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CONVEXITY WITH RESPECT TO min-TYPE FUNCTION

Ivan Ginchev

Abstract: Abstract convex analysis constructs a theory of nonconvex optimization based on suitable extensions of the methods of (usual) convex analysis. One of its problems is to describe and characterize the set of abstract convex functions with respect to given set of elementary functions. In this paper a characterization of a class of abstract convex functions with respect to min-type functions in terms of global calmness is obtained. By analogy with (usual) convex analysis, where convexity (in sense of upper envelope of a set of affine functions) is related to the lower semicontinuity, a question is posed, in how far the global calmness condition can be substituted with lower semicontinuity. As an application of the obtained characterization, it is shown that this is the case when the considered function f is of bounded domain (while an example shows that a similar assertion for functions with unbounded domains is not true). This result stresses the importance of concepts as *convexity at a point* and *uniform lower semicontinuity.* These auxiliary concepts are defined in the paper and some of their properties, related to the proved result, are established.

Key words: *abstract convexity,* min-*type functions, convexity at a point, uniform lower semicontinuity*

Mathematics Subject Classification: *49J52, 49N15*

1 Introduction

Abstract convex analysis, grown after the monographs of Pallaschke, Rolewicz [12], Singer [19], and Rubinov [15], to a mathematical discipline with own face and problems, aims to generalize the results of *convex analysis* to abstract convex functions on the basis of global aspects of the subdifferential (the generalizations of the local aspects lead to *nonsmooth analysis*). Its importance is due to various applications to duality theory, optimization of nonconvex functions etc. After Singer [19] we notice that the problem of constructing a theory of nonconvex optimization, based on suitable extensions of the methods of (usual) convex analysis, has been attacked by several authors independently at about the same time, such as Kutateladze, Rubinov [8]–[9], Dolecki, Kurcyusz [4]–[5], Lindberg [10], Balder [1]. Some of the artifacts of abstract convex analysis have been developed on an earlier stage, say those of generalized convex sets (Danzer, Grünbaum, Klee [3], Fan [6]), of the conjugate of a function associated to arbitrary and not necessary bilinear coupling (Moreau [11], and others), and have been applied to nonconvex optimization, e. g. to develop a unified theory of augmented Lagrangians ([4], [5], [10], [1]).

As Singer [19] underlines, one of the main question in abstract convex analysis is the following: Given an arbitrary set X, which (extended real-valued) functions on X should be called convex? Once the abstract convex functions with respect to a given set of elementary functions H are defined, the problem is to characterize them.

In this paper we occupy with this problem in $\mathbb{R}^n$, when the set of the elementary functions H is the set of abstract affine functions corresponding to the set of abstract linear functions L consisting of the minima of k linear functions. This convexity is called $\mathcal{H}_k$-convexity and is proved to find many applications in optimization [15], say in the theory of extended Lagrange and extended penalty functions, star-shaped analysis, and to numerical methods in global (nonconvex and nonsmooth) optimization as say extensions of the cutting planes and branch-and-bound methods.

As a refinement of the $\mathcal{H}_k$-convexity we introduce in the next section (like in [2]) the notion of the $\mathcal{H}_k^0$-convexity. Theorem 3.1 in Section 3, which is our main result, gives a characterization (sufficient and necessary conditions) for the $\mathcal{H}_k^0$-convexity of a given function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ in terms of (global) calmness when $1 \le k \le n$. The case $k \ge n+1$ is discussed in Theorems 3.6 and 3.7.

Let us stress that for us the $\mathcal{H}_k^0$-convexity is a tool to study some aspects of the $\mathcal{H}_k$-convexity. Theorem 3.1 shows that the $\mathcal{H}_k^0$-convexity obeys (in comparison to $\mathcal{H}_k$-convexity) some analytic comfort. By analogy with (usual) convex analysis, one expects to associate the convexity properties considered here (the ones in terms of upper envelopes) rather to lower semicontinuity than to calmness. Section 4 gives an example showing that the substitution of calmness with lower semicontinuity in Theorem 3.1 makes the thesis false. However Theorem 4.6 shows that for functions with bounded domains the thesis remains true. Its proof requires some notions defined in the paper, namely the ones of *convexity at a point* and of *uniform lower semicontinuity.* Section 5 does few comments, among them concerning the difference between the $\mathcal{H}_k^0$-convexity and the $\mathcal{H}_k$-convexity.

The paper is related to the study on $\mathcal{H}_k$-convexity undertaken by Rubinov and Shveidel in [15], [16], [17] and [18]. The paper does not discuss abstract subdifferentiability associated to $\mathcal{H}_k^0$-convexity, though it would be worth to compare $\mathcal{H}_k^0$-convexity and $\mathcal{H}_k^0$-subdifferentiability. $\mathcal{H}_k^0$-subdifferentiability for positively homogeneous functions is discussed in [2] in the case $k = n$, and in [7] in the case $k < n$.

2 Preliminaries

Denote by $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$. Let X be a given set and H be a set of functions $h : X \to \mathbb{R}$. The set H is called the set of elementary functions. For a function $f : X \to \mathbb{R}_{+\infty}$ the support set of f with respect to H is defined by $\operatorname{supp}(f, H) = \{h \in H \mid h \le f\}$. Here $h \le f$ means $h(x) \le f(x)$ for all $x \in X$. The function f is called H-convex (convex with respect to H) at the point $x^0 \in X$ if $f(x^0) = \sup\{h(x^0) \mid h \in \operatorname{supp}(f, H)\}$. The function f is called H-convex, if it is H-convex at any point $x \in X$.

Often a set L is given, and H is defined by $H = \{h = \ell - c \mid \ell \in L,\ c \in \mathbb{R}\}$. Then L is called the set of abstract linear functions, and H the set of abstract affine functions. When L coincides with the set of the usual linear functions, then H is the set of the usual affine functions, and f is H-convex if and only if it is both convex in the usual sense and lower semicontinuous (lsc). So, the convexity with respect to a set of elementary functions generalizes the usual convexity, and opens the perspectives to extend the various application of usual convexity to abstract convexity (that is H-convexity) with an appropriate set of elementary functions H.

In the sequel we consider the case $X = \mathbb{R}^n$. Then $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ denotes the scalar product of the vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, and $\|x\| = \langle x, x\rangle^{1/2}$ denotes Euclidean norm of x.

We occupy in this paper with (abstract) convexity with respect to min-type functions. For a positive integer k we define the class of abstract linear functions $\mathcal{L}_k$ (min-type functions) as the set of the functionals $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(x) = \min_{1\le i\le k}\langle l^i, x\rangle$ for some $l^1, \dots, l^k \in \mathbb{R}^n$. The respective set of abstract affine functions is denoted by $\mathcal{H}_k := \{h = \ell - c \mid \ell \in \mathcal{L}_k,\, c \in \mathbb{R}\}$. The $\mathcal{H}_k$-convexity is referred to as convexity with respect to min-type functions.

Convexity with respect to min-type functions is studied in [15], [16], [17], [18]. The original aim to characterize the class of $\mathcal{H}_k$-convex functions, undergoes in this paper some change, as explained below. Let us underline that when $k \ge n+1$ the problem to characterize the $\mathcal{H}_k$-convex functions find a satisfactory solution in [15]. So, the interesting case is $k \le n$.

For a given function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ the problem to characterize the $\mathcal{H}_k$-convexity at $x^0 \in \mathbb{R}^n$ meets with some constructive difficulties, which is clarified in [2] for positively homogeneous (PH) functions. It is proved there (Theorem 6.1 in [2]) that every convex-along-lines (CAL) and globally calm PH function is $\mathcal{H}^0_n$-convex. The proof is rather constructive. The property of f to be CAL is necessary for f to be $\mathcal{H}^0_n$-convex. However it is not necessary for f to be $\mathcal{H}_n$-convex. The only thing we can affirm is that for any $\mathcal{H}_n$-convex PH function there exists a line through the origin along which it is CAL. However the constructive approach from Theorem 6.1 in [2] cannot be applied to prove that: *Every globally calm PH function being convex along certain line passing through the origin is $\mathcal{H}_n$-convex.* Moreover, such a claim turns to be wrong as the following example shows:

Example 2.1 ([2]). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1^2/\sqrt{x_1^2 + x_2^2}, & x_1 > 0, \\ 0, & x_1 = 0, \\ -2x_1^2/\sqrt{x_1^2 + x_2^2}, & x_1 < 0. \end{cases}$$

The function f is PH, globally calm, and convex along the line $(x_1, x_2) \mid x_1 = 0$, but it is not $\mathcal{H}_2$-convex.

