems) remain LPs and robust SOCPs (second-order conic programming problems [17]) remain SOCPs etc.

In this paper, from the cardinality of an uncertain set, we consider a two-person game, in which each player attempts to minimize his own cost with each player's own cost matrix uncertain. For example, for a robust inspection game in [1], the payoff matrices for the employee and employer are usually uncertain. We focus on the tractability of the following model, a special case with N=2 in [1]. In this situation, the model essentially reduces to a bimatrix game as follows:

$$\textit{player} \quad \textit{one} \qquad \min_{\mathbf{y}\in\mathcal{Y}} \mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z} \tag{1.1}$$

and

$$\textit{player} \quad \textit{two} \qquad \min_{\mathbf{z}\in\mathcal{Z}} \mathbf{y}^T\tilde{\mathbf{B}}\mathbf{z}, \tag{1.2}$$

where $\mathcal{Y} := \{\mathbf{y} \in \Re^n : \mathbf{y} \geq 0, \mathbf{e}_n^T\mathbf{y} = 1\}$ and $\mathcal{Z} := \{\mathbf{z} \in \Re^m : \mathbf{z} \geq 0, \mathbf{e}_m^T\mathbf{z} = 1\}$ denote mixed strategy sets for players one and two respectively. Let $\tilde{\mathbf{A}} \in D_A$ and $\tilde{\mathbf{B}} \in D_B$ denote uncertain cost matrices of players one and two respectively where D_A and D_B are assumed to be bounded sets, and $\mathbf{e}_n \in \Re^n$ and $\mathbf{e}_m \in \Re^m$ are vectors of all ones. Then the robust counterparts of (1.1) and (1.2) can be stated as

$$\textit{player} \quad \textit{one} \qquad \begin{array}{l} \min \max\limits_{\tilde{\mathbf{A}}\in D_A} \mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z} \\ \text{s.t. } \mathbf{y} \in \mathcal{Y} \end{array} \tag{1.3}$$

and

$$\textit{player} \quad \textit{two} \qquad \begin{array}{l} \min \max\limits_{\tilde{\mathbf{B}}\in D_B} \mathbf{y}^T\tilde{\mathbf{B}}\mathbf{z} \\ \text{s.t. } \mathbf{z} \in \mathcal{Z}. \end{array} \tag{1.4}$$

A pair of strategies $(\mathbf{y}, \mathbf{z})$ is called a *robust optimization equilibria* for problems (1.1) and (1.2) if $\mathbf{y}$ optimizes (1.3) and $\mathbf{z}$ optimizes (1.4) simultaneously. Accordingly, (1.3) and (1.4) are called *robust counterparts* of (1.1) and (1.2). In general, problems (1.3) and (1.4) are semi-infinite programming problems and computationally intractable. How to deal with an uncertain set plays an important role in the solution of these problems. Aghassi and Bertsimas [1] formulated the set of equilibria of an arbitrary robust finite game, with bounded polyhedral uncertain set and no privative information, as the dimension-reducing, component-wise projection of the solution set of a system of linear equalities and inequalities. Hayashi et al. [14] studied the robust optimization equilibria when the uncertain set is a standard ellipsoid under l_2-norm. A standard ellipsoid is an image of an Euclidean ball with the same dimension under a one-to-one affine mapping. In fact, the uncertain set may be general ellipsoids such as flat ellipsoids–usual ellipsoids in proper affine subspaces (such an ellipsoid corresponds to the case of partial uncertainty) or ellipsoidal cylinders–sets of the type sum of a flat ellipsoid and a linear subspace. Taking this into account, we consider a general uncertain set including flat ellipsoids, ellipsoidal cylinders and standard ellipsoids as well. Furthermore, we investigate the robust optimization equilibria when the cost matrix is uncertain under l_2-norm or $l_1 \cap l_\infty$-norm. When the bounded sets D_A and D_B are constraint-wise uncertain, we obtain tractable optimization formulations for (1.3) and (1.4). In particular, when the cost matrices are uncertain under l_2-norm, the robust counterparts can be formulated as a second-order cone complementarity problem (SOCCP) [9, 11] in the following form:

$$\mathcal{K} \ni \mathbf{Gx} + \mathbf{q} \perp \mathbf{Hx} + \mathbf{r} \in \mathcal{K}, \qquad \mathbf{Cx} = \mathbf{d}, \tag{1.5}$$

where $\mathbf{x} \in \Re^{\varsigma+\tau}$, constant matrices $\mathbf{G}$, $\mathbf{H} \in \Re^{\varsigma\times(\varsigma+\tau)}$, $\mathbf{q}$, $\mathbf{r} \in \Re^{\varsigma}$, $\mathbf{C} \in \Re^{\tau\times(\varsigma+\tau)}$ and $\mathbf{d} \in \Re^{\tau}$, $\mathcal{K}$ is a closed convex cone defined by $\mathcal{K} = \mathcal{K}^{\varsigma_1} \times \mathcal{K}^{\varsigma_2} \times \cdots \times \mathcal{K}^{\varsigma_m}$ with ς_j-dimensional second-order cones $\mathcal{K}^{\varsigma_j} = \{(x_1, \mathbf{x}_2) \in \Re \times \Re^{\varsigma_j-1} | \, \|\mathbf{x}_2\|_2 \leq x_1\}$ and $\varsigma = \varsigma_1 + \cdots + \varsigma_m$. When the cost matrices are uncertain under $l_1 \cap l_\infty$-norm, the robust counterparts can be formulated as a mixed complementarity problem (MCP) [10]:

$$\Re^{\varsigma}_{+} \ni \mathbf{Gx} + \mathbf{q} \perp \mathbf{Hx} + \mathbf{r} \in \Re^{\varsigma}_{+}, \qquad \mathbf{Cx} = \mathbf{d}, \tag{1.6}$$

where $\mathbf{G}$, $\mathbf{H}$, $\mathbf{C}$, $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{d}$ are the same as those in (1.5).

