Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 7, No. 3, pp. 531–550, September 2011

EXISTENCE OF STABLE OUTCOME IN A JOB MARKET WITH LINEAR VALUATIONS AND POSSIBLY BOUNDED SALARIES

Yasir Ali and Rashid Farooq

Abstract: We consider a job allocation model where each worker can work for at most one firm and each firm can employ as many workers as it wishes. The preferences of the players of each side are represented by linear and strictly increasing functions which are called valuations of the players. Bounded or unbounded monetary transfers are permitted. We show the existence of a pairwise stable outcome in our model. Our model is a common generalization of the marriage model by Gale and Shapley, assignment game by Shapley and Shubik, and hybrid model by Sotomayor.

Key words: *optimal stable outcome, stable marriage, pairwise stability, job allocation, linear valuations*

Mathematics Subject Classification: *80M50, 91B40, 91B68*

1 Introduction

The concept of two-sided matching markets is well known in mathematical economics. In two-sided matching markets, the set of participants, called the players, is divided into two disjoint subsets; the set of individuals and the set of institutions. Generically, we recognize the individuals as workers and the institutions as firms. The basic problem in such a market is to assign the workers and firms to each other. Each worker has a list of preferences of those firms where he/she is willing to work. Similarly, each firm has preferences over those workers or the group of workers whom the firm wants to employ.

A job allocation is an assignment of workers and firms where each worker is assigned with at most as many firms as he/she wishes and each firm is assigned with at most as many workers as it wishes to employ. A job allocation is stable if all players have acceptable partners and there is no worker-firm pair which is not matched but prefer to be matched to each other rather than staying with their current partners.

The theory of two-sided matching markets was originated by Gale and Shapley [8]. In their pioneering work, they presented the marriage model; a model in which a player on one side is matched with at most one player on the opposite side. The monetary transfer is not permitted in their model. For this reason, the players in this model are called "rigid". Gale and Shapley proposed an algorithm which finds a stable matching. The one-to-one buyer-seller model by Shapley and Shubik [15], known as assignment game, in contrast to Gale and Shapley's marriage model, deals with the players who can trade money, that is, the "flexible" players. Shapley and Shubik showed that the core of the assignment game is a non-empty complete lattice, where the core of a game is defined as the set of undominated outcomes.

The Gale and Shapley marriage model and Shapley and Shubik assignment game has been widely studied and several variations and extensions of these can be found in the literature. Crawford and Knoer [1] developed an algorithm, called the "salary adjustment process" which is a generalization of Gale and Shapley's deferred-acceptance algorithm to the case where money is present, that is, assignment game. The non-emptiness of the core is shown in this model. Kelso and Crawford [10] extended the model of Crawford and Knoer [1] by considering one-to-many job market with money under the gross substitution condition. A generalization of the assignment game is also presented in Demange and Gale [2] where preferences of the players may be represented by any continuous utility functions in the money variable.

Kaneko [9] gave a very general and complicated model. He unified the Gale and Shapley marriage model and the Shapley and Shubik assignment game and established the non-emptiness of the core but could not establish the lattice property. The unification of the marriage model and the continuous model of Demange and Gale [2] can be observed in Roth and Sotomayor [14]. However, the existence of stable outcome is not guaranteed in their model but they investigated the lattice property for payoffs in the core. A one-to-one matching model is proposed in Eriksson and Karlander [3] where they unified the discrete and continuous models. The marriage model becomes a special case of their model if the players, at least on one side, are rigid. The assignment game is obtained when all players are flexible. The existence of stable matching is guaranteed in this model. They further discussed the lattice property of the set of stable outcomes. Some more investigation of their model is also found in Sotomayor [16]. Recently, Sotomayor [17] presented a one-to-one matching model, which is a special case of the model of Eriksson and Karlander [3] in the sense that all players on one side are flexible. On the other side, however, some players are rigid and the remaining flexible. A characterization of the core of this hybrid market can be seen in this paper.

Motivated by the works of Eriksson and Karlander [3] and Sotomayor [16], Fujishige and Tamura [6] proposed a common generalization of the marriage model and the assignment game by utilizing the framework of discrete convex analysis developed by Murota [11, 12, 13]. They further extended their model in [7] by assuming possibly bounded side payments and proved the existence of pairwise stable outcome. The structure of the set of pairwise stable outcome is not discussed in their paper. Very recently, Farooq [4] gave a generalization of the hybrid models of the Eriksson and Karlander [3] and Sotomayor [16]. He proved the existence of a stable outcome for one-to-one matching problem with linear valuations and bounded side payments.

The present work is a generalization of the model of Farooq [4] in the sense that we consider one-to-many matching and possibly bounded salaries. Our model includes, as special cases, the models of Gale and Shapley [8], Shapley and Shubik [15], Eriksson and Karlander [3], Sotomayer [16] and Farooq [4]. The main features of our model are:

— the set of players is partitioned into two sets; the set of firms and set of workers,
— a worker can work for at most one firm,
— each firm has a certain quota to employ workers,
— each worker-firm pair may have lower and upper bounds on the salary,
— the preferences of the players are identified by strictly increasing and linear valuations.*

We remark that the restrictions of boundedness or unboundedness on the salaries do not

*Valuations can be defined in different ways. Here by valuation, we mean estimation of the value of some asset or real property.

impact the algorithm since preferences of the players in our model are represented by strictly increasing and linear valuations. For example, in unrestricted case, the firm's individual rationality constraints induce upper bounds and the worker's individual rationality constraints induce lower bounds for the salaries. Therefore, the possible bounded salaries can not be considered as novelty in our model. Rather, it would help us to understand the comparison of our model and the known models. In the theory of stable matchings, generally it is believed that when one develops an algorithm to show the existence of a stable matching for the one-to-one models (the marriage markets), the same ideas may be adapted to show the existence of a stable matching for one-to-many models (the college admissions markets). However, due to the generality of our model, the simple adaptation of the old algorithm for one-to-one model does not work. A comprehensive work is done in this paper to extend the ideas of the algorithm for one-to-one model (Farooq [4]) to the algorithm for one-to-many model.

We organize the paper as follows. In Section 2, we describe our model and define the pairwise stability. In Section 3, we give a characterization of the pairwise stability. We shall use this characterization to establish an algorithm to obtain a stable outcome in our model. Section 4 deals with a brief description of some existing models and a comparison of our model with the other models. The main part of our work appears in Section 5. In this section, we propose an algorithm which finds a stable job allocation and prove the correctness and termination of the algorithm.

2 Model Description

We consider two finite disjoint sets of players P and Q. Let P and Q be the set of workers and firms, respectively, and $E = P \times Q$. We assume that each worker can work for at most one firm and that each firm can employ as many workers as it wishes. For each $j \in Q$, let $\mu(j)$ denote the maximum number of workers j can employ and $\mu = (\mu(j) \mid j \in Q) \in \mathbf{Z}_+^Q$, where $\mathbf{Z}_+^Q$ is the set of positive vectors of $\mathbf{Z}^Q$.

Assume that each worker-firm pair (i, j) may have lower and upper bounds on the salary, that is, the salaries are possibly bounded. The lower and upper bounds on the salaries are expressed by two vectors $\underline{\pi}$ and $\overline{\pi}$, where $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$, $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$ and $\underline{\pi} \leq \overline{\pi}$.† A vector $s = (s_{ij} \mid (i, j) \in E) \in \mathbf{R}^E$ is called a *feasible salary vector* if $\underline{\pi} \leq s \leq \overline{\pi}$.

We also assume that each worker has a list of preferences of those firms where he/she is willing to work. Similarly, each firm has preferences over those workers whom the firm wants to employ. The preferences of the players are represented by continuous, strictly increasing linear functions, which are called linear valuations in our work. For each $(i, j) \in E$, $\nu_{ij} : \mathbf{R} \to \mathbf{R}$ represents the valuation of a worker i for a monetary transfer from a firm j to i. Similarly, $\nu_{ji} : \mathbf{R} \to \mathbf{R}$ represents the valuation of the firm j for a monetary transfer from a worker i to j.‡

We say that a firm j is *acceptable* to a worker i at $\alpha \in \mathbf{R}$ if $\nu_{ij}(\alpha) \geq 0$. Similarly, a worker i is *acceptable* to a firm j at $\alpha \in \mathbf{R}$ if $\nu_{ji}(\alpha) \geq 0$.

A worker i *prefers* a firm j to a firm j' at $\alpha, \alpha' \in \mathbf{R}$ if $\nu_{ij}(\alpha) > \nu_{ij'}(\alpha')$ and i is *indifferent* between j and j' at $\alpha, \alpha' \in \mathbf{R}$ if $\nu_{ij}(\alpha) = \nu_{ij'}(\alpha')$. Similarly, a firm j *prefers* a worker i to a worker i' at $\alpha, \alpha' \in \mathbf{R}$ if $\nu_{ji}(\alpha) > \nu_{ji'}(\alpha')$ and j is *indifferent* between i and i' at $\alpha, \alpha' \in \mathbf{R}$ if $\nu_{ji}(\alpha) = \nu_{ji'}(\alpha')$.

†For any two vectors $x \in (\mathbf{R} \cup \{-\infty\})^E$ and $y \in (\mathbf{R} \cup \{+\infty\})^E$, we say that $x \leq y$ if $x_{ij} \leq y_{ij}$ for all $(i, j) \in E$.

‡The monetary transfer from a worker to a firm should not be surprising. For example, a worker can agree on the reduction of his/her demanded salary after negotiation with a firm.