So, instead of looking for characterization of the $\mathcal{H}_k$-convexity at $x^0 \in \mathbb{R}^n$, we look for characterization of the $\mathcal{H}^0_k$-convexity defined below. Before giving the formal definition of $\mathcal{H}^0_k$-convexity, let us mention that any $\mathcal{H}^0_k$-convex function at a given point x^0 is also $\mathcal{H}_k$-convex. Hence, every sufficient condition for $\mathcal{H}^0_k$-convexity of f at x^0 is also sufficient for $\mathcal{H}_k$-convexity. The importance of $\mathcal{H}^0_k$-convexity we see in the possibility to characterize it, that is to find a condition both sufficient and necessary (characterization) for a function f to be $\mathcal{H}^0_k$-convex at a given point x^0. The importance of the characterization as a necessary condition for $\mathcal{H}^0_k$-convexity in relation to $\mathcal{H}_k$-convexity is that any function f which either is not $\mathcal{H}_k$-convex, or is $\mathcal{H}_k$-convex but not $\mathcal{H}^0_k$-convex, for sure should break the characterization property. For instance, the function from Example 2.1 being not $\mathcal{H}_2$-convex, is in virtue of Theorem 6.1 in [2] automatically not CAL.

Thus, we will look for characterization of the $\mathcal{H}^0_k$-convexity defined as follows. Let $x^0 \in \mathbb{R}^n$. Define the set of abstract linear functions $\mathcal{L}^0_k(x^0)$ of all $\ell \in \mathcal{L}_k$, $\ell(x) = \min_{1\le i\le k}\langle l^i, x\rangle$, having the property

$$\langle l^1, x^0\rangle = \langle l^2, x^0\rangle = \cdots = \langle l^k, x^0\rangle \quad (= \ell(x^0))\,. \tag{2.1}$$

The respective set of abstract affine functions is $\mathcal{H}^0_k(x^0) = \{h = \ell - c \mid \ell \in \mathcal{L}^0_k(x^0),\, c \in \mathbb{R}\}$. We say that f is $\mathcal{H}^0_k$-convex at x^0 if it is $\mathcal{H}^0_k(x^0)$-convex at x^0. We say that f is *$\mathcal{H}^0_k$-convex* if it is $\mathcal{H}^0_k$-convex at any $x^0 \in \mathbb{R}^n$. Let us underline, that while for a function f the $\mathcal{H}^0_k$-convexity at a given x^0 follows the usual definitions of abstract convex analysis with $\mathcal{H}^0_k(x^0)$ as an underlying set of elementary functions, the definition of $\mathcal{H}^0_k$-convexity of f (without specifying a concrete point x^0) is not based on an underlying set of elementary functions $\mathcal{H}^0_k$

and therefore does not follow the usual definition of abstract convex analysis. The notion of $\mathcal{H}_k^0$-convexity can be considered as some tool to study $\mathcal{H}_k$-convexity. In fact f is $\mathcal{H}_k^0$-convex when it is $\mathcal{H}_k$-convex and for all $x^0 \in \mathbb{R}^n$ the representation $f(x^0) = \sup\{h(x^0) \mid h \in \operatorname{supp}(\mathbb{R}^n, \mathcal{H}_k)\}$ holds also when restricting the choice to $h \in \operatorname{supp}(\mathbb{R}^n, \mathcal{H}_k^0(x^0))$ (observe that $\mathcal{H}_k^0(x^0) \subset \mathcal{H}_k$).

Let us stress the following simple assertion:

Proposition 2.2. *If $k_1 < k_2$ and $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{H}_{k_1}^0$-convex (at $x^0 \in \mathbb{R}^n$), then f is also $\mathcal{H}_{k_2}^0$-convex (at x^0).*

Proof. We have $\mathcal{L}_{k_1}^0(x^0) \subset \mathcal{L}_{k_2}^0(x^0)$. Indeed, let $\ell \in \mathcal{L}_{k_1}^0$ with $\ell(x) = \min_{1 \le i \le k_1} \langle l^i, x\rangle$ satisfying (2.1) with $k = k_1$. Now put $l^i = l^{k_1}$ for $i = k_1 + 1, \dots, k_2$. Obviously, now $\ell(x) = \min_{1 \le i \le k_2} \langle l^i, x\rangle$ and (2.1) holds with $k = k_2$. Therefore $\ell \in \mathcal{L}_{k_2}^0(x^0)$. Similarly, we have $\mathcal{H}_{k_1}^0(x^0) \subset \mathcal{H}_{k_2}^0(x^0)$, which follows from

$$\mathcal{H}_{k_1}^0(x^0) = \{\ell - c \mid \ell \in \mathcal{L}_{k_1}^0(x^0),\, c \in \mathbb{R}\} \subset \{\ell - c \mid \ell \in \mathcal{L}_{k_2}^0(x^0),\, c \in \mathbb{R}\} = \mathcal{H}_{k_2}^0(x^0).$$

Assume that f is $\mathcal{H}_{k_1}^0$-convex at x^0. Then

$$\begin{aligned} f(x^0) &= \sup\{h(x^0) \mid h \in \operatorname{supp}(f, \mathcal{H}_{k_1}^0(x^0))\} \\ &\le \sup\{h(x^0) \mid h \in \operatorname{supp}(f, \mathcal{H}_{k_2}^0(x^0))\} \le f(x^0)\,. \end{aligned}$$

Hence there are equalities everywhere in this chain of inequalities. In particular

$$f(x^0) = \sup\{h(x^0) \mid h \in \operatorname{supp}(f, \mathcal{H}_{k_2}^0(x^0))\}\,,$$

that is f is $\mathcal{H}_{k_2}^0$-convex at x^0. □

In the sequel we use the notion of calmness, originating from [14]. The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called calm at the point $x^0 \in \operatorname{dom} f$ if

$$\operatorname{Calm} f(x^0) := \inf\left\{\frac{f(x) - f(x^0)}{\|x - x^0\|} \mid x \in \mathbb{R}^n,\, x \ne x^0\right\} > -\infty\,.$$

The quantity $\operatorname{Calm} f(x^0)$ is called the calmness of f at x^0.

3 $\mathcal{H}_k^0$-convexity at a Point

Given a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$, a point $x^0 \in \mathbb{R}^n$, a number $c \in \mathbb{R}$, an m-dimensional subspace $L \subset \mathbb{R}^n$, and a vector $\zeta \in \mathbb{R}^n$, we introduce the function

$$f_{x^0,c,L,\zeta}(x) = \begin{cases} c + \langle \zeta,\, z\rangle\,, & x = x^0 + z,\ z \in L\,, \\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

Let us underline that from here on L means, as said above, a linear subspace of $\mathbb{R}^n$. This is a traditional notation in convex analysis [13] for linear subspaces. We do this remark to avoid eventual confusion with the notation L used in Section 2 to denote the set of abstract linear functions. The notation L in this context is traditional in abstract convex analysis [15]. Since in the present paper the two notations are "spacially" separated, the one used up to here, and the other from here on, we think after this warning there is no basis for confusion.