The paper is organized as follows. Section 2 discusses the existence of robust optimization equilibria and investigates the robust counterparts in which players one and two's cost matrices belong to column-wise and row-wise uncertainty respectively with any arbitrary norm. Section 3 considers the robust optimization equilibria under l_2-norm. In this situation, we show that the robust optimization equilibria can be formulated as a second-order cone complementarity problem. In Section 4, we show that the robust counterparts under $l_1 \cap l_\infty$-norm can be converted to a mixed complementarity problem. Some numerical results are presented in Section 5.

2 Existence of Robust Optimization Equilibria and Robust Counterparts under General Norm

To obtain the existence of a robust optimization equilibria, we need the following lemma in [1].

Lemma 2.1 (Theorem 2 [1]). *Any N-person, noncooperative, simultaneous-move, one-shot robust game, in which $N < \infty$, in which player $i \in 1, ..., N$ has $1 < a_i < \infty$ possible actions, in which the uncertain set of payoff matrices $U \subset \Re^{N \prod_{i=1}^{N} a_i}$ is bounded, and in which there is no private information, has an equilibria.*

In our model, we consider two-person, noncooperative, simultaneous-move, one-shot robust game, in which players one and two have n and m possible strategies respectively, in which each player's cost matrix belongs to a bounded set, and in which there is no private information. It follows from Lemma 2.1 that there exists an equilibria for problems (1.3) and (1.4). In other words, there exists a robust optimization equilibria in problems (1.1) and (1.2).

In what follows, we derive the optimization formulations for (1.3) and (1.4) where each player's cost matrix is uncertain. We consider the case where $D_{\mathbf{A}}$ is a column constraint-wise uncertain set and $D_{\mathbf{B}}$ is a row constraint-wise uncertain set [2, 4]. Let

$$D_A^j := \{\tilde{\mathbf{A}}_j^c | \, \tilde{\mathbf{A}}_j^c = \mathbf{A}_j^c + \sum_{l_j=1}^{L_j} \mathbf{r}_j^{l_j} \triangle p_j^{l_j} : \|\triangle \mathbf{p}_j\| \leq \Gamma_j\}, \; j = 1, ..., m \tag{2.1}$$

be the set of all possible realizations of the j-th column in the uncertain cost matrix $\tilde{\mathbf{A}}$, i.e., the projection of $D_{\mathbf{A}} \subseteq \Re^{n\times m} = \Re^n \times \cdots \times \Re^n$ onto the j-th direct factor of the right hand side of the latter relation. In (2.1), $\mathbf{A}_j^c$ is the nominal value for the j-th column of matrix $\tilde{\mathbf{A}}$, $\mathbf{r}_j^{l_j} \in \Re^{n\times 1}$, $l_j = 1, \cdots, L_j$ are directions of data perturbation, $\triangle\mathbf{p}_j = (\triangle p_j^1\ \triangle p_j^2\ \cdots\ \triangle p_j^{L_j})^T$ with $\triangle p_j^{l_j}$, $l_j = 1, \cdots, L_j$ being independent and identically distributed random variables with mean value zero. This implies that $E[\tilde{\mathbf{A}}_j^c]$ (the mean value of $\tilde{\mathbf{A}}_j^c$) equals to $\mathbf{A}_j^c$. L_j may be small, modeling situations involving a small collection of primitive independent uncertainties, or large, potentially as large as the number of entries in the data. In the former case, the elements of $\tilde{\mathbf{A}}_j^c$ are strongly dependent, while in the latter case the elements of $\tilde{\mathbf{A}}_j^c$ are weakly dependent or even independent (when L_j is equal to the number of entries in the data). Γ_j is a parameter controlling the tradeoff between robustness and optimality. We say that the uncertainty $D_\mathbf{A}$ is column constraint-wise, if the uncertain set $D_\mathbf{A}$ is the direct product of the "partial" uncertain sets $D_\mathbf{A}^j$:

$$D_\mathbf{A} = D_\mathbf{A}^1 \times D_\mathbf{A}^2 \times \cdots \times D_\mathbf{A}^m.$$

By construction, we have

$$\max_{\tilde{\mathbf{A}}\in D_A} \mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z} = \max_{\tilde{\mathbf{A}}_j^c\in D_A^j} \mathbf{y}^T(\tilde{\mathbf{A}}_1^c\ \cdots\ \tilde{\mathbf{A}}_m^c)\mathbf{z}.$$

In other words, problem (1.3) remains unchanged when we extend the initial uncertain set $D_\mathbf{A}$ to the direct product $D_\mathbf{A}^1 \times D_\mathbf{A}^2 \times \cdots \times D_\mathbf{A}^m$. The robust counterpart "feels" only the possible realizations of the j-th column, $j = 1, \cdots, m$ and does not feel the dependencies (if any) between these columns in the instances. Similarly, we can define row constraint-wise uncertainty set. Thus, given an arbitrary uncertain set, we can always extend it to a constraint-wise uncertain set resulting in the same robust counterpart.