A set $X = \{(S_j, j) \mid j \in Q\} \subseteq 2^P \times Q$ is called a *job allocation* if

(i) $|S_j| \leq \mu(j)$ for all $j \in Q$.

(ii) $S_j \cap S_{j'} = \emptyset$ for all $j, j' \in Q$ with $j \neq j'$.

In the sequel, whenever we say that $S_j \in X$ (or $j \in X$), we always mean that $(S_j, j) \in X$. For any $j \in Q$, we reserve the notation S_j for X only.

We say that a firm j *employs* a worker i if $i \in S_j$. Obviously, a firm j is *businessless* if $S_j = \emptyset$. Similarly, a worker i is said to be *unemployed* if $i \notin S_j$ for all $j \in Q$. We set $S_i = \{j\}$ if $i \in S_j$ and $S_i = \emptyset$ if i is unemployed. We say that a worker i and a firm j are *matched* if $i \in S_j$.

A quadruple $(X; s, q, r)$ is said to be an *outcome* if X is a job allocation, s is a feasible salary vector and $(q, r) \in \mathbf{R}^P \times \mathbf{R}^Q$ is defined by

$$q_i = \begin{cases} \nu_{ij}(s_{ij}) & \text{if } i \in S_j \text{ for some } j \in Q \\ 0 & \text{otherwise} \end{cases} \qquad (\forall i \in P), \tag{2.1}$$

$$r_j = \begin{cases} \min\{\nu_{ji}(-s_{ij}) \mid i \in S_j\} & \text{if } |S_j| = \mu(j) \\ 0 & \text{otherwise} \end{cases} \qquad (\forall j \in Q), \tag{2.2}$$

where the minimum over an empty set is defined to be 0.

An outcome $(X; s, q, r)$ is *blocked* by a worker-firm pair (i, j) if

$$i \notin S_j \text{ and } \nu_{ij}(s_{ij}) > q_i, \quad \nu_{ji}(-s_{ij}) > r_j. \tag{2.3}$$

The statement in (2.3) is equivalent to saying that i and j are not matched but i prefers j to his/her current employer[§] and j prefers i atleast to one of his worker or still have a vacancy to employ i.

An outcome $(X; s, q, r)$ is *pairwise stable* if the following two conditions are satisfied:

(ps1) $\nu_{ij}(s_{ij}) \geq 0$ and $\nu_{ji}(-s_{ij}) \geq 0$ for all $(i, j) \in E$ with $i \in S_j$.

(ps2) $\nu_{ij}(\alpha) \leq q_i$ or $\nu_{ji}(-\alpha) \leq r_j$ for all $\alpha \in \mathbf{R}$ with $\underline{\pi}_{ij} \leq \alpha \leq \overline{\pi}_{ij}$ and for all $(i, j) \in E$ with $i \notin S_j$.

Condition (ps1)[¶] says that if a firm employs a worker then both are acceptable to each other. Condition (ps2) means $(X; s, q, r)$ is not blocked by any worker-firm pair.

A job allocation X is called *pairwise stable* if $(X; s, q, r)$ is pairwise stable.

3 Characterization

This section is devoted to the characterization of a pairwise stable outcome. We shall use this characterization to devise the algorithm in Section 5.

A *characteristic vector* $\chi_S \in \{0, 1\}^P$ of a set $S \subseteq P$ is defined by

$$\chi_S(k) = \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{if } k \in P \setminus S. \end{cases}$$

Analogously, we can define a characteristic vector $\chi_S \in \{0, 1\}^Q$ of a set $S \subseteq Q$. Obviously, χ_S is the zero vector if $S = \emptyset$.

Next theorem gives a characterization of a pairwise stable outcome. This theorem is a modification of Theorem 2.1 [5] related to our model.

[§] For convenience, we say that a worker is self-employed if he/she is unemployed.

[¶] ps stands for pairwise stability.

Theorem 3.1. *Let X be a job allocation. There exists a feasible salary vector s and a vector (q,r) defined by (2.1) and (2.2) forming a pairwise stable outcome $(X;s,q,r)$ if and only if there exists a feasible salary vector p and vectors $z_P, z_Q \in \{0,1\}^E$ such that*

(ps′1) *for all $i \in P$, χ_{S_i} is an optimal solution of*

$$\begin{array}{lll} \text{maximize}_{k\in Q} & \nu_{ik}(p_{ik})\chi_S(k) & \\ \text{subject to} & S \subseteq Q, & \\ & |S| \leq 1, & \\ & \chi_S(k) \leq z_P(i,k) \quad (k \in Q). & \end{array} \tag{3.1}$$

(ps′2) *for all $j \in Q$, χ_{S_j} is an optimal solution of*

$$\begin{array}{ll} \text{maximize} & \sum_{k\in P} \nu_{jk}(-p_{kj})\chi_S(k) \\ \text{subject to} & S \subseteq P, \\ & |S| \leq \mu(j), \\ & \chi_S(k) \leq z_Q(k,j) \quad (k \in P). \end{array}$$

(ps′3) $z_P \vee z_Q = \mathbf{1}$.‖

(ps′4) $z_P(i,j) = 0 \Rightarrow p_{ij} = \underline{\pi}_{ij}$ *and* $z_Q(i,j) = 0 \Rightarrow p_{ij} = \overline{\pi}_{ij}$.

Proof. ($\Leftarrow$) Suppose that there exist a feasible salary vector p, and z_P, $z_Q \in \{0,1\}^E$ such that the conditions (ps′1)−(ps′4) are satisfied. We prove that $(X;s,q,r)$ satisfies (ps1) and (ps2), where $s = p$ and (q,r) is defined by (2.1) and (2.2). The condition (ps1) is implied by (ps′1) and (ps′2). We only show that (ps2) is also satisfied.

Suppose that (ps2) does not hold. Then there exists $(i,j) \in E$ with $i \notin S_j$ and $\alpha \in \mathbf{R}$ with $\underline{\pi}_{ij} \leq \alpha \leq \overline{\pi}_{ij}$ such that $\nu_{ij}(\alpha) > q_i$ and $\nu_{ji}(-\alpha) > r_j$. If $\nu_{ij}(\alpha) > q_i$ then by (ps′1) and the fact that ν_{ij} is increasing, at least one of the following two cases must hold:

(a1) $z_P(i,j) = 0$ or (a2) $z_P(i,j) = 1$ and $p_{ij} < \alpha$.

Similarly, if $\nu_{ji}(-\alpha) > r_j$ then by the fact that ν_{ji} is increasing, at least one of the following two cases must hold:

(b1) $z_Q(i,j) = 0$ or (b2) $z_Q(i,j) = 1$ and $p_{ij} > \alpha$.

Obviously, (a1) and (b1) can not hold together by (ps′3). If (a1) and (b2) are true then (ps′4) yields that $p_{ij} = \underline{\pi}_{ij}$ which is not possible. With the same argument, (a2) and (b1) can not be true together. The statements (a2) and (b2) are obviously incompatible. Therefore (ps2) must hold.

($\Rightarrow$) Suppose that there exists a feasible salary vector s such that $(X;s,q,r)$ is a pairwise stable outcome, where (q,r) is defined by (2.1) and (2.2). Then (ps1) and (ps2) hold true. We prove that there exist p and $z_P, z_Q \in \{0,1\}^E$ which satisfy (ps′1)−(ps′4). Define z_P and z_Q as follows:

$$z_P(i,j) = \begin{cases} 0 & \text{if } \nu_{ij}(\underline{\pi}_{ij}) > q_i \text{ and } i \notin S_j \\ 1 & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E),$$

$$z_Q(i,j) = \begin{cases} 0 & \text{if } \nu_{ji}(-\overline{\pi}_{ij}) > r_j \text{ and } i \notin S_j \\ 1 & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E).$$

From the definitions of z_P and z_Q, we observe that for any $(i,j) \in E$ with $i \notin S_j$, the following holds:

$$z_P(i,j) = z_Q(i,j) = 1 \Rightarrow \nu_{ij}(\underline{\pi}_{ij}) \leq q_i \text{ and } \nu_{ji}(-\overline{\pi}_{ij}) \leq r_j. \tag{3.2}$$

‖For any $(i,j) \in E$, the (i,j)-th component of $z_P \vee z_Q$ is given by $(z_P \vee z_Q)(i,j) = \max\{z_P(i,j), z_Q(i,j)\}$.