With the function f we relate the following condition:

$$\mathbb{C}(f, x^0, c, L, \zeta): \qquad \inf_{z \in L} \operatorname{Calm} f_{x^0,c,L,\zeta}(x^0 + z) > -\infty\,.$$

Given a function $g : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and a subspace $L \subset \mathbb{R}^n$, we denote by $g|_L$ the restriction of g on L. The dual space of L is denoted by L^* (that is L^* stands for the set of the linear functionals on L). The notation $\partial g|_L(x^0)$ denotes the subdifferential of $g|_L$ at $x^0 \in L$, that is

$$\partial g|_L(x^0) = \{\ell^* \in L^* \mid g|_L(x) \ge \ell^*(x) - \ell^*(x^0) + g|_L(x^0) \text{ for all } x \in L\}\,.$$

The elements $\ell^* \in \partial g|_L(x^0)$ are called subgradients of $g|_L$ at x^0. From the representation $\ell^*(x) = \langle \zeta, x\rangle$ with appropriate $\zeta \in \mathbb{R}^n$ we can identify the functional ℓ^* with the vectors ζ, considering equivalent any two vectors ζ^1, ζ^2, whose difference $\zeta^1 - \zeta^2$ is orthogonal to L (in order that the representation $\ell^*(x) = \langle \zeta, x\rangle$ is unique, we can restrict the considerations only to vectors $\zeta \in L$). On the basis of this identification the subdifferential $\partial g|_L(x^0)$ is considered as a set of vectors $\zeta \in \mathbb{R}^n$. In this sense further we use to write $\zeta \in \partial g|_L(x^0)$.

In the sequel we will also consider the function $f_{x^0,c}$ with $x^0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$, defined by

$$f_{x^0,c}(x) = \begin{cases} c\,, & x = x^0\,, \\ f(x)\,, & \text{otherwise}\,. \end{cases}$$

The following theorem characterizes the $\mathcal{H}_k^0$-convexity at a point x^0 in the case $1 \le k \le n$.

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and $x^0 \in \mathbb{R}^n$. Let k be integer with $1 \le k \le n$. Then f is $\mathcal{H}_k^0$-convex at x^0 if and only if for any $c < f(x^0)$ there exists a $(n+1-k)$-dimensional subspace $L \subset \mathbb{R}^n$ with $x^0 \in L$, and there exists $\zeta \in \partial f_{x^0,c}|_L(x^0)$, such that condition $\mathbb{C}(f, x^0, c, L, \zeta)$ is satisfied.*

Proof. Necessity. Let f be $\mathcal{H}_k^0$-convex at $x^0 \in \mathbb{R}^n$, and let $c < f(x^0)$ be a real. Then there exists $h \in \operatorname{supp}(f, \mathcal{H}_k^0(x^0))$ with $h(x^0 \ge c)$. We can assume without loss of generality that $h(x^0) = c$. Indeed, when $h(x^0) > c$, we can substitute $h(x)$ by $\bar h(x) = h(x) - (c - h(x^0))$. Obviously $\bar h(x^0) = c$ and $\bar h \in \operatorname{supp}(f, \mathcal{L}_k^0(x^0))$, the latter follows from $\bar h(x) < h(x) \le f(x)$. Let $h = \ell - \gamma$, where $\ell \in \mathcal{L}_k^0(x^0)$ and $\gamma \in \mathbb{R}$. From $c = h(x^0) = \ell(x^0) - \gamma$ we get $\gamma = \ell(x^0) - c$, that is $h(x) = \ell(x) - \ell(x^0) + c$. Suppose that $\ell(x) = \min_{1 \le i \le k} \langle l^i, x\rangle$ and (2.1) holds. Actually the equalities

$$\langle l^1, x\rangle = \langle l^2, x\rangle = \ldots \langle l^k, x\rangle \tag{3.1}$$

form a homogeneous linear system with $k-1$ equations, hence of rank at most $k-1$. Accounting, due to (2.1), that x^0 solves this system, we can find a $(n+1-k)$-dimensional subspace $L \subset \mathbb{R}^n$ such that $x^0 \in L$ and any $x \in L$ solves (3.1). Obviously, the restriction of ℓ on L is linear, so we can write $\ell^* := \ell|_L \in L^*$. The inequality $f(x) \ge h(x) = \ell(x) - \ell(x^0) + c$, $x \in \mathbb{R}^n$, restricted to L gives

$$f_{x^0,c}(x) - f_{x^0,c}(x^0) \ge \ell^*(x) - \ell^*(x^0)\,, \quad x \in L\,,$$

whence $\ell^* \in \partial f_{x^0,c}|_L(x^0)$. Using this inequality and the representation $\ell^*(x) = \langle \zeta, x\rangle$, $x \in L$, for $z \in L$ we get

$$\begin{aligned}
\operatorname{Calm} f_{x^0,c,L,\zeta}(x^0+z) &= \inf_{\substack{x\in\mathbb{R}^n\\ x\neq x^0+z}} \frac{f_{x^0,c,L,\zeta}(x) - f_{x^0,c,L,\zeta}(x^0+z)}{\|x-x^0-z\|} \\
\geq \min\left(\inf_{\substack{x\in\mathbb{R}^n\\ x\neq x^0+z}} \frac{f_{x^0,c}(x) - c - \langle\zeta,\, z\rangle}{\|x-x^0-z\|}\right. &, \left. \inf_{\substack{x=x^0+w,\, w\in L\\ w\neq z}} \frac{f_{x^0,c,L,\zeta}(x) - c - \langle\zeta,\, z\rangle}{\|x-x^0-z\|}\right) \\
\geq \min\left(\inf_{\substack{x\in\mathbb{R}^n\\ x\neq x^0+z}} \frac{\ell(x)-\ell(x^0) - \langle\zeta,\, z\rangle}{\|x-x^0-z\|}\right. &, \left. \inf_{\substack{w\in L\\ w\neq z}} \frac{\langle\zeta, w-z\rangle}{\|w-z\|}\right) \\
&\geq \min\left(-\max_{1\leq i\leq k}\|l^i\|\,,\ -\|\zeta\|\right).
\end{aligned} \tag{3.2}$$

The right hand side of this inequality is finite and does not depend on z, whence condition $\mathbb{C}(f,x^0,c,L,\zeta)$ is satisfied.

Sufficiency. Let $c < f(x^0)$. Due to condition $\mathbb{C}(f,x^0,c,L,\zeta)$ there exists a constant $C > 0$ such that

$$\inf_{z\in L} \operatorname{Calm} f_{x^0,L,c,\zeta}(x^0+z) \geq -C > -\infty\,. \tag{3.3}$$

Consider the subspace $M = \{x \in \mathbb{R}^n \mid \langle l,\, x\rangle = 0 \text{ for all } l\in L\}$ of $\mathbb{R}^n$ orthogonal to the subspace L. Since M is a $(k-1)$-dimensional subspace, we can find k vectors $m^1,\dots,m^k$ such that their convex hull S, which is a simplex, contains the ball $B=\{x\in M : \|x\|\leq 1\}$. Let $q(x) = \max_{1\leq i\leq k}\langle m^i, x\rangle$ be the support function of S. Since $S \supset B$ and the support function of B is equal to $\|x\|$, it follows that

$$q(x) = \max_{1\leq i\leq k}\langle m^i, x\rangle \geq \|x\|, \quad x\in M\,. \tag{3.4}$$

Fix $x\in\mathbb{R}^n$ and let $\bar{x}$ be the orthogonal projection of x on L. Then

$$\bar{x} = \sum_{i=1}^{n+1-k} \langle u^i, x\rangle\, u^i\,,$$

where $\{u^1,\dots,u^{n+1-k}\}$ is an orthonormal basis of L. Since $\bar{x} = x^0 + (\bar{x}-x^0) \in x^0 + L$, we have

$$f_{x^0,c,L,\zeta}(\bar{x}) = c + \langle\zeta,\, \bar{x}-x^0\rangle\,. \tag{3.5}$$