Similarly, $D_\mathbf{B}$ is of row constraint-wise uncertainty in the sense that

$$D_\mathbf{B} = D_\mathbf{B}^1 \times D_\mathbf{B}^2 \times \cdots \times D_\mathbf{B}^n$$

and

$$D_B^i := \{\tilde{\mathbf{B}}_i^r |\, \tilde{\mathbf{B}}_i^r = \mathbf{B}_i^r + \sum_{k_i=1}^{K_i} \mathbf{s}_i^{k_i} \triangle q_i^{k_i} : \|\triangle\mathbf{q}_i\| \le \Omega_i\},\ i = 1, ..., n, \tag{2.2}$$

where $\mathbf{B}_i^r$ is the nominal value for the i-th row of matrix $\tilde{\mathbf{B}}$, $\triangle\mathbf{q}_i = (\triangle q_i^1\ \triangle q_i^2\ \cdots\ \triangle q_i^{K_i})^T$ and $\mathbf{s}_i^{k_i} \in \Re^{1\times m}$, $k_i = 1, \cdots, K_i$. $\|\cdot\|$ is an arbitrary vector norm whose dual norm $\|\cdot\|^*$ is given by

$$\|\mathbf{s}\|^* = \max_{\|\mathbf{x}\|\le 1} \mathbf{s}^T\mathbf{x}.$$

It is well known (see, for example [16]) that the dual norm of the dual norm is the original norm. With these notations, we can obtain an equivalent formulation to the inner optimization for problem (1.3). Observe that

$$\begin{aligned}
\max_{\tilde{\mathbf{A}}\in D_A} \mathbf{y}^T\tilde{\mathbf{A}}\mathbf{z} &= \max_{\tilde{\mathbf{A}}_j^c\in D_A^j} \mathbf{y}^T(\tilde{\mathbf{A}}_1^c\ \cdots\ \tilde{\mathbf{A}}_m^c)\mathbf{z} \\
&= \mathbf{y}^T\mathbf{A}\mathbf{z} + \max_{\|\triangle\mathbf{p}_j\|_2\le\Gamma_j} [z_1\mathbf{y}^T(\sum_{l_1=1}^{L_1} \mathbf{r}_1^{l_1}\triangle p_1^{l_1}) + \cdots + z_m\mathbf{y}^T(\sum_{l_m=1}^{L_m} \mathbf{r}_m^{l_m}\triangle p_m^{l_m})] \\
&= \mathbf{y}^T\mathbf{A}\mathbf{z} + \max_{\|\triangle\mathbf{p}_1\|_2\le\Gamma_1} z_1\mathbf{y}^T\mathbf{R}_1\triangle\mathbf{p}_1 + \cdots + \max_{\|\triangle\mathbf{p}_m\|_2\le\Gamma_m} z_m\mathbf{y}^T\mathbf{R}_m\triangle\mathbf{p}_m \\
&= \mathbf{y}^T\mathbf{A}\mathbf{z} + \sum_{j=1}^m z_j\Gamma_j\|\mathbf{R}_j^T\mathbf{y}\|^*,
\end{aligned}$$

where $\mathbf{R}_j = (\mathbf{r}_j^1 \ \mathbf{r}_j^2 \ \cdots \ \mathbf{r}_j^{L_j}) \in \Re^{n\times L_j}$.

Therefore, problem (1.3) can be written as

$$\begin{array}{lll} \textit{player} \quad \textit{one} & \min\limits_{\mathbf{y},\gamma_j} \mathbf{y}^T\mathbf{A}\mathbf{z} + \sum_{j=1}^m \Gamma_j z_j \gamma_j & \\ & \text{s.t.} \quad \|\mathbf{R}_j^T\mathbf{y}\|^* \le \gamma_j, \ j=1,\cdots,m, & (2.3) \\ & \qquad \mathbf{y}\ge 0, \ \mathbf{e}_n^T\mathbf{y}=1. & \end{array}$$

Similar to the above analysis, problem (1.4) can be formulated as

$$\begin{array}{lll} \textit{player} \quad \textit{two} & \min\limits_{\mathbf{z},\sigma_i} \mathbf{y}^T\mathbf{B}\mathbf{z} + \sum_{i=1}^n \Omega_i y_i \sigma_i & \\ & \text{s.t.} \quad \|\mathbf{S}_i^T\mathbf{z}\|^* \le \sigma_i, \ i=1,\cdots,n, & (2.4) \\ & \qquad \mathbf{z}\ge 0, \ \mathbf{e}_m^T\mathbf{z}=1, & \end{array}$$

where $\mathbf{S}_i = (\mathbf{s}_i{}^{1^T} \ \mathbf{s}_i{}^{2^T} \ \cdots \ \mathbf{s}_i{}^{K_i{}^T}) \in \Re^{m\times K_i}$.

