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CONE PREINVEX FUNCTIONS AND APPLICATIONS TO VECTOR OPTIMIZATION*

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Abstract: In this paper, cone preinvex and related functions have been studied. The concept of cone subpreinvex functions is introduced. Some properties of cone subpreinvex functions have been established and their relationships with cone convex, cone subconvex, cone preinvex functions have been explored. Under the condition of cone subpreinvex functions, optimality conditions for vector valued minimization problems are obtained over topological vector space. A Mond-Weir type dual problems are formulated. The duality results are established under the condition of cone subpreinvex functions.

Key words: *vector optimization, cone convexity, cone preinvexity, optimality conditions*

Mathematics Subject Classification: *90C29, 90C46*

1 Introduction

It is well known that convexity plays a central role in mathematical economics, engineering, management sciences and optimization. In recent years, several extensions and generalizations have been developed for classical convexity. An important generalization of convex functions is invex functions introduced by Hanson [5], Hanson pointed that, under the assumption of invexity, the Kuhn-Tucker conditions are sufficient for optimality of nonlinear programming problems. Ben-Israel and Mond [1] introduced a class of convex functions, which is called the preinvex function. Weir and Mond [14] and Weir and Jeyakumar [15] have studied the basic properties of preinvex functions and their applications in optimization. More recently, properties and applications of generalized preinvexity and generalized invexity were studied by many authors for example, [4, 18, 7] and references therein. In the study of multiobjective programming problems, cone convex [3], cone convexlike [6], cone subconvexlike [8], cone strictly convexlike, cone generalized convexlike, cone generalized subconvexlike [16], cone subconvex [9], cone semistrictly convex [13] have been introduced one after the other. Under these generalized cone convexity assumptions, alternative theorems, optimality conditions for minimizing problems and duality results have been obtained. Many results of generalized cone convexity for vector valued have been extended to set-valued maps, many characters of efficient solution and weakly efficient solution were studied under generalized convexity for set-valued maps, see, for example, [10, 2, 11, 17, 12] and references therein.

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Motivated by these ideas, in the present paper, cone subpreinvex vector-valued functions is introduced. Some examples are given to illustrate that cone subpreinvex functions is extension for cone convex functions and it is differ from the known cone convexity. We obtain optimality conditions for a vector minimization problem in terms of Gâteaux derivatives of the functions. At the end, we associate a Mond-Weir type dual and establish a duality result.

2 Preliminaries

Throughout this paper, let X, Y, Z be real topological vector spaces. Let X^*,Y^*,Z^* be the dual spaces of X, Y, Z, respectively. The apex of all cones considered in this paper will be at the origin. For a cone $C \subset Y$, we set

$$C^* = \{y^* \in Y^* : \langle y^*, y\rangle \geq 0, \forall y \in C\},$$

$$C^{*i} = \{y^* \in Y^* : \langle y^*, y\rangle > 0, \forall y \in C \setminus \{0_Y\}\},$$

where $\langle \cdot, \cdot\rangle$ denotes the dual product between Y and Y^*. Consider the vector-valued map $f : \Gamma \subseteq X \to Y$. Let $\Gamma \subseteq X$ be a nonempty set and $C \subset Y$ be an convex cones with nonempty interior. Let $\eta : X \times X \to X$ be a vector-valued function.

Definition 2.1 ([3]). The function f is said to be C-convex on convex set Γ, if $\forall \alpha \in (0, 1)$, $\forall x, y \in \Gamma$ such that

$$\alpha f(x) + (1 - \alpha)f(y) - f(\alpha x + (1 - \alpha)y) \in C.$$

Definition 2.2 ([8]). The function f is said to be C-subconvexlike on nonempty set $\Gamma \subset X$, if $\exists \theta \in intC$, $\forall \alpha \in (0, 1)$, $\forall x_1, x_2 \in \Gamma$, $\forall \varepsilon > 0$, $\exists x_3$ such that

$$\varepsilon\theta + \alpha f(x_1) + (1 - \alpha)f(x_2) - f(x_3) \in C.$$

Definition 2.3 ([10, 2]). Let Γ be a nonempty set in X, and consider the set-valued map $F : \Gamma \to 2^Y$.