Since $\bar{x}\in L$, from (3.3) we have

$$f_{x^0,c,L,\zeta}(x) - f_{x^0,c,L,\zeta}(\bar{x}) \geq -C\ \|x-\bar{x}\|\,.$$

Due to (3.4) and $x-\bar{x}\in M$ we get

$$\|x-\bar{x}\| \leq \max_{1\leq i\leq k}\langle m^i, x-\bar{x}\rangle\,,$$

so that

$$f_{x^0,c,L,\zeta}(x) - f_{x^0,c,L,\zeta}(\bar{x}) \geq -C\,\|x-\bar{x}\| \geq -C\ \max_{1\leq i\leq k}\langle m^i, x-\bar{x}\rangle\,.$$

Since $m^i\in M$, $i=1,\dots,k$, and $\bar{x}$ belongs to the subspace L being orthogonal to M, it follows that $\langle m^i,\bar{x}\rangle = 0$ for $i=1,\dots,k$. Using these equalities and (3.5) we obtain

$$f_{x^0,c}(x) \geq f_{x^0,c,L,\zeta}(x) = \big(f_{x^0,c,L,\zeta}(x) - f_{x^0,c,L,\zeta}(\bar{x})\big) + f_{x^0,c,L\zeta}(\bar{x})$$

$$\geq -C \max_{1\leq i\leq k} \langle m^i, x\rangle + c + \langle \zeta,\, \bar{x} - x^0\rangle$$

$$\geq -C \max_{1\leq i\leq k} \langle m^i, x\rangle + c + \sum_{i=1}^{n+1-k} \langle u^i,\, x\rangle \langle \zeta,\, u^i\rangle - \langle \zeta,\, x^0\rangle\,,$$

or equivalently

$$f_{x^0,c}(x) - f_{x^0,c}(x^0) \geq \min_{1\leq i\leq k} \langle -C\, m^i + \sum_{i=1}^{n+1-k} \langle \zeta,\, u^i\rangle\, u^i\,,\, x\,\rangle - \langle \zeta,\, x^0\rangle\,. \tag{3.6}$$

Here we have used the inequality $f_{x^0,c}(x) \geq f_{x^0,c,L,\zeta}(x)$ which needs to be explained only when $x = x^0 + z$, $z \in L$. Then this inequality reduces to

$$f_{x^0,c}(x^0+z) - f_{x^0,c}(x^0) \geq \langle \zeta,\, z\rangle\,, \quad z \in L\,,$$

which is true by definition, since $\zeta \in \partial f_{x^0,c}|_L(x^0)$ is a subgradient of the function $f_{x^0,c}|_L$.

Put now

$$l^i = -C\, m^i + \sum_{i=1}^{n+1-k} \langle \zeta,\, u^i\rangle\, u^i\,, \quad i = 1,\ldots,k\,,$$

and observe that these vectors do not depend on x (from here on x could be considered an arbitrary vector). Define the functional $\ell : \mathbb{R}^n \to \mathbb{R}$ by $\ell(x) = \min_{1\leq i\leq k}\langle l^i, x\rangle$. We have obviously

$$\langle l^i, x^0\rangle = \langle -C\, m^i\,,\, x^0\,\rangle + \sum_{i=1}^{n+1-k} \langle \zeta,\, u^i\rangle \langle u^i,\, x^0\rangle$$

$$= \langle \sum_{i=1}^{n+1-k} \langle \zeta,\, u^i\rangle\, u^i\,,\, x^0\,\rangle = \langle \bar{\zeta},\, x^0\rangle\,, \quad i = 1,\ldots,k\,,$$

where $\bar{\zeta} = \sum_{i=1}^{n+1-k}\langle \zeta,\, u^i\rangle\, u^i$ is the orthogonal projection of ζ on L. These equalities show that $\ell \in \mathcal{L}^0_k(x^0)$ and

$$\ell(x^0) = \langle \bar{\zeta},\, x^0\rangle = \langle \zeta,\, x^0\rangle - \langle \zeta - \bar{\zeta},\, x^0\rangle = \langle \zeta,\, x^0\rangle\,.$$

Now inequality (3.6) can be written as

$$f(x) \geq \ell(x) - \ell(x^0) + c\,, \quad \forall\, x \in \mathbb{R}^n\,,$$

which shows that the function $h = \ell - \ell(x^0) + c \in \mathcal{H}^0_k(x^0)$ belongs to $\operatorname{supp}(f, \mathcal{H}^0_k(x^0))$. With account that $h(x^0) = c$ and $c < f(x^0)$ arbitrary, we see that $f(x^0) = \sup\{h(x^0) \mid h \in \operatorname{supp}(f, \mathcal{H}^0_k(x^0))\}$, that is f is $\mathcal{H}^0_k$-convex at x^0. □

In the previous theorem the hypotheses involve the non-emptyness of the subdifferential $\partial g(x^0)$ where $g = f|_{x^0,c}$. The question, when this subdifferential is not empty, leads to the notion of a function convex at a point, which will be introduced next.

Let L be a m-dimensional linear space, $g : L \to \mathbb{R}_{+\infty}$ a given function, and $x^0 \in L$. We will say that g is convex at x^0, if $g(x^0) = (\operatorname{conv} g)(x^0)$. The equality $\operatorname{conv} g = \operatorname{conv}\{g_x \mid x \in L\}$, where $g_x(y) = g(x)$ for $y = x$ and $g_x(y) = +\infty$ for $y \in L \setminus \{x\}$, represents $\operatorname{conv} g$ as a convex hull of the family of convex functions $\{f_x \mid x \in L\}$. Hence according to Theorem 5.6 in [13] we have $(\operatorname{conv} g)(x) = \inf\{\sum_{y\in L} \lambda_y g(y) \mid \sum_{y\in L} \lambda_y y = x\}$, where the infimum is taken over all representation of x as a convex combination of $y \in L$, such that only finitely many coefficients λ_y are non-zero. The formula is also valid if one actually restricts y to lie in $\operatorname{dom} g$. Further, due to Carathéodory Theorem (Theorem 17.1 in [13]), we can confine to convex combinations of at most $m+1$ elements. Therefore it holds:

Proposition 3.2. *The function $g : L \to \mathbb{R}_{+\infty}$ (L is m-dimensional linear space) is convex at $x^0 \in L$ if and only if $g(x^0) = \inf \{\sum_{i=1}^{m+1} \lambda_{y^i} g(y^i) \mid \sum_{i=1}^{m+1} \lambda_{y^i} y^i = x^0\}$.*

Now we discuss the non-emptyness of the subdifferential $\partial g(x^0)$.

Proposition 3.3. *Let $g : L \to \mathbb{R}_{+\infty}$ (L is m-dimensional linear space) be a proper function (that is $\operatorname{dom} f \neq \emptyset$) and let $\partial g(x^0) \neq \emptyset$ for some $x^0 \in L$. Then $x^0 \in \operatorname{dom} g$, g is convex at x^0, and $\operatorname{conv} g$ is lsc at x^0. In such a case it holds $\partial g(x^0) = \partial(\operatorname{conv} g)(x^0) = \partial(\operatorname{cl} \operatorname{conv} g)(x^0)$.*

Proof. Let $\zeta \in \partial g(x^0)$ and $y \in \operatorname{dom} g$. Then

$$g(x^0) \le g(y) - \langle \zeta, y\rangle + \langle \zeta, x^0\rangle < +\infty\,.$$

Varying y in the above inequality and applying convex combinations we get

$$g(x^0) \le \sum_y \lambda_y g(y) - \langle \zeta, \sum_y \lambda_y y\rangle + \langle \zeta, x^0\rangle\,. \tag{3.7}$$

When $\sum_y \lambda_y y = x^0$, this gives $g(x^0) \le \sum_y \lambda_y g(y)$, which proves that g is convex at x^0. Inequality (3.7) shows that $\partial g(x^0) \subset \partial(\operatorname{conv} g)(x^0)$. The opposite inclusion is obvious, whence $\partial g(x^0) = \partial(\operatorname{conv} g)(x^0)$. Hence, the convex function $\operatorname{conv} g$ is subdifferentiable at x^0. Then according to Corollary 23.5.2 in [13] $\operatorname{conv} g$ is lsc at x^0 and $\partial(\operatorname{conv} g)(x^0) = \partial(\operatorname{cl} \operatorname{conv} g)(x^0)$. □

Now we consider conditions implying $\partial g(x^0) \neq \emptyset$. According to Proposition 3.3, convexity of g at x^0 is necessary for this.