$\boxed{3}$ Uncertain Cost Matrix under l_2-Norm

The choice of a norm in (2.3) and (2.4) is very important. An uncertain set with l_2-norm forms a relatively wide family, possesses a very nice analytical structure, and can be used to approximate well many cases of complicated convex sets. Under l_2-norm, (2.3) and (2.4) turn to be the following SOCPs:

$$\begin{array}{lll} \textit{player} \quad \textit{one} & \min\limits_{\mathbf{y},\gamma_j} \mathbf{y}^T\mathbf{A}\mathbf{z} + \sum_{j=1}^m \Gamma_j z_j \gamma_j & \\ & \text{s.t.} \quad \begin{pmatrix} \gamma_j \\ \mathbf{R}_j^T\mathbf{y} \end{pmatrix} \in \mathcal{K}^{L_j+1}, \ j=1,\cdots,m, & (3.1) \\ & \qquad \mathbf{y}\ge 0, \ \mathbf{e}_n^T\mathbf{y}=1 & \end{array}$$

and

$$\begin{array}{lll} \textit{player} \quad \textit{two} & \min\limits_{\mathbf{z},\sigma_i} \mathbf{y}^T\mathbf{B}\mathbf{z} + \sum_{i=1}^n \Omega_i y_i \sigma_i & \\ & \text{s.t.} \quad \begin{pmatrix} \sigma_i \\ \mathbf{S}_i^T\mathbf{z} \end{pmatrix} \in \mathcal{K}^{K_i+1}, \ i=1,\cdots,n, & (3.2) \\ & \qquad \mathbf{z}\ge 0, \ \mathbf{e}_m^T\mathbf{z}=1. & \end{array}$$

Let us consider (3.1) first. It is an SOCP whose KKT conditions can be stated as

$$\mathcal{K}^{L_j+1} \ni \begin{pmatrix} \Gamma_j z_j \\ \mathbf{u}_j \end{pmatrix} \perp \begin{pmatrix} \gamma_j \\ \mathbf{R}_j^T\mathbf{y} \end{pmatrix} \in \mathcal{K}^{L_j+1}, \ j=1,\cdots,m,$$

$$\Re_+^n \ni \mathbf{y} \perp \mathbf{A}\mathbf{z} + \mathbf{e}_n\xi - \mathbf{R}_1\mathbf{u}_1 - \cdots - \mathbf{R}_m\mathbf{u}_m \in \Re_+^n, \quad \mathbf{e}_n^T\mathbf{y}=1,$$

where $\mathbf{u}_j \in \Re^{L_j}$, $\gamma_j \in \Re$, $j=1,\cdots,m$ and $\xi\in\Re$ are the Lagrangian multipliers. Similarly, the KKT conditions for (3.2) can be written as

$$\mathcal{K}^{K_i+1} \ni \begin{pmatrix} \Omega_i y_i \\ \mathbf{t}_i \end{pmatrix} \perp \begin{pmatrix} \sigma_i \\ \mathbf{S}_i^T\mathbf{z} \end{pmatrix} \in \mathcal{K}^{K_i+1}, \ i=1,\cdots,n,$$

$$\Re_+^m \ni \mathbf{z} \perp \mathbf{B}^T\mathbf{y} + \mathbf{e}_m\eta - \mathbf{S}_1\mathbf{t}_1 - \cdots - \mathbf{S}_n\mathbf{t}_n \in \Re_+^m, \quad \mathbf{e}_m^T\mathbf{z}=1,$$

where $\mathbf{t}_i \in \Re^{K_i}$, $\sigma_i\in\Re$, $i=1,\cdots,n$ and $\eta\in\Re$ are the Lagrangian multipliers.

Let $\bar{\Gamma}_j$ and $\bar{\Omega}_i$ be two vectors containing mostly zeros except Γ_j and Ω_i at the j-th and i-th components respectively. Let

$$\mathbf{x} = (\mathbf{y}^T \ \mathbf{z}^T \ \mathbf{u}_1^T \ \cdots \ \mathbf{u}_m^T \ \mathbf{t}_1^T \ \cdots \ \mathbf{t}_n^T \ \gamma_1 \cdots \gamma_m \ \sigma_1 \ \cdots \ \sigma_n \ \xi \ \eta)^T.$$

Let

$$\mathbf{G} = \left(\begin{array}{ccccccccccccccccc}
\mathbf{I}_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
\mathbf{R}_1^T & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
 & & & & & & & \vdots & & & & & & & & \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 \\
\mathbf{R}_m^T & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
0 & \mathbf{I}_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & \mathbf{S}_1^T & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
 & & & & & & & \vdots & & & & & & & & \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & \mathbf{S}_n^T & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
\end{array} \right),$$

$$\mathbf{H} = \left(\begin{array}{ccccccccccccccccc}
0 & \mathbf{A} & -\mathbf{R}_1 & \cdots & -\mathbf{R}_m & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \mathbf{e}_n & 0 \\ \hline
0 & \bar{\Gamma}_1^T & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \mathbf{I}_{L_1} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
 & & & & & & & \vdots & & & & & & & & \\ \hline
0 & \bar{\Gamma}_m^T & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & \mathbf{I}_{L_m} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
\mathbf{B}^T & 0 & 0 & \cdots & 0 & -\mathbf{S}_1 & \cdots & -\mathbf{S}_n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \mathbf{e}_m \\ \hline
\bar{\Omega}_1^T & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & \mathbf{I}_{K_1} & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
 & & & & & & & \vdots & & & & & & & & \\ \hline
\bar{\Omega}_n^T & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \mathbf{I}_{K_n} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline
\end{array} \right).$$