The statement (3.2) together with (ps2) implies that

$$\exists \gamma_{ij} \in [\underline{\pi}_{ij}, \overline{\pi}_{ij}] \text{ such that } \nu_{ij}(\gamma_{ij}) \leq q_i \text{ and } \nu_{ji}(-\gamma_{ij}) \leq r_j \quad \text{for all } (i,j) \in E \text{ with } z_P(i,j) = z_Q(i,j) = 1 \text{ and } i \notin S_j. \tag{3.3}$$

Now define the salary vector $p \in \mathbf{R}^E$ as follows:

$$p_{ij} = \begin{cases} \underline{\pi}_{ij} & \text{if } z_P(i,j) = 0 \\ \overline{\pi}_{ij} & \text{if } z_Q(i,j) = 0 \\ \gamma_{ij} & \text{if } z_P(i,j) = z_Q(i,j) = 1 \text{ and } i \notin S_j \\ s_{ij} & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E), \tag{3.4}$$

where γ_{ij}, for $(i,j) \in E$ with $z_P(i,j) = z_Q(i,j) = 1$ and $i \notin S_j$, is defined by (3.3). The definitions of z_P and z_Q along with (ps2) imply (ps$'$3) and the definition of p implies (ps$'$4). Next we prove (ps$'$2). Let $j \in Q$ and $S \subseteq P$ is such that $|S| \leq \mu(j)$ and $\chi_S(k) \leq z_Q(k,j)$ for all $k \in P$. For any $i \in S \setminus S_j$, it is enough to show that

$$\nu_{ji}(-p_{ij}) \leq r_j. \tag{3.5}$$

Since $i \in S$ we must have $z_Q(i,j) = 1$. If $z_P(i,j) = 1$ then (3.3) and the definition of p implies (3.5). If $z_P(i,j) = 0$ then the definitions of z_P and p give $\nu_{ij}(p_{ij}) > q_i$. This along with (ps2) gives (3.5). Analogously, we can prove (ps$'$1). This completes the proof. □

4 Comparison of our Model with Existing Models

In this section, we compare our model with few existing models that are directly related to our model. With each $(i,j) \in E$, we associate a pair (β_{ij}, β_{ji}) of real numbers. We say that $j \in Q$ is *acceptable* to $i \in P$ if $\beta_{ij} \geq 0$. Similarly, i is *acceptable* to j if $\beta_{ji} \geq 0$. A (one-to-one) matching is a subset of E such that each player appears at most once.

In the pioneering work by Gale and Shapley [8] in two sided matching markets, an algorithm is proposed which produces a stable matching. The main characteristic of their algorithm is that when men are proposer, it gives a unique stable matching and each man has the best partner that he can have in any other stable matching. However, if the role of the sexes are reversed, the algorithm yields a unique stable matching where each woman has the best partner that she can have in any other stable matching.

In the one-to-one buyer-seller model by Shapley and Shubik [15], known as assignment game, they proved non constructively the existence of a stable outcome and showed that the set of stable outcomes and the core of the game are the same. By obvious reasons, the players in the marriage model are called "rigid" and the players in the assignment game are called "flexible".

It is very natural to think of a single market which consists of both rigid and flexible players. Eriksson and Karlander [3] proposed a mixed market model (the RiFle assignment game) consisting of both rigid players and flexible players. They proved the existence of stable outcome. However, their proofs do not hold for all continuous markets (the assignment game due to Shapley and Shubik [15]). Sotomayor [16] also considered the mixed market model that contains both the marriage model and the assignment game as special cases. Her model is a generalization of the RiFle assignment game in the sense that her proofs hold for both discrete and continuous markets. Mainly using her terminologies, we describe this hybrid model mathematically.

We think of the sets P and Q as sets of workers and firms, respectively. We partition the players into two classes R and F where R is the set of rigid players and F is the set of flexible players. Define two subsets R^* and F^* of E by:

$$\begin{aligned} R^* &= \{(i,j) \in E \mid i \in R \text{ or } j \in R\}, \\ F^* &= \{(i,j) \in E \mid i,j \in F\}. \end{aligned}$$

R^* and F^* are called the sets of rigid and flexible pairs, respectively. A matching X is called pairwise stable if there exists $(q,r) \in \mathbf{R}^P \times \mathbf{R}^Q$ such that

(h1) $q_i + r_j = \beta_{ij} + \beta_{ji}$ for all $(i,j) \in X$.

(h2) $q_i = \beta_{ij}$ and $r_j = \beta_{ji}$ for all $(i,j) \in X \cap R^*$.

(h3) $q \geq 0, r \geq 0$ and $q_i = 0$ (resp. $r_j = 0$) if i (resp. j) is unmatched.

(h4) $q_i + r_j \geq \beta_{ij} + \beta_{ji}$ for all $(i,j) \in F^*$.

(h5) $q_i \geq \beta_{ji}$ or $r_j \geq \beta_{ji}$ if $(i,j) \in R^*$.

Sotomayor [16] proved the existence of the stable outcome in this model. Further she proved that the core is a complete lattice. One can easily see that if $F^* = \emptyset$ then the above model coincides with marriage model by Gale and Shapley [8]. Also, if $R^* = \emptyset$ then it would coincide with the assignment game by Shapley and Shubik [15].

Let us assume that $\mu = (1, \ldots, 1) \in \mathbf{Z}_+^Q$ and define the linear valuations in a special way as follows:

$$\nu_{ij}(s_{ij}) = \beta_{ij} + s_{ij}, \quad \nu_{ji}(-s_{ij}) = \beta_{ji} - s_{ij} \qquad (\forall (i,j) \in E),$$

where $s_{ij} \in \mathbf{R}$ with $\underline{\pi}_{ij} \leq s_{ij} \leq \overline{\pi}_{ij}$ and $\beta_{ij}, \beta_{ji} \in \mathbf{R}$. If we fix $\underline{\pi} = \overline{\pi} = \mathbf{0}$ then the marriage model due to Gale and Shapley [8] becomes a special case of our model. If we let $\underline{\pi} = (-\infty, \ldots, -\infty)$ and $\overline{\pi} = (+\infty, \ldots, +\infty)$ then we get the assignment game due to Shapley and Shubik [15]. Now we assume that $\underline{\pi} \in (\mathbf{R} \cup \{-\infty\})^E$ and $\overline{\pi} \in (\mathbf{R} \cup \{+\infty\})^E$, that is, the set of pairs is partitioned randomly in to the set of rigid pairs and flexible pairs. This then shows that hybrid model of Eriksson and Karlander [3] and Sotomayor [16] is a special case of our model.

5 An algorithm for Finding a Stable Job Allocation

This section deals with finding a stable job allocation for our model described in Section 2. We do this by establishing an algorithm which is an extended version of the algorithm proposed by Farooq [4]. His algorithm works when each worker can work for at most one firm and each firm can employ at most one worker, and the salaries are bounded. The algorithm proposed in this section includes the cases where firms can employ as many workers as they wish and when the salaries have no bounds. At the end of this section, we show that the algorithm works correctly and terminates after a finite number of iterations.

Let us define the valuations ν_{ij} and ν_{ji} as follows:

$$\nu_{ij}(x) = \alpha_{ij}x + \beta_{ij}, \quad \nu_{ji}(x) = \alpha_{ji}x + \beta_{ji} \qquad (\forall (i,j) \in E), \tag{5.1}$$

where α_{ij} and α_{ji} are given positive real numbers and, β_{ij} and β_{ji} are any given real numbers.

Initially, we define the salary vector $p \in \mathbf{R}^E$ by

$$p_{ij} := \begin{cases} \overline{\pi}_{ij} & \text{if } \nu_{ji}(-\overline{\pi}_{ij}) \geq 0 \text{ and } \overline{\pi}_{ij} < +\infty \\ \max\{\underline{\pi}_{ij}, \frac{\beta_{ji}}{\alpha_{ji}}\} & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E). \tag{5.2}$$

Then $\underline{\pi}_{ij} \leq \max\{\underline{\pi}_{ij}, \frac{\beta_{ji}}{\alpha_{ji}}\} \leq p_{ij} \leq \overline{\pi}_{ij}$ for all $(i,j) \in E$. That is, p is a feasible salary vector. Now define $z_P \in \{0,1\}^E$ as follows:

$$z_P(i,j) = \begin{cases} 1 & \text{if } \nu_{ji}(-p_{ij}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E). \tag{5.3}$$

We next define $z_0 \in \{0,1\}^E$ by

$$z_0(i,j) = \begin{cases} 0 & \text{if } \nu_{ij}(p_{ij}) \leq 0 \\ 1 & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E), \tag{5.4}$$

and fix

$$\tilde{z}_P = z_P \wedge z_0.^{**} \tag{5.5}$$

Note that any $(i,j) \in E$ with $\tilde{z}_P(i,j) = 1$ implies that $\nu_{ij}(p_{ij}) > 0$ and $\nu_{ji}(-p_{ij}) \geq 0$, that is, i and j are mutually acceptable. Furthermore, define $\hat{z}_P \in \{0,1\}^E$ by

$$\hat{z}_P(i,j) = \begin{cases} 1 & \text{if } \tilde{z}_P(i,j) = 1 \text{ and } \nu_{ij}(p_{ij}) = \\ & \max\{\nu_{ij'}(p_{ij'}) \mid j' \in Q, \tilde{z}_P(i,j') = 1\} \\ 0 & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E). \tag{5.6}$$

Thus for any $(i,j) \in E$ with $\hat{z}_P(i,j) = 1$, we see that the worker i and the firm j are mutually acceptable and i prefers j to all those firms which accept i. Also for any $j \in Q$ and $S \in 2^Q$, define $r_j^S \in \mathbf{R}$ by

$$r_j^S = \min\{\nu_{ji}(-p_{ij}) \mid i \in S\}. \tag{5.7}$$

Let $S \in 2^P$, $z_Q \in \{0,1\}^E$ and p be a feasible salary vector. Then the 3-tuple (S, z_Q, p) is said to be a *best choice* for $j \in Q$ if it satisfies the following property:

(BC) $\forall i \in P \setminus S$ with $z_Q(i,j) = 1$, the following hold:

(a) $\nu_{ji}(-p_{ij}) \leq r_j^S$.

(b) $|S| \leq \mu(j)$ and if $\nu_{ji}(-p_{ij}) > 0$ then $|S| = \mu(j)$.