Proposition 3.4. *Let $g : L \to \mathbb{R}_{+\infty}$ (L is m-dimensional linear space) be convex at x^0. If $x^0 \notin \operatorname{dom} g$, then $\partial g(x^0) = \emptyset$. If $x^0 \in \operatorname{ri} \operatorname{dom} g$ (ri stands for relative interior), then $\partial g(x^0) \neq \emptyset$.*

Proof. Like in Proposition 3.3, we see that convexity of g at x^0 implies $\partial g(x^0) = \partial(\operatorname{conv} g)(x^0)$. Now the thesis follows from Theorem 23.4 in [13] applied to the convex function $\operatorname{conv} g$. □

The function $g : \mathbb{R} \to \mathbb{R}_{+\infty}$, given by $g(x) = -\sqrt{1 - x^2}$ for $|x| \le 1$ and $g(x) = +\infty$ for $|x| > 1$, is convex and lsc, but not subdifferentiable at ± 1. So, at the points of the relative boundary of $\operatorname{dom} g$, the convexity and the lower semicontinuity does not imply subdifferentiability. However, with regard to the hypotheses of Theorem 3.1 where the function $g = f_{x^0,c}|_L$ is of importance, the following holds:

Proposition 3.5. *Let $g : L \to \mathbb{R}_{+\infty}$ (L is m-dimensional linear space) be convex at $x^0 \in L$. Let $c < g(x^0)$. Then $g_{x^0,c}$ is convex at x^0. If in addition $x^0 \in \operatorname{dom} g$ and $\operatorname{conv} g$ is lsc at x^0, then $\partial g_{x^0,c}(x^0) \neq \emptyset$. The same is true, if the condition "$x^0 \in \operatorname{dom} g$" is substituted by "$g$ bounded from below".*

Proof. Consider a convex combination $\sum_y \lambda_y y = x^0$. One of the following cases can have place:

1^0. $\lambda_{x^0} = 1$. Then $\lambda_y = 0$ for $y \neq x^0$ and

$$\sum_y \lambda_y\, g_{x^0,c}(y) = g_{x^0,c}(x^0) = c\,.$$

2^0. $\lambda_{x^0} < 1$. Then

$$\begin{aligned}\sum_y \lambda_y\, g_{x^0,c}(y) &= \sum_{y\neq x^0} \lambda_y\, g_{x^0,c}(y) + \lambda_{x^0} c \\ &= (1-\lambda_{x^0}) \sum_{y\neq x^0} \frac{\lambda_y}{1-\lambda_{x^0}}\, g(y) + \lambda_{x^0} c \\ &\geq (1-\lambda_{x^0}) + \lambda_{x^0} c = c\,.\end{aligned}$$

We have used the convex combination

$$\sum_{y\neq x^0} \frac{\lambda_y}{1-\lambda_{x^0}}\, y = \frac{1}{1-\lambda_{x^0}}\, \left(x^0 - \lambda_{x^0}\, x^0\right) = x^0\,.$$

In both cases we obtain $g_{x^0,c}(x^0) = c \leq \sum_y \lambda_y g_{x^0,c}(y)$. Therefore $g_{x^0,c}$ is convex at x^0.

Let $x^0 \in \operatorname{dom} g$ and $h := \operatorname{cl}\operatorname{conv} g$. From the hypotheses we have $h(x^0) = g(x^0) < +\infty$. The epigraph $H_+ := \operatorname{epi} h = \{(x,r) \in L\times\mathbb{R} \mid h(x) \geq r\}$ is closed and convex, and does not contain the point (x^0, c). Put $H_- = \{(x^0,c)\}$. The Separation Theorem guarantees the existence of $(\xi,\eta) \in L\times\mathbb{R}$ and $\gamma\in\mathbb{R}$, such that

$$\langle \xi, x\rangle + \eta\, r \geq \gamma \quad \text{for} \quad (x,r) \in H_+\,, \tag{3.8}$$

$$\langle \xi, x\rangle + \eta\, r \leq \gamma \quad \text{for} \quad (x,r) \in H_-\,. \tag{3.9}$$

Since $(x^0, g(x^0))$ satisfies (3.8), and with account of (3.9), we get $\eta \neq 0$. Further, since $(x^0, r) \in H_+$ for all $r \geq g(x^0)$, we get $\eta > 0$. Dividing by η and putting $\zeta = -\xi/\eta$ and $r = g(x)$ in (3.9), we get

$$\langle -\zeta, x\rangle + g(x) \geq \frac{\gamma}{\eta} \geq \langle -\zeta, x^0\rangle + c\,, \tag{3.10}$$

which shows that $\zeta \in \partial g_{x^0,c}(x^0)$.

Let us have now g bounded from below instead of $x^0 \in \operatorname{dom} g$. Suppose that $g(x) \geq \mu > -\infty$ for all $x\in L$. Obviously then also $h(x) \geq \mu$, $\forall x \in L$. We may assume that $c > \mu$. For $\delta > 0$ define the set

$$H_\delta = \left\{ (x,r) \in L\times\mathbb{R} \mid \|x - x^0\| \leq \delta\, \frac{c-r}{c-\mu}\,,\ r \leq c \right\}.$$

We claim that for some $\delta > 0$ the convex set H_δ does not intersect the set $H_+ = \operatorname{epi} h$. If this were not the case, we would have a sequence $x^n \in L$ with $h(x^n) < c$ and

$$\|x^n - x^0\| \leq \frac{1}{n}\ \frac{c - h(x^n)}{c-\mu} \leq \frac{1}{n}\,.$$

This shows that $x^n \to x^0$. From the closedness of h we get $g(x^0) = h(x^0) \leq \liminf_n h(x^n) \leq c$. This contradicts the hypothesis $c < g(x^0)$. Put now $H_- = H_\delta$ for $\delta > 0$ chosen so that H_δ and H_+ are disjoint. The Separation Theorem gives the existence of $(\xi,\eta) \in L\times\mathbb{R}\setminus\{(0,\,0)\}$ and $\mu\in\mathbb{R}$ such that (3.8) and (3.9) hold. If $e \in L$, $\|e\| \leq \delta$, then $\left(\frac{c-r}{c-\mu}\, e + x^0,\, r\right) \in H_-$ for all $r \leq c$. According to (3.9) we have

$$\frac{c-r}{c-\mu}\, \langle \xi, e\rangle + \langle \xi,\, x^0\rangle + \eta\, r \leq \gamma\,. \tag{3.11}$$

From (3.11) we get $\eta \neq 0$. Otherwise $\xi \neq 0$, and with regard that also $\|-e\| \le \delta$, we should have both

$$\frac{c-r}{c-\mu}\,\langle \xi, e\rangle + \langle \xi, x^0\rangle \le \gamma \quad \text{and} \quad -\,\frac{c-r}{c-\mu}\,\langle \xi, e\rangle + \langle \xi, x^0\rangle \le \gamma$$

for all $r \le c$, which is not possible if e is chosen so that $\langle \xi, e\rangle \neq 0$. Having obtained $\eta \neq 0$, we claim that also $\eta > 0$. To show this observe that $(x^0, r) \in H_-$ for all $r \le c$, which leads to the inequality $\langle \xi, x^0\rangle + \eta\, r \le \gamma$, which cannot be true for all $r \le c$ if it were $\eta < 0$.

Take in (3.9) the point $(x, r) = (x^0, c)$ and put $\zeta = -\xi/\eta$. Dividing by η we get the inequality (3.10) which shows that $\zeta \in \partial g_{x^0,c}(x^0)$. $\square$

In the next two theorems we characterize $\mathcal{H}_k^0$-convexity at a point x^0 in the case $k \ge n+1$. Theorem 3.6 deals with the case $x^0 = 0$ and Theorem 3.7 with the case $x^0 \neq 0$.