Let $K = K_1 + \cdots + K_n$, $L = L_1 + \cdots + L_m$. Then problems (3.1) and (3.2) can be formulated as the KKT system which is an SOCCP (1.5) with $\varsigma = 2m+2n+K+L$, $\tau = 2$, $\mathbf{q}$, $\mathbf{r}$ being two ς-dimensional zeros vectors, and $\mathcal{K} = \Re_+^n \times \mathcal{K}^{L_1+1} \times \cdots \mathcal{K}^{L_m+1} \times \Re_+^m \times \mathcal{K}^{K_1+1} \times \cdots \mathcal{K}^{K_n+1}$,

$$\mathbf{C} = \begin{pmatrix} e_n^T & 0 & 0 & \cdots & 0 & 0\cdots0 & 0 & \cdots & 0 & 0\cdots0 & 0 & 0 \\ 0 & e_m^T & 0 & \cdots & 0 & 0\cdots0 & 0 & \cdots & 0 & 0\cdots0 & 0 & 0 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The above arguments have shown that the following theorem.

Theorem 3.1. *Let the uncertain cost matrix sets D_A and D_B be given by (2.1) and (2.2) respectively under l_2-norm. Then solving a robust optimization equilibria for problems (1.1) and (1.2) can be converted to solving an SOCCP as above.*

As discussed in [8], when all the data entries of the problem have independent random perturbation, we can further reduce the size of the robust model. Essentially, we can express the model of uncertainty in the form of (2.1), for which $\mathbf{r}_j^{l_j}$ contains mostly zeros except at the entries corresponding to the data element, that is, $\mathbf{r}_j^1 = (r_j^1 \ 0 \cdots 0)^T, \cdots, \mathbf{r}_j^{L_j} = (0 \cdots 0 \ r_j^{L_j})^T$, and for which $\triangle p_j^{l_j}$ is the independent random variable associated with the

l_j-th data element. Then $L_j,\ j=1,\cdots,m$ equal to n. Following a similar argument to problem (2.2), we can show that when $r_j^1=\cdots=r_j^n$ and $s_i^1=\cdots=s_i^m$, problems (2.1) and (2.2), under l_2-norm, reduce to a special case in [14].

$\boxed{4}$ Uncertain Cost Matrix under $l_1 \cap l_\infty$-Norm

The uncertain set with l_2-norm provides a reasonable approximation to more complicated uncertain sets. However, a practical drawback of such an approach is that it leads to nonlinear models, which are relatively expensive in computation. In what follows, we investigate the case in which $D_{\mathbf{A}}$ and $D_{\mathbf{B}}$ are constraint-wise uncertain sets under $l_1 \cap l_\infty$-norm, that is, $\|\mathbf{x}\|_{1\cap\infty}=\max\{\frac{1}{\Gamma}\|\mathbf{x}\|_1,\ \|\mathbf{x}\|_\infty\}$ with $\Gamma>0$ (see [8]). In this situation, the approximation to an uncertain set is relatively reasonable (see [6, 7, 8]). Furthermore, the robust counterparts turn to be linear programming problems and their KKT conditions can be written as an MCP. To obtain tractable formulations for problems (2.3) and (2.4) under $l_1 \cap l_\infty$-norm, we need to investigate the dual norm of $l_1 \cap l_\infty$-norm. To this end, Bertsimas et al. [6] defined a different norm, called D-norm. Specifically, for $\mathbf{x}=(x_1,\cdots,x_\nu)^T\in\Re^\nu$ and $p\in[1,\nu]$, the D-norm is defined by

$$|||\mathbf{x}|||_p=\max_{\{S\cup\{t\}|S\subseteq N,|S|\le\lfloor p\rfloor,t\in N\backslash S\}}\{\sum_{j'\in S}|x_{j'}|+(p-\lfloor p\rfloor)|x_t|\},\tag{4.1}$$

where N denotes the set of indices j', $j'=1,\cdots,\nu$ with $x_{j'}$ subject to parameter uncertainty. The following result can easily be obtained from [6].

Lemma 4.1. (a) *The dual norm of the norm* $|||\cdot|||_p$ *is given by*

$$|||\mathbf{s}|||_p^*=\max\{\frac{1}{p}\|\mathbf{s}\|_1,\ \|\mathbf{s}\|_\infty\}.\tag{4.2}$$

(b) *The inequality* $|||\mathbf{x}|||_p\le\gamma$ *with* $\mathbf{x}\ge 0$ *is equivalent to*

$$p\theta+\sum_{j'=1}^{\nu}t_{j'}\le\gamma,\ t_{j'}+\theta\ge x_{j'},\ t_{j'}\ge 0,\ \forall\, j'=1,\cdots,\nu,\ \theta\ge 0.\tag{4.3}$$