To find a matching, initially we define a vector $z_Q \in \{0,1\}^E$ and a vector $\tilde{\mu} \in \mathbf{Z}_+^Q$ as follows:

$$z_Q(i,j) := \begin{cases} 1 & \text{if } p_{ij} < \overline{\pi}_{ij} \text{ or} \\ & [\overline{\pi}_{ij} = \underline{\pi}_{ij} \text{ and } \nu_{ji}(-\overline{\pi}_{ij}) < 0] \\ 0 & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E), \tag{5.8}$$

$$\tilde{\mu} = (\tilde{\mu}(j) = 0 \mid j \in Q). \tag{5.9}$$

**For any $(i,j) \in E$, the (i,j)-th component of $z_P \wedge z_0$ is given by $(z_P \wedge z_0)(i,j) = \min\{z_P(i,j), z_0(i,j)\}$.

Now, for each $j \in Q$, define η_j by

$$\eta_j = \{S \in 2^P \mid (S, z_Q, p) \text{ satisfies (BC) for } j, \tilde{\mu}(j) \le |S| \text{ and } \chi_S(i) \le \hat{z}_P(i,j) \quad (\forall i \in P)\} \tag{5.10}$$

and let

$$\eta = \cup_{j \in Q}\{\eta_j\}. \tag{5.11}$$

We further define a set Γ as follows:

$$\Gamma = \cup_{j \in Q}\{\eta_j \times \{j\}\}. \tag{5.12}$$

Find a matching $X = \{(S_j, j) \mid j \in Q\}^{\dagger\dagger}$ in the bipartite graph $(\eta, Q; \Gamma)$ which satisfies the following:

$$S_j \cap S_{j'} = \emptyset \text{ for all } S_j, S_{j'} \in X \text{ with } j \ne j', \tag{5.13}$$

$$\sum_{(i,j)\in E} \nu_{ji}(-p_{ij})\chi_{S_j}(i) \text{ is maximum among the matchings satisfying (5.13)}, \tag{5.14}$$

$$\sum_{(i,j)\in E} (\ln \alpha_{ji} - \ln \alpha_{ij})\chi_{S_j}(i) \text{ is maximum among the matchings satisfying (5.13) and (5.14)}. \tag{5.15}$$

Then obviously X is a job allocation. Let S_P be the set of all workers which are employed by some firm, that is,

$$S_P = \cup_{j \in Q} S_j. \tag{5.16}$$

We redefine $z_Q \in \{0,1\}^E$ and $\tilde{\mu} \in \mathbf{Z}_+^Q$ by

$$z_Q(i,j) := \begin{cases} 1 & \text{if } p_{ij} < \overline{\pi}_{ij} \text{ or } i \in S_j \\ z_Q(i,j) & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E), \tag{5.17}$$

$$\tilde{\mu} = (\tilde{\mu}(j) = |S_j| \mid j \in Q). \tag{5.18}$$

Since $\chi_{S_j}(i) \le \hat{z}_P(i,j)$, for all $(i,j) \in E$, the following holds:

(ps$'$1$_w$) If $S_P \ne \emptyset$ then, for all $i \in S_P$, χ_{S_i} is an optimal solution of (3.1).

By the definitions of p and z_Q, $\nu_{ji}(-p_{ij}) \le 0$ or $i \in S_j$ for all $(i,j) \in E$ with $z_Q(i,j) = 1$. Thus (ps$'$2) is satisfied. Also, for any $(i,j) \in E$, if $z_P(i,j) = 0$ then $\nu_{ji}(-p_{ij}) < 0$. In this case, (5.2) implies $p_{ij} = \underline{\pi}_{ij}$ and hence $z_Q(i,j) = 1$ by (5.17). Now, if $z_Q(i,j) = 0$ then (5.17) implies that $p_{ij} = \overline{\pi}_{ij}$. Therefore, $\nu_{ji}(-p_{ij}) \ge 0$ by (5.2). Definition (5.3) yields $z_P(i,j) = 1$. Hence (ps$'$3) and (ps$'$4) hold. Therefore, the quadruple $(X; p, z_P, z_Q)$ satisfies (ps$'$2)$-$(ps$'$4).

Thus we have found a quadruple $(X; p, z_P, z_Q)$ which satisfies (ps$'1_w$), (ps$'$2)$-$(ps$'$4). Our main purpose is to find a quadruple $(X; p, z_P, z_Q)$ which satisfies (ps$'$1)$-$(ps$'$4). We observe that if $S_P = P$ then (ps$'1_w$) and (ps$'$1) coincide. Also, if $\tilde{z}_P(i,j) = 0$ for all $(i,j) \in E$ with $i \in P \setminus S_P$ then again (ps$'1_w$) and (ps$'$1) coincide. Now, if $\tilde{z}_P(i_0,j) = 1$ for some $(i_0,j) \in E$

††For the sake of convenience, X is represented in this form. For instance, if some $j \in Q$ is not matched in X then we can always add a pair (S_j, j) in X with $S_j = \emptyset$.

with $i_0 \in P \setminus S_P$ then we modify p as well as X, z_P and z_Q in such a way that (ps$'1_w$), (ps$'$2)$-$(ps$'$4) are preserved. We express this modification procedure here.

Define a set $E_P \subseteq E$ by

$$E_P = \{(i,j) \in E \mid \hat{z}_P(i,j) = 1\}. \tag{5.19}$$

Construct a directed graph $T = (\{i_0\} \cup E_P, A)$ with arc set A consisting of three disjoint sets of arcs A_0, A_1 and A_2 defined by

$$\begin{aligned} A_0 &:= \{(i_0,(i_0,j)) \mid (i_0,j) \in E_P\}, \\ A_1 &:= \{((i,j),(k,j)) \in E_P \times E_P \mid i \notin S_j, k \in S_j, \\ &\qquad \nu_{jk}(-p_{kj}) = r_j^{S_j} = \nu_{ji}(-p_{ij})\}, \\ A_2 &:= \{((i,j),(i,k)) \in E_P \times E_P \mid i \in S_j, j \neq k\}. \end{aligned} \tag{5.20}$$

For any $((i,j),(i,k)) \in A_2$ note that since $i \in S_j$, therefore, $i \notin S_k$. We assign a weight to each arc of A by defining the weight function $w : A \to \mathbf{R}$ as follows:

$$w(a) = \begin{cases} \ln \alpha_{i_0 j} & \text{if } a = (i_0,(i_0,j)) \in A_0, \\ -\ln \alpha_{ji} + \ln \alpha_{jk} & \text{if } a = ((i,j),(k,j)) \in A_1, \\ -\ln \alpha_{ij} + \ln \alpha_{ik} & \text{if } a = ((i,j),(i,k)) \in A_2. \end{cases} \tag{5.21}$$

The graph T now satisfies the following lemma.

Lemma 5.1 (Farooq [4]). *T has no negative cycle with respect to the weight function w.*

By Lemma 5.1, we can find shortest distances from i_0 to all vertices of T. Define $d : E \to \mathbf{R} \cup \{+\infty\}$ where $d(i,j)$ denotes the shortest distance from i_0 to (i,j) with respect to w in the graph T if $(i,j) \in E_P$, otherwise take $d(i,j) = +\infty$. For any parameter $\varepsilon \geq 0$, we define $p(\varepsilon)$ as follows:

$$p_{ij}(\varepsilon) = p_{ij} - \varepsilon \exp(-d(i,j)) \quad (\forall (i,j) \in E). \tag{5.22}$$

The definition (5.22) states that for any $(i,j) \in E$, p_{ij} is decreased by a parameter $\varepsilon > 0$ if there is a path from i_0 to (i,j) in the graph T, that is, $d(i,j) < +\infty$ and remains unchanged otherwise. We denote the set of all reachable vertices from i_0 in the graph T by $R(i_0)$, that is,

$$R(i_0) := \{(i,j) \in E \mid d(i,j) < +\infty\}. \tag{5.23}$$

Observe that $R(i_0) \subseteq E_P$. The following lemmas give few characteristics of the graph T. To avoid the confusion, we again mention that $S_j \in X$ and r_j is defined by (2.2) for any $j \in Q$.

Lemma 5.2. *Assume that $(i,j) \in R(i_0)$. If $i \notin S_j$ then $0 \leq \nu_{ji}(-p_{ij}) \leq r_j$.*

We omit the proof of the above lemma since it is an easy consequence of Lemma 4.1 of Farooq [4].

Lemma 5.3 (Farooq [4]). *If $(i,j) \in R(i_0)$ then $(i,k) \in R(i_0)$ for all $(i,k) \in E_P$. Furthermore, $\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon))$ for any $\varepsilon \geq 0$.*

Lemma 5.4 (Farooq [4]). *Assume that $(i,j) \in R(i_0)$ with $i \notin S_j$ and there exists $k \in S_j$ such that $r_j^{S_j} = \nu_{jk}(-p_{kj})$. Then the following statements hold:*

(i) *$\nu_{ji}(-p_{ij}(\varepsilon)) \leq \nu_{jk}(-p_{kj}(\varepsilon))$ for a sufficiently small $\varepsilon \geq 0$.*

(ii) $\nu_{ji}(-p_{ij}(\varepsilon)) \leq \nu_{jk}(-p_{kj}(\varepsilon))$ *for all* $\varepsilon \geq 0$ *if* $((i,j),(k,j)) \in A_1$.

(iii) $\nu_{ji}(-p_{ij}(\varepsilon)) = \nu_{jk}(-p_{kj}(\varepsilon))$ *for all* $\varepsilon \geq 0$ *if the arc* $((i,j),(k,j))$ *lies on a shortest path from* i_0 *to* (k,j).