Theorem 3.6. *The function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{H}_k^0$-convex at 0 with $k = n+1$ if and only if for all $c < f(0)$ it holds $\operatorname{Calm} f_{0,c}(0) > -\infty$ (the necessity is true for arbitrary positive integer k). For $k > n+1$, the function f is $\mathcal{H}_k^0$-convex at 0 if and only if it is $\mathcal{H}_{n+1}^0$-convex at 0.*

Proof. Necessity. Let f be $\mathcal{H}_k^0$-convex at 0 with k arbitrary positive integer, and let $c < f(0)$. Then there exists $h \in \mathcal{H}_k^0(0)$ such that $h \in \operatorname{supp}(f, \mathcal{H}_k^0(0))$ and $h(0) > c$. Without loss of generality we can assume $h(0) = c$ (this was explained in the proof of Theorem 3.1). Then $h(x) = \ell(x) - \ell(0) + c = \ell(x) + c$ for some $\ell \in \mathcal{L}_k^0(0)$. Let $\ell(x) = \min_{1\le i\le k}\langle l^i, x\rangle$. Then

$$\operatorname{Calm} f_{0,c}(0) = \inf_{x\neq 0} \frac{f_{0,c}(x) - f_{0,c}(0)}{\|x\|} \ge \inf_{x\neq 0}\frac{h(x)-c}{\|x\|} = \inf_{x\neq 0}\frac{\ell(x)}{\|x\|}$$

$$= \inf_{x\neq 0}\frac{\min_{1\le i\le k}\langle l^i, x\rangle}{\|x\|} \ge -\max_{1\le i\le k}\|l^i\| > -\infty\,.$$

Sufficiency Consider the case $k = n+1$. Let $c < f(0)$ and $C > 0$ be such that $\operatorname{Calm} f_{0,c}(0) > -C$. Take the vectors $m^1, \dots, m^{n+1}$ such that their convex hull S contains the ball $B = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$. Let $q(x) = \max_{1\le i\le k}\langle m^i, x\rangle$ be the support function of S. Since $S \supset B$ and the support function of B is equal to $\|x\|$ it follows that

$$q(x) = \max_{1\le i\le k}\langle m^i, x\rangle \ge \|x\|, \quad x \in \mathbb{R}^n\,.$$

Put $l^i = -Cm^i$, $i = 1, \dots, n+1$, and $\ell(x) = \min_{1\le i\le n+1}\langle l^i, x\rangle$. Obviously

$$\langle l^1, 0\rangle = \langle l^2, 0\rangle = \cdots = \langle l^{n+1}, 0\rangle = 0 \quad (= \ell(0))\,,$$

whence $\ell \in \mathcal{L}_{n+1}^0(0)$. Now

$$\ell(x) = \min_{1\le i\le n+1}\langle l^i, x\rangle = -C \max_{1\le i\le n+1}\langle m^i, x\rangle \le -C\,\|x\| \le f_{0,c}(x) - c \le f(x) - c\,.$$

This shows that $h = \ell + c \in \operatorname{supp}(f, \mathcal{H}_{n+1}^0(0))$. Since $c < f(0)$ is arbitrary, $h(0) = \sup\{h(0) \mid h \in \operatorname{supp}(f, \mathcal{H}_{n+1}^0(0))\}$, that is f is $\mathcal{H}_{n+1}^0$-convex at 0.

Let now $k > n+1$. If f is $\mathcal{H}_k^0$-convex at 0, then, as proved in the necessity, the function $f_{0,c}$ is calm at 0 for any $c < f(0)$. This implies, as proved in the sufficiency, that f is $\mathcal{H}_{n+1}^0$-convex at 0. Conversely, if f is $\mathcal{H}_{n+1}^0$-convex at 0, then on the basis of Proposition 2.2 it is also $\mathcal{H}_k^0$-convex at 0. $\square$

Theorem 3.7. *For $k \geq n+1$, the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{H}_k^0$-convex at $x^0 \neq 0$, if and only if it is $\mathcal{H}_n^0$-convex at x^0.*

Proof. Necessity Let f be $\mathcal{H}_k^0$-convex at x^0, and let $c < f(x^0)$. Then there exists $h \in \operatorname{supp}(f, \mathcal{H}_k^0(x^0))$ with $h(x^0) \geq c$. Without loss of generality we can assume $h(x^0) = c$. Let $h(x) = \ell(x) - \ell(x^0) + c$, where $\ell \in \mathcal{L}_k^0(x^0)$. Let $\ell(x) = \min_{1 \leq i \leq k} \langle l^i, x\rangle$ with $\langle l^i, x\rangle = \ell(x^0)$ for $i = 1, \dots, k$. Put $L = \{tx^0 \mid t \in \mathbb{R}\}$ and consider the function $f_{x^0,c,L,\zeta}$ defined with $\zeta \in \mathbb{R}^n$ such that $\langle \zeta, x\rangle = \ell(x)$ for $x \in L$ (observe that $\ell|_L$ is a linear functional). Applying the estimations (3.2) from the necessity of Theorem 3.1 with $z \in L$ we get

$$\operatorname{Calm} f_{x^0,c,L,\zeta}(x^0 + z) \geq \min\left(-\max_{1 \leq i \leq k} \|l^i\|\,,\ -\|\zeta\|\right),$$

where the right hand side is finite and does not depend on z. Therefore condition $\mathbb{C}(f, x^0, c, L, \zeta)$ is satisfied. Now on the basis of the sufficiency of Theorem 3.1, we get that f is $\mathcal{H}_n^0$-convex at x^0. Conversely, if f is $\mathcal{H}_n^0$-convex at x^0, then on the basis of Proposition 2.2 it is also $\mathcal{H}_k^0$-convex at x^0 with $k \geq n+1$. □

4 $\mathcal{H}_k^0$-convexity and Lower Semicontinuity

The following proposition associates the $\mathcal{H}_k^0$-convexity and lower semicontinuity.

Proposition 4.1. *If the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{H}_k^0$-convex at $x^0 \in \mathbb{R}^n$ then it is lsc at x^0.*

Proof. The function f is lsc at x^0, since it holds $f(x^0) = \sup\{h(x^0) \mid h \in \operatorname{supp}(f, \mathcal{H}_k^0(x^0)\}$ and the functions in $\mathcal{H}_k^0(x^0)$ are continuous. □

In this paper we deal mainly with the case $k \leq n$. To avoid confusion due to this fact, let us specially underline that Proposition 4.1 is true for arbitrary positive integer k.

Now it is natural to pose the question, whether a lsc function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is $\mathcal{H}_k^0$-convex. From convex analysis we know that this is the case when $k = 1$ (each lsc convex function is an upper envelope of affine functions). However the following example shows that the things look different when $k \geq 2$.

Example 4.2. The function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \begin{cases} -\sqrt{|x_1 x_2|}\,, & x_1 \geq 0, \\ \sqrt{|x_1 x_2|}\,, & x_1 < 0, \end{cases}$$

is continuous (hence lsc), but it is not $\mathcal{H}_2^0$-convex at the nonzero points of the coordinate axes.