Consider the case where D_A and D_B are bounded uncertain sets under $l_1 \cap l_\infty$-norm. In other words, the norm in expression (2.1) is given by (4.2) with $p=\Gamma_j$, namely,

$$\|\triangle\mathbf{p}_j\|=|||\triangle\mathbf{p}_j|||_{\Gamma_j}^*=\max\{\frac{1}{\Gamma_j}\|\triangle\mathbf{p}_j\|_1,\ \|\triangle\mathbf{p}_j\|_\infty\}$$

for $j=1,\cdots,m$. Following from Lemma 4.1 (a) and the dual norm of the dual norm is the original norm, the dual norm of the above is $\|\cdot\|^*=|||\cdot|||_{\Gamma_j}$. Then, by (4.1) and Lemma 4.1 (b), the constraints

$$\|\mathbf{R}_j^T\mathbf{y}\|^*\le\gamma_j,\ j=1,\cdots,m\tag{4.4}$$

in (2.3) under $l_1 \cap l_\infty$-norm are equivalent to

$$\Gamma_j\theta_j+\sum_{l_j=1}^{L_j}w_{l_j}^j\le\gamma_j,\ \ (\mathbf{r}_j^{l_j})^T\mathbf{y}\le w_{l_j}^j+\theta_j,\ \ \forall\, l_j=1,\cdots,L_j,$$

$$\mathbf{w}^j=(w_1^j,\cdots,w_{L_j}^j)^T\in\Re_+^{L_j},\ \ \theta_j\in\Re_+,\ \ j=1,\cdots,m,$$

which can be rewritten as

$$\Gamma_j\theta_j + \mathbf{e}_{L_j}^T\mathbf{w}^j \le \gamma_j,\ \mathbf{R}_j^T\mathbf{y} \le \mathbf{w}^j + \mathbf{e}_{L_j}\theta_j,\ \theta_j \in \Re_+,\ \mathbf{w}^j \in \Re_+^{L_j},\ j = 1, \cdots, m.$$

Let $\Gamma = \mathrm{diag}(\Gamma_j),\ j = 1, \cdots, m,\ \mathbf{R} = \begin{pmatrix} \mathbf{R}_1 & \mathbf{R}_2 & \cdots & \mathbf{R}_m \end{pmatrix} \in \Re^{n\times L}$,

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{pmatrix},\ \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{pmatrix},\ \mathbf{w} = \begin{pmatrix} \mathbf{w}^1 \\ \mathbf{w}^2 \\ \vdots \\ \mathbf{w}^m \end{pmatrix} \text{ and } \mathbf{M} = \begin{pmatrix} \mathbf{e}_{L_1}^T & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{e}_{L_2}^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathbf{e}_{L_m}^T \end{pmatrix}.$$

Then problem (2.3) can be written as the following minimization problem over $(\mathbf{y}\ \gamma\ \theta\ \mathbf{w}) \in \Re^n \times \Re^m \times \Re^m \times \Re^L$:

$$\begin{aligned} &\min \mathbf{y}^T\mathbf{A}\mathbf{z} + (\Gamma\mathbf{z})^T\gamma \\ &\text{s.t.}\quad \Gamma\theta + \mathbf{M}\mathbf{w} \le \gamma, \\ &\qquad\ \mathbf{R}^T\mathbf{y} \le \mathbf{w} + \mathbf{M}^T\theta, \\ &\qquad\ \mathbf{e}_n^T\mathbf{y} = 1, \\ &\qquad\ \mathbf{y} \in \Re_+^n,\ \theta \in \Re_+^m,\ \mathbf{w} \in \Re_+^L. \end{aligned} \tag{4.5}$$

Problem (4.5) is a linear programming whose KKT conditions are

$$\begin{gathered} \Re_+^m \ni \mathbf{h}_1 \perp \gamma - \Gamma\theta - \mathbf{M}\mathbf{w} \in \Re_+^m, \quad \Re_+^L \ni \mathbf{g}_1 \perp -\mathbf{R}^T\mathbf{y} + \mathbf{M}^T\theta + \mathbf{w} \in \Re_+^L, \\ \Re_+^m \ni \theta \perp \Gamma\mathbf{h}_1 - \mathbf{M}\mathbf{g}_1 \in \Re_+^m, \quad \Re_+^L \ni \mathbf{w} \perp \mathbf{M}^T\mathbf{h}_1 - \mathbf{g}_1 \in \Re_+^L, \\ \Re_+^n \ni \mathbf{y} \perp \mathbf{A}\mathbf{z} + \mathbf{R}\mathbf{g}_1 + \mathbf{e}_n\xi \in \Re_+^n, \quad \Gamma\mathbf{z} - \mathbf{h}_1 = 0, \quad \mathbf{e}_n^T\mathbf{y} = 1, \end{gathered} \tag{4.6}$$

where $\xi \in \Re$, $\mathbf{g}_1 \in \Re^L$ and $\mathbf{h}_1 \in \Re^m$ are the Lagrangian multipliers.