Our aim is to propose an algorithm which finds a stable job allocation. In each iteration of the algorithm, we shall modify p by a parameter ε in such a way that the conditions (ps$'1_w$), (ps$'$2)$-$(ps$'$4) are preserved. The cases which may arise by modifying p are discussed below:

Case 1. For any $(i,j) \in R(i_0)$, we have $\nu_{ij}(p_{ij}) \geq \nu_{ik}(p_{ik})$ for all $(i,k) \in E$. If $(i,j),(i,k) \in R(i_0)$ then $\nu_{ij}(p_{ij}) = \nu_{ik}(p_{ik})$. Lemma 5.3 implies that $\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon))$ for all $\varepsilon \geq 0$. If $(i,j) \in R(i_0)$ and $(i,k) \notin R(i_0)$ then $\nu_{ij}(p_{ij}) > \nu_{ik}(p_{ik})$. Hence, we can find $\varepsilon > 0$ such that $\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon)) = \nu_{ik}(p_{ik})$. In this way, a new element (i,k) is added in E_P which may augment S_k. This, however, depends upon whether $|S_k| = \mu(k)$ or $|S_k| < \mu(k)$.

Case 2. Since valuations are strictly increasing, for any $(i,j) \in R(i_0)$ we can always find an $\varepsilon \geq 0$ such that $\nu_{ij}(p_{ij}(\varepsilon)) = 0$ or $p_{ij}(\varepsilon) = \underline{\pi}_{ij} > -\infty$. As mentioned earlier, we want (ps$'$1) and (ps$'1_w$) to coincide. If $\tilde{z}_P$ is decreased, (ps$'$1) and (ps$'1_w$) comes closer to each other. Now if $\nu_{ij}(p_{ij}(\varepsilon)) = 0$ then we can decrease z_0. Consequently, $\tilde{z}_P$ will decrease. If $p_{ij}(\varepsilon) = \underline{\pi}_{ij}$ then decreasing $p(\varepsilon)$ further would mean that it is no longer a feasible salary vector. Also, in this case, we can switch $z_P(i,j)$ to zero and $z_Q(i,j)$ to 1. Thus (ps$'$3) and (ps$'$4) are preserved, and $\tilde{z}_P$ will decrease.

Case 3. Let $(i,j) \in R(i_0)$ and $((i,j),(k,j)) \in A_1$. Without loss of generality, assume that $((i,j),(k,j))$ lies on the shortest path from i_0 to (k,j). Suppose that there exists $k' \in S_j \setminus \{k\}$. Then $r_j^{S_j} \leq \nu_{jk'}(-p_{k'j})$.

(i) If $r_j^{S_j} = \nu_{jk'}(-p_{k'j})$ then $((i,j),(k',j)) \in A_1$. By the construction of graph T, $((i,j),(k',j))$ lies on the shortest path from i_0 to (k',j). Therefore, for any $\varepsilon \geq 0$, $\nu_{jk}(-p_{kj}(\varepsilon)) = \nu_{jk'}(-p_{k'j}(\varepsilon))$ by Lemma 5.4.

(ii) If $r_j^{S_j} < \nu_{jk'}(-p_{k'j})$ then again by the construction of the graph T, $(k',j) \notin R(i_0)$. Hence, one can find $\varepsilon > 0$ such that $\nu_{jk}(-p_{kj}(\varepsilon)) = \nu_{jk'}(-p_{k'j}(\varepsilon))$. By Lemma 5.4, we get $\nu_{ji}(-p_{ij}(\varepsilon)) = \nu_{jk}(-p_{kj}(\varepsilon))$. Thus any $\varepsilon' > \varepsilon$ yields $\nu_{ji}(-p_{ij}(\varepsilon')) = \nu_{jk}(-p_{kj}(\varepsilon')) > \nu_{jk'}(-p_{k'j})$. But $\nu_{jk'}(-p_{k'j}) = \nu_{jk'}(-p_{k'j}(\varepsilon'))$ since $(k',j) \notin R(i_0)$. This shows that (ps$'$2) does not hold. Therefore, $p(\varepsilon)$ cannot be decreased any more.

Case 4. Let $(i,j) \in R(i_0)$ with $i \notin S_j$ and there exists $k \in S_j$ such that $r_j^{S_j} = \nu_{jk}(-p_{kj})$. From (2.2) and (5.7), it is easy to see that $r_j \leq r_j^{S_j}$. Then $\nu_{ji}(-p_{ij}) \leq r_j^{S_j}$ by Lemma 5.2.

(i) If $\nu_{ji}(-p_{ij}) = \nu_{jk}(-p_{kj})$ then $((i,j),(k,j)) \in A_1$. Lemma 5.4 implies that $\nu_{ji}(-p_{ij}(\varepsilon)) \leq \nu_{jk}(-p_{kj}(\varepsilon))$ for any $\varepsilon \geq 0$.

(ii) If $\nu_{ji}(-p_{ij}) < \nu_{jk}(-p_{kj})$ and $d(i,j) \geq d(k,j)$ then $\nu_{ji}(-p_{ij}(\varepsilon)) < \nu_{jk}(-p_{kj}(\varepsilon))$ for any $\varepsilon \geq 0$.

(iii) If $\nu_{ji}(-p_{ij}) < \nu_{jk}(-p_{kj})$ and $d(i,j) < d(k,j)$ then one can find an $\varepsilon > 0$ such that $\nu_{ji}(-p_{ij}(\varepsilon)) = \nu_{jk}(-p_{kj}(\varepsilon))$. Then $\nu_{ji}(-p_{ij}(\varepsilon')) > \nu_{jk}(-p_{kj}(\varepsilon'))$ for any $\varepsilon' > \varepsilon$. This, however, shows that (ps$'$2) does not hold. Therefore, in this case we cannot decrease $p(\varepsilon)$ any more.

With the discussion above, we define the following parameters.

$$\begin{aligned}
\varepsilon_1 &= \max\{\varepsilon \geq 0 \mid \nu_{ij}(p_{ij}(\varepsilon)) \geq \nu_{ik}(p_{ik}) \quad \forall (i,j) \in R(i_0),\\
&\quad \forall (i,k) \in E \text{ with } z_P(i,k) = 1 \text{ and } |S_k| < \mu(k)\},\\
\varepsilon_2 &= \max\{\varepsilon \geq 0 \mid \nu_{ij}(p_{ij}(\varepsilon)) \geq \nu_{ik}(p_{ik}) \quad \forall (i,j) \in R(i_0),\\
&\quad \forall (i,k) \in E \setminus R(i_0) \text{ with } z_P(i,k) = 1 \text{ and } |S_k| = \mu(k)\},\\
\varepsilon_3 &= \max\{\varepsilon \geq 0 \mid \nu_{ij}(p_{ij}(\varepsilon)) \geq 0 \quad \forall (i,j) \in R(i_0)\},\\
\varepsilon_4 &= \max\{\varepsilon \geq 0 \mid p_{ij}(\varepsilon) \geq \underline{\pi}_{ij} \quad \forall (i,j) \in R(i_0)\},\\
\varepsilon_5 &= \max\{\varepsilon \geq 0 \mid \nu_{ji}(-p_{ij}(\varepsilon)) \leq \nu_{jk}(-p_{kj}) \quad \forall (i,j) \in R(i_0),\\
&\quad \forall (k,j) \in E_P \text{ with } i,k \in S_j \text{ and } \nu_{ji}(-p_{ij}) < \nu_{jk}(-p_{kj})\},\\
\varepsilon_6 &= \max\{\varepsilon \geq 0 \mid \nu_{ji}(-p_{ij}(\varepsilon)) \leq \nu_{jk}(-p_{kj}(\varepsilon)) \quad \forall (i,j) \in R(i_0),\\
&\quad \forall (k,j) \in E_P \text{ with } i \notin S_j, k \in S_j, d(i,j) < d(k,j)\\
&\quad \text{ and } \nu_{ji}(-p_{ij}) < \nu_{jk}(-p_{kj})\},
\end{aligned} \tag{5.24}$$

where the maximum over an empty set is defined to be $+\infty$. Observe that the existence of ε_1 and ε_2 is due to Case 1, ε_3 and ε_4 is due to Case 2, ε_5 is due to Case 3 and ε_6 is due to Case 4. To modify the salary vector p we determine the parameter $\varepsilon \geq 0$ by

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}. \tag{5.25}$$

Note that ε is well defined by the definition of ε_3. For the sake of convenience, we assume that ε_1 and ε_2 exist for $(\hat{i},\hat{j}),(\hat{i},\hat{k}) \in E$, ε_3 and ε_4 exist for $(\hat{i},\hat{j}) \in E$ and, ε_5 and ε_6 exist for $(\hat{i},\hat{j}),(\hat{k},\hat{j}) \in E$. By the above definition of ε, if $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$ then $\nu_{\hat{i}\hat{j}}(p_{\hat{i}\hat{j}}(\varepsilon)) = \nu_{\hat{i}\hat{k}}(p_{\hat{i}\hat{k}}(\varepsilon))$. Similarly, if $\varepsilon = \varepsilon_3$ then $\nu_{\hat{i}\hat{j}}(p_{\hat{i}\hat{j}}(\varepsilon)) = 0$ and if $\varepsilon = \varepsilon_4$ then $p_{\hat{i}\hat{j}}(\varepsilon) = \underline{\pi}_{\hat{i}\hat{j}}$. Also, if $\varepsilon = \varepsilon_5$ or $\varepsilon = \varepsilon_6$ then $\nu_{\hat{j}\hat{i}}(-p_{\hat{i}\hat{j}}(\varepsilon)) = \nu_{\hat{j}\hat{k}}(-p_{\hat{k}\hat{j}}(\varepsilon))$.