(i) The set Γ is said to be η-invex if there exists a function $\eta : X \times X \to X$ such that $x_2 + \alpha\eta(x_1, x_2) \in \Gamma$, $\forall x_1, x_2 \in \Gamma$, $\forall \alpha \in (0, 1)$.

(ii) The set-valued map F is said to be C-preinvex on the η-invex set Γ iff, $\forall x_1, x_2 \in \Gamma$ and $\forall \alpha \in (0, 1)$, we have

$$\alpha F(x_1) + (1 - \alpha)F(x_2) - F(x_2 + \alpha\eta(x_1, x_2)) \subseteq C.$$

When the set-valued map $F : \Gamma \to 2^Y$ becomes the vector-valued map $f : \Gamma \to Y$, we get the definition of C-preinvex functions for vector-valued map.

Definition 2.4 ([9]). The function f is said to be C-subconvex on convex set Γ, if $\exists \theta \in intC$, $\forall \alpha \in (0, 1)$, $\forall \varepsilon > 0$, such that

$$\varepsilon\theta + \alpha f(x_1) + (1 - \alpha)f(x_2) - f(\alpha x_1 + (1 - \alpha)x_2) \in C, \quad \forall x_1, x_2 \in \Gamma.$$

It is obvious that

C-convexity⇒ C-preinvexity⇒ C-subconvexlikeness,

C-convexity⇒ C-subconvexity⇒ C-subconvexlikeness.

It is noticed that the Definition of C-subconvex function[9] is introduced under the condition that C is a convex cone, but when C is a closed convex cone, by ε is arbitrary, C-subconvex function[9] is also C-convex function[3].

3 Cone Subpreinvex Functions

In this section, we will introduce a new class of cone convex functions, which named cone subpreinvex functions.

Definition 3.1. The function $f : \Gamma \subseteq X \to Y$ is said to be C-subpreinvex with respect to η and θ on η-invex set Γ, if $\exists \theta \in intC$, $\forall x, y \in \Gamma$, $\forall \alpha \in (0, 1)$, such that

$$\alpha(1-\alpha)\theta + \alpha f(x) + (1-\alpha)f(y) - f(y + \alpha\eta(x, y)) \in C.$$

When $\eta(x, y) = x - y$, it is called C-subconvex function. (C-subconvex function defined in this paper differs from C-subconvex function[9]).

It is clear that

$$C\text{-convexity} \Rightarrow C\text{-subconvexity}(Def.3.1) \Rightarrow C\text{-subpreinvexity}.$$

But the converse implication is not true, which can be seen from the following examples.

Example 3.2. Let $f : R \to R^2$ and η be functions defined by

$$f(x) = (-|x|, 0), \quad C = R^2_+$$

$$\eta(x,y) = \begin{cases} x - y, & x \geq 0, y \geq 0 \ \ or \ \ x \leq 0, y \leq 0 \\ y - x, & x > 0, y < 0 \ \ or \ \ x < 0, y > 0. \end{cases}$$

Then f is C-subpreinvex function with respect to $\eta(x, y)$ and $\theta = (1, 1) \in intC$, but f is not C-subconvex function.

Because for $\forall \theta = (\theta_1, \theta_2) \in intC$, taking $x = \theta_1$, $y = -2\theta_1$, $\alpha = \frac{1}{2}$, we have

$$\begin{aligned} & \alpha(1-\alpha)\theta + \alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y) \\ = & \frac{1}{4}(\theta_1, \theta_2) + (-\frac{\theta_1}{2}, 0) + (-\theta_1, 0) - (-\frac{\theta_1}{2}, 0) \\ = & (-\frac{3\theta_1}{4}, \frac{\theta_2}{4}) \\ \notin & C. \end{aligned}$$

Example 3.3. Let $f : [0, 1] \to R^2$ be defined as

$$f(x) = (x^2, -x^2), \quad C = \{(x, y) : -x \leq y, y \geq 0\}.$$

Then f is C-subpreinvex function with respect to $\eta(x, y) = x - y$ and $\theta = (1, 1) \in intC$, i.e. f is C-subconvex function with respect to the same θ, but f is not C-convex function.