Similarly, problem (2.4) under $l_1 \cap l_\infty$-norm can be written as the following minimization problem over $(\mathbf{z}\ \sigma\ \delta\ \mathbf{v}) \in \Re^m \times \Re^n \times \Re^n \times \Re^K$:

$$\begin{aligned} &\min \mathbf{y}^T\mathbf{B}\mathbf{z} + (\Omega\mathbf{y})^T\sigma \\ &\text{s.t.}\quad \Omega\delta + \mathbf{N}\mathbf{v} \le \sigma, \\ &\qquad\ \mathbf{S}^T\mathbf{z} \le \mathbf{v} + \mathbf{N}^T\delta, \\ &\qquad\ \mathbf{e}_m^T\mathbf{z} = 1, \\ &\qquad\ \mathbf{z} \in \Re_+^m,\ \delta \in \Re_+^n,\ \mathbf{v} \in \Re_+^K, \end{aligned} \tag{4.7}$$

where $\Omega = \mathrm{diag}(\Omega_i),\ i = 1, \cdots, n,\ \mathbf{S} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 & \cdots & \mathbf{S}_n \end{pmatrix} \in \Re^{m\times K}$,

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix},\quad \sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix},\quad \mathbf{v} = \begin{pmatrix} \mathbf{v}^1 \\ \mathbf{v}^2 \\ \vdots \\ \mathbf{v}^n \end{pmatrix} \text{ and } \mathbf{N} = \begin{pmatrix} \mathbf{e}_{K_1}^T & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{e}_{K_2}^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathbf{e}_{K_n}^T \end{pmatrix}.$$

The KKT conditions of problem (4.7) can be stated as

$$\begin{gathered} \Re_+^n \ni \mathbf{h}_2 \perp \sigma - \Omega\delta - \mathbf{N}\mathbf{v} \in \Re_+^n, \quad \Re_+^K \ni \mathbf{g}_2 \perp -\mathbf{S}^T\mathbf{z} + \mathbf{N}^T\delta + \mathbf{v} \in \Re_+^K, \\ \Re_+^n \ni \delta \perp \Omega\mathbf{h}_2 - \mathbf{N}\mathbf{g}_2 \in \Re_+^n, \quad \Re_+^K \ni \mathbf{v} \perp \mathbf{N}^T\mathbf{h}_2 - \mathbf{g}_2 \in \Re_+^K, \\ \Re_+^m \ni \mathbf{z} \perp \mathbf{B}^T\mathbf{y} + \mathbf{S}\mathbf{g}_2 + \mathbf{e}_m\eta \in \Re_+^m, \quad \Omega\mathbf{y} - \mathbf{h}_2 = 0, \quad \mathbf{e}_m^T\mathbf{z} = 1, \end{gathered} \tag{4.8}$$

where $\eta \in \Re$, $\mathbf{g}_2 \in \Re^K$ and $\mathbf{h}_2 \in \Re^n$ are the Lagrangian multipliers.

Let $\mathbf{x} = (\mathbf{y}^T \ \gamma \ \theta \ \mathbf{w}^T \ \mathbf{h}_1^T \ \mathbf{g}_1^T \ \xi \ \mathbf{z}^T \ \sigma \ \delta \ \mathbf{v}^T \ \mathbf{h}_2^T \ \mathbf{g}_2^T \ \eta)^T$,

$$\mathbf{G} = \begin{pmatrix} \mathbf{I}_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I}_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_K & 0 \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{R} & \mathbf{e}_n & \mathbf{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Gamma & -\mathbf{M} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{M}^T & -\mathbf{I}_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_m & -\Gamma & -\mathbf{M} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{R}^T & 0 & \mathbf{M}^T & \mathbf{I}_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{B}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{S} & \mathbf{e}_m \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Omega & -\mathbf{N} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{N}^T & -\mathbf{I}_K & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I}_n & -\Omega & -\mathbf{N} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{S}^T & 0 & \mathbf{N}^T & \mathbf{I}_K & 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} \mathbf{e}_n^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{I}_m & 0 & 0 & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{e}_m^T & 0 & 0 & 0 & 0 & 0 & 0 \\ \Omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{I}_n & 0 & 0 \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Let $\mathbf{q}$, $\mathbf{r}$ be two ς-th zeros vectors, then combining (4.6) and (4.8), we obtain an MCP (1.6) where $\varsigma = 3m + 3n + 2K + 2L$, $\tau = m + n + 2$.

Therefore, we have the following theorem.

Theorem 4.2. *Let the uncertain cost matrix sets D_A and D_B be given respectively by (2.1) and (2.2) under $l_1 \cap l_\infty$- norm. Then solving a robust optimization equilibria for problems (1.1) and (1.2) can be converted to solving an MCP as above.*

Remark 4.3. In our model, we only considered the case $\mathbf{R}_j^T \mathbf{y} \geq 0$, $j = 1, \cdots, m$ in (4.4). The case $\mathbf{R}_j^T \mathbf{y} \leq 0$ can be investigated in a similar way.