Next we describe the algorithm which finds a stable outcome in a finite number of iterations.

Algorithm `Job_Allocation`

Step 0: Initially, define $p, z_P, z_0, \tilde{z}_P, \hat{z}_P$ by (5.2)−(5.6) and $z_Q, \tilde{\mu}, \eta_j\ (j \in Q), \eta, \Gamma$ by (5.8)−(5.12).

Step 1: Find a matching X in the bipartite graph $(\eta, Q; \Gamma)$ satisfying (5.13)−(5.15). Define S_P by (5.16) and update z_Q and $\tilde{\mu}$ by (5.17) and (5.18).

Step 2: If $P = S_P$ or for any $i \in P \setminus S_P$ we have $\tilde{z}_P(i,j) = 0$, for all $(i,j) \in E$, then stop.

Step 3: Let $i_0 \in P \setminus S_P$ such that $\tilde{z}_P(i_0, j) = 1$ for some $j \in Q$. Define E_P by (5.19).

Step 4: Construct a directed graph $T = (\{i_0\} \cup E_P, A)$ with arc set A consisting of A_0, A_1 and A_2 defined by (5.20). Define weight function w by (5.21). Find the shortest distances $d(i,j)$ from i_0 to all vertices (i,j) of T with respect to w and put $d(i,j) = +\infty$ if (i,j) is not a vertex of T. Determine ε by (5.25), find $p(\varepsilon)$ by (5.22) and define $R(i_0)$ by (5.23).

Step 5: **(a)** If $\varepsilon = \varepsilon_1$ then set $\tilde{\mu}(\hat{k}) := \tilde{\mu}(\hat{k}) + 1$ and go to Step 7; else go to (b).

(b) If $\varepsilon = \varepsilon_2$ then go to Step 7; else go to (c).

(c) If $\varepsilon = \varepsilon_3$ then set $z_0(\hat{i}, j) := 0$ for all $j \in Q$ and go to Step 7; else go to (d).

(d) If $\varepsilon = \varepsilon_4$ then set $z_P(\hat{i},\hat{j}) := 0$ and $z_Q(\hat{i},\hat{j}) := 1$ and go to Step 7; else go to (e).

(e) If $\varepsilon = \varepsilon_5$ then go to Step 7; else go to (f).

(f) If $\varepsilon = \varepsilon_6$ then, for each $j \in Q$, define $r_j^{S_j}$ by (5.7) for $p(\varepsilon)$. Construct a directed graph $T = (\{i_0\} \cup E_P, A)$ with arc set A consisting of A_0, A_1 and A_2 defined by (5.20) for $p(\varepsilon)$. Define weight function w by (5.21) and $R(i_0)$ by (5.23). If X satisfies (5.14) and (5.15) then put $p = p(\varepsilon)$ and go to Step 6; else go to Step 7.

Step 6: Update z_Q by (5.17). Find the shortest distances $d(i,j)$ from i_0 to all vertices (i,j) of T with respect to w and put $d(i,j) = +\infty$ if (i,j) is not a vertex of T. Determine ε by (5.25) and find $p(\varepsilon)$ by (5.22). Go to Step 5.

Step 7: Put $p := p(\varepsilon)$. Update $\tilde{z}_P$ and $\hat{z}_P$ by (5.5)–(5.6) and define η_j $(j \in Q), \eta$ and Γ by (5.10)–(5.12). Go to Step 1.

In the sequel, we shall use the notation [Step AA $\to$ Step BB] which means the `Job_Allocation` goes from Step AA to Step BB.

The following lemma describes the important features of `Job_Allocation`.

Lemma 5.5. *In each iteration of* `Job_Allocation`*, the following statements hold:*

(i) *z_P and z_0 decrease or remain the same. z_Q and $\tilde{\mu}$ increase or remain the same.*

(ii) *If $\varepsilon \in \{\varepsilon_1, \varepsilon_2\}$ or $\varepsilon = \varepsilon_5 < \min\{\varepsilon_3, \varepsilon_4\}$ or [Step 5 (f) $\to$ Step 7] is executed then $\hat{z}_P$ increases or remains the same. In particular, $\hat{z}_P$ increases if $\varepsilon = \varepsilon_1 > 0$ or $\varepsilon = \varepsilon_2 < \varepsilon_1$.*

(iii) *If $\varepsilon = \varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$ or $\varepsilon = \varepsilon_4 < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ then $\hat{z}_P$ decreases.*

(iv) *If [Step 5 (f) $\to$ Step 6] is executed then the graph $(\eta, Q; \Gamma)$ is preserved.*

(v) *If $\hat{z}_P(i,j)$ turns to 0 from 1 at Step 7, for some $(i,j) \in E$, then it never changes its orientation in the subsequent iterations.*

(vi) *If $\hat{z}_P(i,j)$ turns to 1 from 0 at Step 7, for some $(i,j) \in E$, then $p_{ij}(\varepsilon)$ is the initial value defined in* (5.2).

(vii) *For any $(i,j) \in E$, if $p_{ij}(\varepsilon) < p_{ij}$ at Step 4 or at Step 6 then $\tilde{\mu}(j) = \mu(j)$.*

Proof. The following inequality holds for any $(i,j) \in R(i_0)$:

$$\nu_{ij}(p_{ij}(\varepsilon)) \geq \nu_{ik}(p_{ik}(\varepsilon)) \qquad (\forall (i,k) \in E \setminus R(i_0) \text{ with } z_P(i,k) = 1) \tag{5.26}$$

since $\varepsilon \leq \min\{\varepsilon_1, \varepsilon_2\}$. If $\varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$ then

$$\nu_{ij}(p_{ij}(\varepsilon)) > \nu_{ik}(p_{ik}(\varepsilon)) \qquad (\forall (i,k) \in E \setminus R(i_0) \text{ with } z_P(i,k) = 1). \tag{5.27}$$

Also, for all $(i,j), (i,k) \in R(i_0)$, Lemma 5.3 yields that

$$\nu_{ij}(p_{ij}(\varepsilon)) = \nu_{ik}(p_{ik}(\varepsilon)). \tag{5.28}$$

For all $(i,j) \in R(i_0)$, (5.26) and (5.28) imply that

$$\nu_{ij}(p_{ij}(\varepsilon)) \geq \nu_{ik}(p_{ik}(\varepsilon)) \qquad (\forall (i,k) \in E \text{ with } z_P(i,k) = 1). \tag{5.29}$$

(i) z_0 decreases at Step 5 (c) if [Step 5 (c) $\to$ Step 7] is executed, else it is not updated. Similarly, z_P decreases at Step 5 (d) if [Step 5 (d) $\to$ Step 7] is executed, else it is not updated. This implies the first part of the assertion.

In each iteration at Step 1, $\tilde{\mu}$ is updated by (5.18) for the current matching where for each $j \in Q$, $\tilde{\mu}(j)$ is the number of workers employed by j. $\tilde{\mu}(\hat{k})$ increases at Step 5 (a) if [Step 5 (a) $\rightarrow$ Step 7] is executed. In the next iteration, $\tilde{\mu}(j)$ is the lower bound of the number of workers employed by j, for each $j \in Q$. Hence $\tilde{\mu}$ increases or remains the same. Next, we see that in each iteration, z_Q is updated at Step 1 and in some iterations it is updated at Step 6 as well. In either case, we use (5.17) to update z_Q. From (5.17), it is obvious to see that z_Q increases or remains the same.

(ii) Observe that $\varepsilon = \varepsilon_1$ if [Step 5 (a) $\rightarrow$ Step 7] is executed. If $\varepsilon = \varepsilon_1 = 0$ then $p_{ij}(\varepsilon) = p_{ij}$, for all $(i,j) \in E$, and hence $\hat{z}_P$ remains the same. If $\varepsilon = \varepsilon_1 > 0$ then $\nu_{\hat{i}\hat{j}}(p_{\hat{i}\hat{j}}) > \nu_{\hat{i}\hat{k}}(p_{\hat{i}\hat{k}})$ and $\nu_{\hat{i}\hat{j}}(p_{\hat{i}\hat{j}}(\varepsilon)) = \nu_{\hat{i}\hat{k}}(p_{\hat{i}\hat{k}}(\varepsilon))$. This together with the inequality (5.29) implies that $\hat{z}_P$ increases. Analogously, we can prove that $\hat{z}_P$ increases when [Step 5 (b) $\rightarrow$ Step 7] is executed. Note that in this case $0 < \varepsilon = \varepsilon_2 < \varepsilon_1$. Next, when [Step 5 (e) $\rightarrow$ Step 7] or [Step 5 (f) $\rightarrow$ Step 7] is executed then $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$. From (5.27) and (5.28), it is clear that $\hat{z}_P$ remains the same.

(iii) [Step 5 (c) $\rightarrow$ Step 7] is executed when $\varepsilon = \varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$ and [Step 5 (d) $\rightarrow$ Step 7] is executed when $\varepsilon = \varepsilon_4 < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. Note that by (i), z_0 and z_P remain the same or decrease. If [Step 5 (c) $\rightarrow$ Step 7] is executed then z_0 decreases at Step 5 (c) and if [Step 5 (d) $\rightarrow$ Step 7] is executed then z_P decreases at Step 5 (d). In either case, (5.27) and (5.29) imply that $\hat{z}_P$ decreases at Step 7.