Because for $x = 0$, $y = 1$, $\alpha = \frac{1}{2}$, we have

$$\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y) = (\frac{1}{4}, -\frac{1}{4}) \notin C.$$

From this example we can see $\forall\ \theta = (\theta_1, \theta_2) \in intC$, there exist $x = 0$, $y = 1$, $\alpha = \frac{1}{2}$ and $\varepsilon = \frac{1}{8\theta_2} > 0$, such that

$$\varepsilon\theta + \alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y) = (\frac{\theta_1}{8\theta_2}, \frac{1}{8}) + (\frac{1}{4}, -\frac{1}{4}) = (\frac{\theta_1}{8\theta_2} + \frac{1}{4}, -\frac{1}{8}) \notin C.$$

Therefore, f is not C-subconvex function defined by Hu and Ling [9]. On the other hand, by Definitions 2.4 and 3.1, it is obvious that every C-subconvex function defined by Hu and Ling [9] must be C-subconvex function defined in this paper. Thus, C-subpreinvexity defined in Definition 3.1 is a new class of cone convexity.

Example 3.4. Let us set

$$\Gamma = \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 > 1\} \cup \{(0,1),(1,0)\},$$

$$f(x_1, x_2) = (x_1, x_2), \quad \forall (x_1, x_2) \in \Gamma, \quad C = R^2_+.$$

Then f is C-subconvexlike function because $f(\Gamma) + intC$ is convex set. But f is not C-subconvex function as Γ is not convex set.

In the following we assume that X, Y and Z are Hausdorff topological vector spaces, $C \subseteq Y$ is a pointed closed convex cone with nonempty interior. we will give some properties of cone preinvex function in terms of G*âteaux* derivative, which is defined below.

Definition 3.5. A mapping $f : X \to Y$ is said to be G*âteaux* differentiable at $\bar{x} \in X$ if for any $v \in X$,

$$\lim_{t \to 0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t} \tag{3.1}$$

exists. (3.1) is denoted by $f'_{\bar{x}}(v)$.

From Definition 3.5, it is easy to know that $f'_{\bar{x}}(0) = 0$, and $f'_{\bar{x}}(\alpha v) = \alpha f'_{\bar{x}}(v)$, for any real number α.

Theorem 3.6. *Let Γ be a η-invex subset of X and $f : \Gamma \subseteq X \to Y$ be Gâteaux differentiable, if f is C-subpreinvex function with respect to η and θ on Γ, then*

$$-2\theta + f'_y(\eta(x,y)) + f'_x(\eta(y,x)) \in -C, \quad \forall x, y \in \Gamma,$$

where $f'_y(\eta(x,y))$ is the Gâteaux derivative of f at y in the direction $\eta(x,y)$.

Proof. Suppose f is C-subpreinvex function with respect to η and θ on Γ, then $\forall x, y \in \Gamma$, $\forall \alpha \in (0,1)$,

$$\alpha(1-\alpha)\theta + \alpha f(x) + (1-\alpha) f(y) - f(y + \alpha \eta(x,y)) \in C.$$

Hence,

$$(1-\alpha)\theta + f(x) - f(y) - \frac{f(y + \alpha\eta(x,y)) - f(y)}{\alpha} \in C.$$

Taking limit as $\alpha \to 0^+$, we have

$$\theta + f(x) - f(y) - f'_y(\eta(x,y)) \in C. \tag{3.2}$$

Similarly, we can get

$$\theta + f(y) - f(x) - f'_x(\eta(y,x)) \in C. \tag{3.3}$$

Adding (3.2) and (3.3), we have

$$-2\theta + f'_y(\eta(x,y)) + f'_x(\eta(y,x)) \in -C.$$

□

4 Optimality Condition

In this section, we consider the following optimization problem.