To fix the attention, consider the point $x^0 = (a, 0)$, $a > 0$. Let $c < f(x^0) = 0$. Now $L = \{tx^0 \mid t \in \mathbb{R}\}$ is the unique 1-dimensional space containing x^0. If $\zeta \in \partial f_{x^0,c}(x^0)$, $\zeta \in L$, then $\zeta = 0$. Indeed, for $t \neq 0$, we should have $f_{x^0,c}(x^0 + tx^0) - f_{x^0,c}(x^0) \geq \langle \zeta, tx^0\rangle$, or equivalently $-c \geq t\langle \zeta, x^0\rangle$. For $t > 0$, dividing by t and letting $t \to +\infty$, we get $0 \geq \langle \zeta, x^0\rangle$. For $t < 0$, dividing by t and letting $t \to -\infty$, we get $0 \leq \langle \zeta, x^0\rangle$. Hence $\langle \zeta, x^0\rangle = 0$. When $\zeta \in L$ this gives $\zeta = 0$. Fix $x_2 > c^2/a$ and let $t > 0$. Now

$$\operatorname{Calm} f_{x^0,c,L,\zeta}(x^0 + tx^0) \leq \frac{f_{x^0,c,L,\zeta}((1+t)a, x_2/(1+t)) - f_{x^0,c,L,\zeta}((1+t)a, 0)}{x_2/(1+t)}$$

$$= (1+t)\,\frac{-\sqrt{ax_2}-c}{x_2} \to -\infty \quad \text{as} \quad t\to\infty\,.$$

Therefore condition $\mathbb{C}(f,x^0,c,L,\zeta)$ is not satisfied and according to Theorem 3.1 f is not $\mathcal{H}_2^0$-convex at x^0.

Thus, the lower semicontinuity alone does not imply the $\mathcal{H}_k^0$-convexity, but it could do this when combined with some additional condition. Theorem 4.6 below supports this claim.

We will need the notion of uniform lower semicontinuity. The function $g:\mathbb{R}^n\to\mathbb{R}_{+\infty}$ will be called uniformly lsc on $D\subset \operatorname{dom} f$ if for all $x\in D$, and for all $\varepsilon>0$, there exists $\delta>0$ possibly depending on ε but not depending on x, such that $y\in\mathbb{R}^n$ and $\|y-x\|<\delta$ implies $g(y)>g(x)-\varepsilon$. Obviously, if f is uniformly lsc on D, then f is lsc on D.

The following propositions present some properties of the uniform lower semicontinuity.

Proposition 4.3. *Let $g:\mathbb{R}^n\to\mathbb{R}_{+\infty}$ be uniformly lsc on $D\subset\operatorname{dom} f$. Then the restriction $g|_D$ of g on D is continuous.*

Proof. Since f is lsc on D, in order to show its continuity it is enough to show that f is also upper semicontinuous (usc) on f. Let $x^0\in D$ and $x^n\to x^0$ where $x^n\in D$. Suppose on the contrary, that $\gamma:=\limsup_n g(x^n)>g(x^0)$. Taking a subsequence, we may assume that $g(x^n)\to\gamma$. Put $\varepsilon=(\gamma-g(x^0))/2>0$. Then there exists $\delta>0$ such that $x\in D$ and $\|y-x\|<\delta$ implies $g(y)>g(x)-\varepsilon$. Diminishing eventually δ, we may assume that $\|x^n-x^0\|<\delta$ implies $g(x^n)>\gamma-\varepsilon$. Fix now x^n, such that $\|x^n-x^0\|<\delta$. From the uniform lower semicontinuity we should have $g(x^0)>g(x^n)-\varepsilon$, and therefore

$$g(x^0)>g(x^n)-\varepsilon>\gamma-2\varepsilon=g(x^0)\,,$$

a contradiction. □

Proposition 4.4. *Let $g:\mathbb{R}^n\to\mathbb{R}_{+\infty}$ be lsc on $D\subset\operatorname{dom} f$, and let D be compact. If the restriction $g|_D$ of g on D is continuous, then g is uniformly lsc on D.*

Proof. Suppose on the contrary, that g is not uniformly lsc on D. Choose a sequence $\delta_n\to 0^+$. Then there should exist $\varepsilon_0>0$ and sequences $x^n\in D$, $y^n\in\mathbb{R}^n$, such that $\|y^n-x^n\|<\delta_n$ but $g(y_n)\le g(x^n)-\varepsilon_0$. From the compactness of D, passing to a subsequence, we may assume that $x^n\to x^0\in D$. Now we have also $y^n\to x^0$. Taking limits in the inequality $g(y_n)\le g(x^n)-\varepsilon_0$ and using the continuity of $g|_D$ we get $\liminf_g(y^n)\le\limsup_n g(y^n)\le g(x^0)-\varepsilon_0$. This contradicts however the lower semicontinuity of g at x^0. □

In the proof of Theorem 4.6 we will need the following proposition.

Proposition 4.5. *Let $g:L\to\mathbb{R}_{+\infty}$ (L is m-dimensional linear space) be lsc. Suppose also that g is convex at x^0 and $\operatorname{dom} g$ is bounded. Then the function $\operatorname{conv} g$ is lsc at x^0.*

Proof. Assume, on the contrary, that $\operatorname{conv} g$ is not lsc at x^0. Then there exists $c<f(x^0)$ and a convex combination

$$\sum_{i=1}^{m+1}\lambda_{\nu,i}\,y^{\nu,i}=x^\nu \tag{4.1}$$

with $y^{\nu,i}\in\operatorname{dom} g$ and $x^\nu\to x^0$, such that

$$\sum_{i=1}^{m+1}\lambda_{\nu,i}\,g(y^{\nu,i})<c+\frac{1}{\nu}\,.$$

With account that $\operatorname{dom} g$ is bounded, passing to a subsequence, we may assume that $\lim_\nu \lambda_{\nu,i} \to \lambda_i$ and $\lim_\nu y^{\nu,i} = y^i$ for $i = 1, \dots, m+1$. A passing to a limit in (4.1) gives a representation of x^0 as a convex combination $\sum_{i=1}^{m+1} \lambda_i y^i = x^0$. From the lower semicontinuity of g we have $g(y^i) \le \liminf_\nu g(y^{\nu,i})$ for $i = 1, \dots, m+1$. Now we get

$$\sum_{i=1}^{m+1} \lambda_i \, g(y^i) \le \sum_{i=1}^{m+1} \lambda_{\nu,i} \, \liminf_\nu g(y^{\nu,i})$$
$$\le \liminf_\nu \sum_{i=1}^{m+1} \lambda_{\nu,i} \, g(y^{\nu,i}) \le \liminf_\nu \left(c + \frac{1}{\nu} \right) = c \, .$$

This chain of inequalities contradicts however the convexity of g at x^0, according to which we should have $\sum_{i=1}^{m+1} \lambda_i g(y^i) \ge g(x^0) > c$. □

Theorem 4.6. *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be lsc with bounded domain $\operatorname{dom} f$. Let $1 \le k \le n$, and suppose that for any $x^0 \in \mathbb{R}^n$ there exists a $(n+1-k)$-dimensional subspace $L \subset \mathbb{R}^n$ with $x^0 \in L$, such that the restriction $f|_L$ is convex at x^0. Then f is $\mathcal{H}_k^0$-convex.*

Proof. Fix $x^0 \in \mathbb{R}^n$ and let $c < f(x^0)$. Put $B_\sigma = \{x \in \mathbb{R}^n \mid \|x\| \le \sigma\}$. Let $\sigma_0 > 0$ be such that $\operatorname{dom} f \subset B_{\sigma_0}$ and $x^0 \in B_{\sigma_0}$. and let $\sigma_0 < \sigma_1$. From the compactness of B_{σ_0} and the lower semicontinuity of f we have

$$\inf_{x \in \mathbb{R}^n} f(x) = \inf_{x \in B_{\sigma_0}} f(x) > -\infty \, .$$

Therefore there exists μ such that $-\infty < \mu < c$ and $\mu \le \inf_{x \in \mathbb{R}^n} f(x)$.