5 Numerical Experiments

In the previous sections, we have shown that some robust optimization equilibria problems for bimatrix games can be formulated as SOCCPs or MCPs with different norms. In this section, we present two numerical examples for robust optimization equilibria. The algorithm we adopt is based on the methods proposed in [13]. For simplicity, we study the case where the two players' cost matrices are uncertain under l_2-norm. Moreover, we assume that $L_j = K_i$=3 for all $i, j = 1, 2, 3$, and assume that $\mathbf{R}_1 = \mathbf{S}_1 = \mathbf{I}_3$, $\mathbf{R}_2 = \mathbf{S}_2 = 2\mathbf{I}_3$, and

$\mathbf{R}_3 = \mathbf{S}_3 = 3\mathbf{I}_3$. This implies that all data entries of the problems have independent random perturbations [8]. First we consider the bimatrix game with cost matrices:

$$\mathbf{A}_1 = \begin{pmatrix} -1 & 8 & 3 \\ 10 & -1 & 4 \\ 3 & 10 & 1 \end{pmatrix}, \qquad \mathbf{B}_1 = \begin{pmatrix} 6 & -4 & 0 \\ -1 & 7 & 5 \\ 3 & 1 & 4 \end{pmatrix}.$$

In practical applications, the above data such as L_j, $\mathbf{S}_i$ and cost matrices etc. are obtained by statistics or sampling or other estimations. Denote $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ and $\Omega = (\Omega_1, \Omega_2, \Omega_3)$. When $\Gamma = \Omega = 0$, we obtain the solution $\bar{y} = (0.4815, 0.1852, 0.3333)$ and $\bar{z} = (0.1699, 0.2628, 0.5673)$. Essentially, this solution is a Nash equilibria. In this case, the costs for players one and two are 3.7052 and -1.5928 respectively. From table one, we observe that the precise estimation (Nash equilibria) may be inaccurate in some cases. On the other hand, as Γ_j and Ω_i increase to 6 for all i and j, the corresponding robust solutions deviate from the precise estimation and in these situations, the two players costs are reduced. Subsequently, as Γ_j and Ω_i continue to increase, the two players' cost also increase.

Table one: Robust optimization equilibria for various Γ and Ω

Γ	Ω	$\bar{y}_r$	$\bar{z}_r$	$\bar{y}_r^T A_1 \bar{z}_r$	$\bar{y}_r^T B_1 \bar{z}_r$
(0.5,1,3)	(0.5,1,3)	(0.5473,0.1253,0.3274)	(0.2527,0.2448,0.5026)	3.4338	-2.1582
(1,2,6)	(1,2,6)	(0.5826,0.1491,0.2683)	(0.3024,0.2559,0.4417)	3.0374	-2.4656
(6,6,6)	(6,6,6)	(0.5835,0.2008,0.2157)	(0.3341,0.3000,0.3659)	2.4249	-2.4861
(6,12,15)	(6,12,15)	(0.4925,0.2603,0.2472)	(0.3449,0.3297,0.3254)	2.4336	-1.7881
(15,15,15)	(15,15,15)	(0.4495,0.2818,0.2687)	(0.3437,0.3362,0.3201)	2.5692	-1.4624

Next we consider the bimatrix game with cost matrices:

$$\mathbf{A}_2 = \begin{pmatrix} -16 & 20 & 10 \\ 11 & -9 & 40 \\ -15 & -10 & -27 \end{pmatrix}, \qquad \mathbf{B}_2 = \begin{pmatrix} -14 & -40 & -18 \\ -11 & 10 & 50 \\ 36 & 16 & 40 \end{pmatrix}.$$

Similarly, when $\Gamma = \Omega = 0$, we obtain that the Nash equilibria are $\bar{y} = (0.4144, 0.2808, 0.3048)$ and $\bar{z} = (0, 1, 0)$ and the corresponding costs for players one and two are 2.7128 and -8.8912 respectively. From table two, we see that the two players' costs don't vary significantly when Γ_j and Ω_i increase gradually to 1, 2 and 6 for $j = 1, 2, 3$ and $i = 1, 2, 3$ respectively, which shows that in this case, the model possesses high robustness. Subsequently, as Γ_j and Ω_i continue to increase from 1, 2 and 6 to 6, 12 and 15 for $j = 1, 2, 3$ and $i = 1, 2, 3$ respectively, player one's cost increases while player two's cost decreases. When Γ_j and Ω_i increase from 6, 12 and 15 for $j = 1, 2$ and $i = 1, 2$ respectively to 15, player one's cost decreases while player two's cost increases.

Table two: Robust optimization equilibria for various Γ and Ω

Γ	Ω	$\bar{y}_r$	$\bar{z}_r$	$\bar{y}_r^T A_2 \bar{z}_r$	$\bar{y}_r^T B_2 \bar{z}_r$
(0.5,1,3)	(0.5,1,3)	(0.4145,0.2808,0.3047)	(0,1,0)	2.7158	-8.8968
(1,2,6)	(1,2,6)	(0.4148,0.2806,0.3046)	(0,1,0)	2.7246	-8.9124
(6,6,6)	(6,6,6)	(0.4293,0.2732,0.2975)	(0,1,0)	3.1522	-9.6800
(6,12,15)	(6,12,15)	(1,0,0)	(0,1,0)	20	-40
(15,15,15)	(15,15,15)	(0.3089,0.3110,0.3801)	(0.0057,0.9916,0.0027)	-0.4454	-3.0359

The two tables above indicate two aspects. One is that precise estimation (Nash equilibria) may be inaccurate in some cases so the robustness should be considered under uncertainty. The other is that as shown in the two tables, the players' costs in many cases increase as the parameters increase. However, it may happen that the two players simultaneously obtain relatively low costs when the size of the parameters Γ and Ω increases to a certain degree and then as the parameters continue to increase, one player's cost increases accompanied by the decrease of the other's cost. Therefore, it shows that the parameters Γ and Ω play an important role in controlling the robustness and optimality. However, how to choose an appropriate parameter is significantly a hard work.

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