(iv) If [Step 5 (f) $\rightarrow$ Step 6] is executed then $\varepsilon = \varepsilon_6 < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$. Therefore at Step 6, $\tilde{\mu}$ remains unchanged. Also by (ii), $\hat{z}_P$ remains the same or increases. But from the definitions of ε_1 to ε_6, we observe that $\hat{z}_P$ increases if and only if $\varepsilon = \varepsilon_1 > 0$ or $\varepsilon = \varepsilon_2 < \varepsilon_1$. Therefore $\hat{z}_P$ also remains unchanged when [Step 5 (f) $\rightarrow$ Step 6] is executed. Since $\varepsilon = \varepsilon_6 < \varepsilon_5$, the definitions of ε_5 and ε_6 together with Lemma 5.4 imply the (a) of (BC). The (b) of (BC) holds since $\varepsilon < \varepsilon_1$. Thus $(\eta, Q; \Gamma)$ is preserved at Step 6.

(v) If $\varepsilon = \varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$ or $\varepsilon = \varepsilon_4 < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ then $\hat{z}_P$ decreases. In other cases, it increases or remains the same. $\hat{z}_P$ decreases if z_0 decreases at Step 5 (c) or z_P decreases at Step 5 (d). In each iteration at Step 7, we have $\hat{z}_P \leq z_0$ and $\hat{z}_P \leq z_P$. Since z_0 and z_p decrease or remain the same, therefore, the decreased components of $\hat{z}_P$ can never increase.

(vi) If $\hat{z}_P(i,j)$ is 0 at Step 5 then $(i,j) \notin R(i_0)$. (5.22) gives $p_{ij}(\varepsilon) = p_{ij}$ and by (v), p_{ij} must be the initial value defined by (5.2). This proves the assertion.

(vii) This follows from the fact that $\varepsilon \leq \varepsilon_1$ and by (5.18). ☐

Lemma 5.6. *In each iteration of* `Job_Allocation`*, if there exists a matching X in the bipartite graph $(\eta, Q; \Gamma)$ at Step 1 then (S_j, z_Q, p) satisfies* (BC) *for all $j \in Q$, where z_Q is the vector updated at Step 1.*

Proof. We prove it by induction. In the first iteration at Step 1, the assertion obviously holds. Suppose that there exists a matching X in the bipartite graph $(\eta, Q; \Gamma)$ at Step 1 in the t-th iteration, $t \geq 2$, such that (S_j, z_Q, p) satisfies (BC) for all $j \in Q$, where z_Q is the vector updated at Step 1. Also, for convenience, we denote the vectors/sets calculated for $p(\varepsilon)$ at Step 7 by $z_P(\varepsilon), z_0(\varepsilon), \tilde{z}_P(\varepsilon), \hat{z}_P(\varepsilon), \eta_j(\varepsilon)$, for all $j \in Q$, $\eta(\varepsilon)$ and $\Gamma(\varepsilon)$.

Let $S \in 2^P$ and $j' \in Q$ is such that $\chi_S(i) \leq \hat{z}_P(\varepsilon)(i,j')$ for all $i \in P$ and $|S_{j'}| \leq |S|$. Suppose that $(S, z_Q, p(\varepsilon))$ satisfies (BC) for j'. To prove the assertion, it is equivalent to show that $(S, z_Q(\varepsilon), p(\varepsilon))$ satisfies (BC) for j', where $z_Q(\varepsilon)$ is defined by

$$z_Q(\varepsilon)(i,j) := \begin{cases} 1 & \text{if } j = j' \text{ and } [p_{ij}(\varepsilon) < p_{ij} \text{ or } i \in S] \\ z_Q(i,j) & \text{otherwise} \end{cases} \qquad (\forall (i,j) \in E).$$

If $p_{ij'}(\varepsilon) = p_{ij'}$, for all $(i,j') \in E$, then the following holds:

$$[z_Q(i,j') = 0, \quad z_Q(\varepsilon)(i,j') = 1] \Longrightarrow i \in S.$$

In this case, it is obvious to see that $(S, z_Q(\varepsilon), p(\varepsilon))$ satisfies (BC) for j'. If there exists $(i,j') \in E$ such that $p_{ij'}(\varepsilon) < p_{ij'}$ then

$$|S| = |S_{j'}| = \mu(j') \tag{5.30}$$

by Lemma 5.5 (vii). In this case, firstly, we show that $r^S_{j'}(\varepsilon) \geq r^{S_{j'}}_{j'}(\varepsilon)$, where $r^S_{j'}(\varepsilon)$ and $r^{S_{j'}}_{j'}(\varepsilon)$ are calculated by (5.7) for $p(\varepsilon)$. On contrary, suppose that $r^S_{j'}(\varepsilon) < r^{S_{j'}}_{j'}(\varepsilon)$. Since $\varepsilon \leq \varepsilon_5$, we have

$$\nu_{j'i}(-p_{ij'}(\varepsilon)) \geq r^{S_{j'}}_{j'}(\varepsilon) \quad (\forall i \in S_{j'}). \tag{5.31}$$

Let $i' \in S$ is such that $r^S_{j'}(\varepsilon) = \nu_{j'i'}(-p_{i'j'}(\varepsilon))$. From (5.31), we get $i' \notin S_{j'}$. Then (5.30) yields that there exists $\tilde{i} \in S_{j'}$ such that $\tilde{i} \notin S$. Definition (5.17) gives $z_Q(\tilde{i},j') = 1$ and (5.31) yields $\nu_{j'\tilde{i}}(-p_{\tilde{i}j'}(\varepsilon)) \geq r^{S_{j'}}_{j'}(\varepsilon)$. But $r^S_{j'}(\varepsilon) < r^{S_{j'}}_{j'}(\varepsilon)$, that is, $\nu_{j'\tilde{i}}(-p_{\tilde{i}j'}(\varepsilon)) > r^S_{j'}(\varepsilon)$. This contradicts that $(S, z_Q, p(\varepsilon))$ satisfies (BC) for j'. Therefore $r^S_{j'}(\varepsilon) \geq r^{S_{j'}}_{j'}(\varepsilon)$. Next, we prove that $(S, z_Q(\varepsilon), p(\varepsilon))$ satisfies (BC) for j'. It suffices to prove (a) of (BC).

Let $i \notin S$ such that $z_Q(\varepsilon)(i,j') = 1$. On contrary, suppose that

$$\nu_{j'i}(-p_{ij'}(\varepsilon)) > r^S_{j'}(\varepsilon). \tag{5.32}$$

Since $(S, z_Q, p(\varepsilon))$ satisfies (BC) for j', we must have $z_Q(i,j') = 0$. This implies that $i \notin S_{j'}$ and $p_{ij'}(\varepsilon) < p_{ij'}$, that is, $(i,j') \in R(i_0)$. By Lemma 5.2, we have $\nu_{j'i}(-p_{ij'}) \leq r_{j'}$. But $r_{j'} \leq r^{S_{j'}}_{j'}$ by definition. Lemma 5.4 and the definition of ε_6 give

$$\nu_{j'i}(-p_{ij'}(\varepsilon)) \leq r^{S_{j'}}_{j'}(\varepsilon). \tag{5.33}$$

Since $r^{S_{j'}}_{j'}(\varepsilon) \leq r^S_{j'}(\varepsilon)$, the inequalities (5.32) and (5.33) contradict. Hence $(S, z_Q(\varepsilon), p(\varepsilon))$ satisfies (BC) for j'. □

Lemma 5.7. *In each iteration of* `Job_Allocation`*, there exists a matching at Step 1 in the bipartite graph* $(\eta, Q; \Gamma)$ *satisfying* (5.13)−(5.15).

Proof. The initial selection of η_j, for all $j \in Q$, by (5.10) implies that there exists a matching at Step 1 in the first iteration satisfying (5.13). Hence, one can find a matching satisfying (5.13)−(5.15). We suppose that there exists a matching $X = \{(S_j, j) \mid j \in Q\}$ in the bipartite graph $(\eta, Q; \Gamma)$ at Step 1 satisfying (5.13)−(5.15) in the t-th iteration, $t \geq 2$. To avoid any confusion, we specify that z_Q is the vector after update at Step 1. The vectors/sets calculated for $p(\varepsilon)$ at Step 7 are denoted by $z_P(\varepsilon), z_0(\varepsilon), \tilde{z}_P(\varepsilon), \hat{z}_P(\varepsilon), \eta_j(\varepsilon)$, for all $j \in Q$, $\eta(\varepsilon)$ and $\Gamma(\varepsilon)$.

Claim: For all $j \in Q$, $(S_j, z_Q, p(\varepsilon))$ satisfies (BC).