$$(MP) \qquad \begin{array}{ll} \min & f(x), \\ \text{s. t.} & -g(x) \in D. \end{array}$$

where $f : X \to Y$ and $g : X \to Z$, $C \subseteq Y$, $D \subseteq Z$ are pointed closed convex cones with nonempty interiors, Y, Z are ordered Hausdorff topological vector spaces with the order defined by cones C and D, respectively. We denote the feasible set of (MP) by S, i.e.,

$$S := \{x \in X : -g(x) \in D\}.$$

Definition 4.1. Let $e \in intC$, $\varepsilon \geq 0$.
(1) A point $\bar{x} \in S$ is said to be a εe-efficient solution of (MP) iff there exists no $x \in S$ such that

$$f(\bar{x}) - f(x) \in C \setminus \{0\} + \varepsilon e.$$

(2) A point $\bar{x} \in S$ is said to be a εe-weakly efficient solution of (MP) iff there exists no $x \in S$ such that

$$f(\bar{x}) - f(x) \in intC + \varepsilon e.$$

When $\varepsilon = 0$, (1) and (2) is said to be efficient solution of (MP) and weakly efficient solution of (MP), respectively.

Now we present the following sufficient optimality conditions.

Theorem 4.2. *Let $f : X \to Y$ and $g : X \to Z$ be Gâteaux differentiable at $\bar{x} \in S$, and let f be C-subpreinvex with respect to η and θ, g be D-preinvex with respect to η on X. If there exist $\bar{\lambda} \in C^{*i}$, $\bar{\mu} \in D^*$ such that*

$$\langle \bar{\lambda}, f'_{\bar{x}}(x)\rangle + \langle \bar{\mu}, g'_{\bar{x}}(x)\rangle = 0, \quad \forall\, x \in X, \tag{4.1}$$

$$\langle \bar{\mu}, g(\bar{x})\rangle = 0. \tag{4.2}$$

Then $\bar{x}$ is a θ-efficient solution of (MP).

Proof. Supposing that $\bar{x}$ is not a θ-efficient solution of (MP). Then there exists $x \in S$ such that

$$f(\bar{x}) - f(x) \in C \setminus \{0\} + \theta. \tag{4.3}$$

Since f is C-subpreinvex with respect to η and θ, therefore, for any $\alpha \in (0, 1)$,

$$\alpha(1-\alpha)\theta + \alpha f(x) + (1-\alpha)f(\bar{x}) - f(\bar{x} + \alpha\eta(x, \bar{x})) \in C,$$

that is

$$(1-\alpha)\theta + f(x) - f(\bar{x}) - \frac{f(\bar{x} + \alpha\eta(x, \bar{x})) - f(\bar{x})}{\alpha} \in C.$$

Because f is Gâteaux differentiable, taking limit as $\alpha \to 0^+$, we have

$$\theta + f(x) - f(\bar{x}) - f'_{\bar{x}}(\eta(x, \bar{x})) \in C.$$

Hence

$$\langle \bar{\lambda}, \theta\rangle + \langle \bar{\lambda}, f(x) - f(\bar{x})\rangle - \langle \bar{\lambda}, f'_{\bar{x}}(\eta(x, \bar{x}))\rangle \geq 0. \tag{4.4}$$

Similarly, as g is C-preinvex with respect to η, then for any $\alpha \in (0,1)$,

$$\langle \bar{\mu}, g(x) - g(\bar{x}) \rangle - \langle \bar{\mu}, g'_{\bar{x}}(\eta(x,\bar{x})) \rangle \geq 0. \tag{4.5}$$

Adding (4.4) and (4.5), by (4.1), (4.2) and $x \in S$, we obtain

$$\langle \bar{\lambda}, \theta + f(x) - f(\bar{x}) \rangle \geq 0$$

Since $\bar{\lambda} \in C^{*i}$, then

$$f(\bar{x}) - f(x) \notin C \setminus \{0\} + \theta,$$

which is contradiction to (4.3). □

From Definition 4.1, we know every εe-efficient solution of (MP) is also εe-weakly efficient solution of (MP). Therefore, we get the following result.