Choose the $(n+1-k)$-dimensional space L containing x^0 so that $f|_L$ is convex at x^0. Since f is lsc, also $f|_L$ is lsc. From Proposition 4.5 we get that $\operatorname{conv} f|_L$ is lsc at x^0. Let $c < \bar{c} < f(x^0)$. Since $f|_L$ is bounded from below, according to Proposition 3.5 there exists $\zeta \in \partial(f|_L)_{x^0,\bar{c}}(x^0) = \partial f_{x^0,c}|_L(x^0)$. Obviously, it holds also $\zeta \in \partial f|_{x^0,c}(x^0)$. Now we will show that condition $\mathbb{C}(f, x^0, c, L, \zeta)$ is satisfied. Observe first, that the function $f_{x^0,\bar{c},L,\zeta}$ is lsc. This follows straightforward from the facts that: $L \subset \mathbb{R}^n$ is closed, $f_{x^0,\bar{c},L,\zeta}(x) \le f(x)$ for all $x \in \mathbb{R}^n$, $f_{x^0,\bar{c},L,\zeta}$ restricted to $\mathbb{R}^n \setminus L$ coincides with the lsc function f, and $f_{x^0,\bar{c},L,\zeta}$ restricted to L coincides with the continuous function $x \to \bar{c} + \langle \zeta, \, x - x^0 \rangle$. From Proposition 4.4 it follows that $f_{x^0,\bar{c},L,\zeta}$ is uniformly lsc on $L \cap B_{\sigma_1}$. We will show that $f_{x^0,\bar{c},L,\zeta}$ is uniformly lsc on L, for which it is enough to show that it is uniformly lsc on $L \subset B_{\sigma_1}$. This is however obvious, since for $x \in L \setminus B_{\sigma_1}$ and $\|y - x\| < \sigma_1 - \sigma_0$ we have

$$f_{x^0,\bar{c},L,\zeta}(y) - f_{x^0,\bar{c},L,\zeta}(x) = \begin{cases} \langle \zeta, y - x \rangle \, , & y \in L \, , \\ +\infty \, , & y \notin L \, , \end{cases}$$

(now for every $\varepsilon > 0$ and $0 < \delta < \min(\sigma_1 - \sigma_2, \, \varepsilon / \|\zeta\|)$ the conditions $x \in L \setminus \sigma_1$ and $\|y - x\| < \delta$ imply $f_{x^0,\bar{c},L,\zeta}(y) - f_{x^0,\bar{c},L,\zeta}(x) \ge -\varepsilon$).

The proved uniform lower semicontinuity implies that there exists $\delta > 0$, such that $x \in L$ implies

$$f_{x^0,\bar{c},L,\zeta}(y) - f_{x^0,\bar{c},L,\zeta}(x) > -(\bar{c} - c) \quad \text{when} \quad \|y - x\| \le \delta \, . \tag{4.2}$$

When $\|y - x\| \le \delta$ and $y \in L$ inequality (4.2) gives:

$$f_{x^0,c,L,\zeta}(y) - f_{x^0,c,L,\zeta}(x) = \langle \zeta, y - x \rangle \ge -\|\zeta\| \, \|y - x\| \, . \tag{4.3}$$

When $\|y - x\| \le \delta$ and $y \notin L$ inequality (4.2) gives:

$$f_{x^0,c,L,\zeta}(y) - f_{x^0,c,L,\zeta}(x) > 0 \, . \tag{4.4}$$

When $\|y-x\| > \delta$ and $y \in L$ we have inequality (4.3).
When $\|y-x\| > \delta$ and $y \in B_{\sigma_0} \setminus L$ we have

$$\begin{aligned} f_{x^0,c,L,\zeta}(y) - f_{x^0,c,L,\zeta}(x) &\geq \mu - c - \langle \zeta,\, x - x^0 \rangle \\ &\geq -\,\frac{c-\mu}{\delta}\, \|y-x\| - \frac{2\sigma_0\, \|\zeta\|}{\delta}\, \|y-x\| \,. \end{aligned} \tag{4.5}$$

When $\|y-x\| > \delta$ and $y \in (\mathbb{R}^n \setminus B_{\sigma_0}) \setminus L$ we have

$$f_{x^0,c,L,\zeta}(y) - f_{x^0,c,L,\zeta}(x) = +\infty \,. \tag{4.6}$$

Combining relations (4.3)–(4.6) we obtain

$$\operatorname{Calm} f_{x^0,c,L,\zeta}(x) \geq -\, \max\left(\|\zeta\|\,,\, \frac{c-\mu}{\delta} + \frac{2\sigma_0\, \|\zeta\|}{\delta} \right) .$$

The right hand side of this inequality does not depend on x. Therefore condition $\mathbb{C}(f, x^0, c, L, \zeta)$ has place. □

The following example is a straightforward application of Theorem 4.6 (observe that the hypotheses of this theorem are satisfied).

Example 4.7. Let $D \subset \mathbb{R}^2$ be a bounded set. Define the function $g : \mathbb{R}^2 \to \mathbb{R}_{+\infty}$ by

$$g(x_1, x_2) = \begin{cases} f(x_1, x_2)\,, & (x_1, x_2) \in D\,, \\ +\infty\,, & \text{otherwise}\,, \end{cases}$$

where f is the function from Example 4.2. Then g is $\mathcal{H}_2^0$-convex (recall that the function f in Example 4.2 was not $\mathcal{H}_2^0$-convex).

As Example 4.2 shows, the boundedness of the domain $\operatorname{dom} f$ is essential for the validity of Theorem 4.6.

5 Comments

As it was mentioned in the beginning, we introduced the $\mathcal{H}_k^0$-convexity with the aim to study $\mathcal{H}_k$-convexity. This is because $\mathcal{H}_k^0$-convexity admits a characterization through a constructive approach as in Theorem 3.1. The $\mathcal{H}_k^0$-convexity implies $\mathcal{H}_k$-convexity, hence the sufficient conditions for $\mathcal{H}_k^0$-convexity are also sufficient for $\mathcal{H}_k$-convexity. The necessary conditions for $\mathcal{H}_k^0$-convexity alone are not necessary for $\mathcal{H}_k$-convexity. For instance, as it was shown the convexity at x^0 of the restriction $f|_L$ on some $(n+1-k)$-dimensional subspace L is a necessary condition for the $\mathcal{H}_k^0$-convexity at x^0, but it is not necessary for the $\mathcal{H}_k$-convexity at x^0. To stress the differences between $\mathcal{H}_k^0$-convexity and $\mathcal{H}_k$-convexity we turn to Example 4.2. As it shown, the function f there is not $\mathcal{H}_2^0$-convex at the non zero points of the coordinate axes. At the same time, at these points f is $\mathcal{H}_2$-convex. Let us e. g. demonstrate the $\mathcal{H}_2$-convexity at the point $x^0 = (1,\, 0)$. For $s > 0$ define the functionals $\ell_s \in \mathcal{H}_2$ by

$$\ell_s(x_1, x_2) = \min\big((-\sqrt{s} + 2s\sqrt{s})\, x_1 - 2\sqrt{s}\, x_2,\; (-\sqrt{s} - 2s\sqrt{s})\, x_1 + 2\sqrt{s}\, x_2 \big) \,.$$

It can be shown that $\ell_s \leq f$. Accounting that also $\ell_s(x^0) = \ell_s(1,0) = -\sqrt{s} - 2s\sqrt{s} \to 0 = f(x^0)$ as $s \to 0^+$, we get that f is $\mathcal{H}_2$-convex at x^0. This example shows that the problem to characteriza $\mathcal{H}_k$-convexity in general meets with difficulties, and is still an open problem.

Theorem 4.6 characterizes the $\mathcal{H}_k^0$-convex functions with bounded domains (when $\operatorname{dom} f$ is bounded then it can be shown that the conditions of Theorem 4.6 are not only sufficient, but also necessary for the $\mathcal{H}_n^0$-convexity of f). When $\operatorname{dom} f$ is unbounded, the hypotheses of Theorem 4.6 do not characterize the $\mathcal{H}_n^0$-convexity of f, as seen from Example 4.2. The characterization of $\mathcal{H}_n^0$-convexity in terms of lower semicontinuity for functions with unbounded domain is also an open problem.

The notion of uniform continuity finds numerous applications in mathematical analysis. The introduced here notion of uniform lower semicontinuity, due to Proposition 4.3, seems to be of less importance. Nevertheless, as seen from the proof of Theorem 4.6, it can find some applications.

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Ivan Ginchev
University of Insubria, Department of Economics, Via Monte Generoso 71, 21100 Varese, Italy
E-mail address: vechnig@yahoo.com