Corollary 4.3. *Let $f : X \to Y$ and $g : X \to Z$ be Gâteaux differentiable at $\bar{x} \in S$, and let f be C-subpreinvex with respect to η and θ, g be D-preinvex with respect to η on X. If there exist $\bar{\lambda} \in C^{*i}$, $\bar{\mu} \in D^*$ such that (4.1) and (4.2) hold, then $\bar{x}$ is a θ-weakly efficient solution of (MP).*

5 Duality

We consider the following Mond-Weir type dual for the problem (MP).

$$(MD) \qquad \begin{array}{ll} \max & f(y), \\ \text{s. t.} & \langle \lambda, f'_y(x) \rangle + \langle \mu, g'_y(x) \rangle = 0, \quad \forall\, x \in X, \\ & \langle \mu, g(y) \rangle \geq 0, \quad y \in X, \\ & 0 \neq \lambda \in C^*, \quad \mu \in D^*. \end{array}$$

Where f', g' is Gâteaux differential of f and g, respectively.

Theorem 5.1. *Let $\bar{x}$ be a feasible solution for (MP) and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be feasible for (MD). Let f be C-subpreinvex function with respect to η and θ on X, g be D-preinvex function with respect to the same η on X. Then*

$$f(\bar{y}) - f(\bar{x}) \notin intC + \theta.$$

Proof. Since g is D-preinvex function with respect to η on X, then for $\bar{x},\ \bar{y} \in X$, $\forall \alpha \in (0,1)$,

$$\alpha g(\bar{x}) + (1-\alpha) g(\bar{y}) - g(\bar{y} + \alpha \eta(\bar{x},\bar{y})) \in D,$$

hence

$$g(\bar{x}) - g(\bar{y}) - \frac{g(\bar{y} + \alpha\eta(\bar{x},\bar{y})) - g(\bar{y})}{\alpha} \in D.$$

Because D is a closed cone, by Gâteaux differentiability of g, letting $\alpha \to 0^+$, we have

$$g(\bar{x}) - g(\bar{y}) - g'_{\bar{y}}(\eta(\bar{x},\bar{y})) \in D,$$

which implies

$$\langle \bar{\mu}, g(\bar{x}) \rangle - \langle \bar{\mu}, g(\bar{y}) \rangle - \langle \bar{\mu}, g'_{\bar{y}}(\eta(\bar{x},\bar{y})) \rangle \geq 0. \tag{5.1}$$

Because f is C-subpreinvex function with respect to η and θ on X, then for any $\alpha \in (0,1)$,

$$\alpha(1-\alpha)\theta + \alpha f(\bar{x}) + (1-\alpha)f(\bar{y}) - f(\bar{y} + \alpha\eta(\bar{x},\bar{y})) \in C.$$

Since C is a closed cone, by Gâteaux differentiability of f, taking limit as $\alpha \to 0^+$, we have

$$\theta + f(\bar{x}) - f(\bar{y}) - f'_{\bar{y}}(\eta(\bar{x},\bar{y})) \in C.$$

Hence

$$\langle\bar{\lambda},\theta\rangle + \langle\bar{\lambda}, f(\bar{x}) - f(\bar{y})\rangle - \langle\bar{\lambda}, f'_{\bar{y}}(\eta(\bar{x},\bar{y}))\rangle \geq 0. \tag{5.2}$$

Adding (5.1) and (5.2), by feasibility of $\bar{x}$ and $(\bar{y},\bar{\lambda},\bar{\mu})$ for (MP) and (MD), respectively, we have

$$\langle\bar{\lambda},\theta + f(\bar{x}) - f(\bar{y})\rangle \geq 0,$$

Therefore,

$$f(\bar{y}) - f(\bar{x}) \notin intC + \theta.$$

□

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