[Proof of Claim] Let $(i,j) \in E$ is such that $i \notin S_j$ and $z_Q(i,j) = 1$. (S_j, z_Q, p) satisfies (BC) by Lemma 5.6, hence $\nu_{ji}(-p_{ij}) \leq r^{S_j}_j$. Since $\varepsilon \leq \min\{\varepsilon_5, \varepsilon_6\}$ and by Lemma 5.4, we get

$$\nu_{ji}(-p_{ij}(\varepsilon)) \leq r^{S_j}_j(\varepsilon), \tag{5.34}$$

where $r^{S_j}_j(\varepsilon)$ is calculated by (5.7) for $p(\varepsilon)$. Now let $\nu_{ji}(-p_{ij}(\varepsilon)) > 0$. If $\nu_{ji}(-p_{ij}) > 0$ then $|S_j| = \mu(j)$. If $\nu_{ji}(-p_{ij}) \leq 0$ then $p_{ij}(\varepsilon) < p_{ij}$, that is, $(i,j) \in R(i_0)$, and Lemma 5.5 (vii) implies that $|S_j| = \mu(j)$. This together with (5.34) implies that $(S_j, z_Q, p(\varepsilon))$ satisfies (BC). [end of proof of Claim]

We first consider the case when [Step 5 (a) → Step 7] is executed. Set $\widehat{S}_{\hat{k}} = S_{\hat{k}} \cup \{\hat{i}\}$. Then by Lemma 5.5 (ii), we have $\chi_{\widehat{S}_{\hat{k}}}(i) \leq \hat{z}_P(\varepsilon)(i, \hat{k})$ for all $i \in P$. We show that $\widehat{S}_{\hat{k}} \in \eta_{\hat{k}}(\varepsilon)$. By the above Claim, $(S_{\hat{k}}, z_Q, p(\varepsilon))$ satisfies (BC) for $\hat{k}$. Then obviously $(\widehat{S}_{\hat{k}}, z_Q, p(\varepsilon))$ satisfies (BC) for $\hat{k}$. Also $\tilde{\mu}(\varepsilon)(\hat{k}) \leq |\widehat{S}_{\hat{k}}| \leq \mu(\hat{k})$. Hence $\widehat{S}_{\hat{k}} \in \eta_{\hat{k}}(\varepsilon)$.

Now if $\hat{i} = i_0$ then $\widehat{X} = \{(S_j, j) \mid j \in Q \setminus \{\hat{k}\}\} \cup \{(\widehat{S}_{\hat{k}}, \hat{k})\}$ is a matching satisfying (5.13). Suppose $\hat{i} \neq i_0$ and, without loss of generality, assume that $\hat{i} \in S_{\hat{j}}$. Then there exists a shortest path S from i_0 to $(\hat{i}, \hat{j})$ in T denoted by

$$S = (i_0, j_0), (i_1, j_0), \ldots, (i_s, j_s) = (i_s, \hat{j}), (i_{s+1}, \hat{j}) = (\hat{i}, \hat{j}) \tag{5.35}$$

such that $((i_h, j_h), (i_{h+1}, j_h)) \in A_1$ for $h = 0, 1, \ldots, s-1$ and $((i_h, j_{h-1}), (i_h, j_h)) \in A_2$ for $h = 1, \ldots, s$. Then by the construction of the graph T, $i_h \notin S_{j_h}$ and $i_{h+1} \in S_{j_h}$ for all $h = 0, 1, \ldots, s$. Now define

$$\widehat{S}_{j_h} = \{S_{j_h} \setminus \{i_{h+1}\}\} \cup \{i_h\} \quad (\forall h = 0, 1, \ldots, s). \tag{5.36}$$

Also for any $i \in P$, Lemma 5.5 (ii) gives $\chi_{\widehat{S}_{j_h}}(i) \leq \hat{z}_P(\varepsilon)(i, j_h)$ for all $h = 0, 1, \ldots, s$. We prove that $(\widehat{S}_{j_h}, z_Q, p(\varepsilon))$ satisfies (BC), for all $h = 0, 1, \ldots, s$. From the above Claim, $(S_{j_h}, z_Q, p(\varepsilon))$ satisfies (BC). Since $((i_h, j_h), (i_{h+1}, j_h))$, for all $h = 0, 1, \ldots, s$, lies on the shortest path from i_0 to $(\hat{i}, \hat{j})$, by Lemma 5.4 and the fact that $\varepsilon \leq \min\{\varepsilon_5, \varepsilon_6\}$, we have

$$r_{j_h}^{\widehat{S}_{j_h}}(\varepsilon) = \nu_{j_h i_h}(-p_{i_h j_h}(\varepsilon)) = \nu_{j_h i_{h+1}}(-p_{i_h j_{h+1}}(\varepsilon)) = r_{j_h}^{S_{j_h}}(\varepsilon) \quad (\forall h = 0, 1, \ldots, s),$$

which shows that $(\widehat{S}_{j_h}, z_Q, p(\varepsilon))$ satisfies (BC) for $j_h, h = 0, 1, \ldots, s$. Let $\widehat{Q} = \{j_h \mid h = 0, 1, \ldots, s\} \cup \{\hat{k}\}$. Then note that $\widehat{X} = \{(S_j, j) \mid j \in Q \setminus \widehat{Q}\} \cup \{(\widehat{S}_j, j) \mid j \in \widehat{Q}\}$ is a matching in the bipartite graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$ satisfying (5.13) and hence one can find a matching at Step 1 in $(t+1)$-th iteration satisfying (5.13)–(5.15). The case when [Step 5 (b) → Step 7] or [Step 5 (e) → Step 7] is executed, Lemma 5.5 and the above Claim guarantee that X is a matching in the bipartite graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$ satisfying (5.13). Next, we deal the case when [Step 5 (c) → Step 7] is executed. By Lemma 5.3 and since $\varepsilon = \varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$, we get

$$\chi_{S_j}(i) \leq \hat{z}_P(\varepsilon)(i, j) \quad (\forall (i, j) \in E \text{ with } i \neq \hat{i}). \tag{5.37}$$

If $\hat{i} = i_0$ then from (5.37) and by the above Claim, obviously X is a matching in the bipartite graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$ satisfying (5.13). If $\hat{i} \neq i_0$ then by the modification at Step 5 (c), we get $\hat{z}_P(\varepsilon)(\hat{i}, j) = 0$ for all $j \in Q$. Obviously, X is not a matching in the graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$ and we need some manipulation. Since there exists $j \in Q$ such that $\hat{i} \in S_j$, without loss of generality, we assume that $\hat{i} \in S_{\hat{j}}$. Then there exists a shortest path S from i_0 to $(\hat{i}, \hat{j})$ in the graph T denoted by (5.35). Defining the sets $\widehat{S}_{j_h}$, for all $h = 0, 1, \ldots, s$, by (5.36), analogously we can show that $(\widehat{S}_{j_h}, z_Q, p(\varepsilon))$ satisfies (BC) for $j_h, h = 0, 1, \ldots, s$. Let $\widehat{Q} = \{j_h \mid h = 0, 1, \ldots, s\}$ and set

$$\widehat{X} = \{(S_j, j) \mid j \in Q \setminus \widehat{Q}\} \cup \{(\widehat{S}_j, j) \mid j \in \widehat{Q}\}.$$

Observe that $\widehat{X}$ is a matching in the graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$ satisfying (5.13).

Finally, we consider the case when [Step 5 (d) → Step 7] is executed. Lemma 5.3 and the inequality $\varepsilon = \varepsilon_4 < \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ yield (5.37). Now if $\hat{i} \notin S_{\hat{j}}$ then inequality (5.37) holds for all $(i, j) \in E$. Above Claim implies that X is a matching in the graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$. If $\hat{i} \in S_{\hat{j}}$ then just like the previous case, we can find a matching $\widehat{X}$ in the bipartite graph $(\eta(\varepsilon), Q; \Gamma(\varepsilon))$ satisfying (5.13). This completes the proof. □

Lemma 5.8. *In each iteration of* `Job_Allocation`*, the quadruple* $(X; p, z_P, z_Q)$ *at Step 1 satisfies (ps′*1_w*), (ps′2)–(ps′4).*

Proof. Since in each iteration, $\varepsilon \leq \varepsilon_3$ and for any $j \in Q$, $\chi_{S_j}(i) \leq \hat{z}_P(i, j)$ for all $i \in P$. Therefore, the definition of $\hat{z}_P$ implies that (ps′1_w) holds. It is obvious to see that if (S_j, z_Q, p), for all $j \in Q$, satisfies (BC) then (ps′2) holds, where z_Q is updated at Step 1. By the Lemma 5.6, (S_j, z_Q, p) satisfies (BC) for all $j \in Q$ at Step 1, therefore (ps′2) always holds at Step 1. As discussed earlier that the initial selection of z_P and z_Q by (5.3) and (5.17) implies that (ps′3) and (ps′4) hold. Now z_P decreases or remains the same and z_Q increases or remains the same in each iteration by Lemma 5.5. Whenever some component of z_P is decreased at Step 5 (d), the corresponding components of z_Q and p are 1 and the lower bound, respectively. Therefore, (ps′3) and (ps′4) hold in each iteration. □

Next is our main result which shows that if `Job_Allocation` terminates then we get a pairwise stable job allocation.

Theorem 5.9. *If* `Job_Allocation` *terminates then the output* $(X; p, z_P, z_Q)$ *satisfies (ps′1)–(ps′4) and hence* X *is pairwise stable.*

Proof. By Theorem 3.1, X is pairwise stable if $(X; p, z_P, z_Q)$ satisfies (ps′1)−(ps′4). By Lemma 5.8, (ps′1_w), (ps′2)−(ps′4) are satisfied at Step 1 in each iteration. If `Job_Allocation` terminates at Step 2, we observe that $P = S_P$ or for all $(i, j) \in E$ with $i \in P \setminus S_P$, we have $\tilde{z}_P(i, j) = 0$. This means that (ps′1_w) and (ps′1) coincide at termination, that is, $(X; p, z_P, z_Q)$ satisfies (ps′1)−(ps′4). □

In the rest of the work, we shall prove that the `Job_Allocation` terminates after a finite number of iterations.

Let S, $S' \in 2^P$ such that $S \cap S' = \emptyset$ and $|S| = |S'| \neq 0$. We say that $j \in Q$ *replaces* S *by* S' in the t-th iteration of `Job_Allocation`, $t \geq 2$, if

(i) $S \subseteq S_j$ and $S' \cap S_j = \emptyset$ in $(t-1)$-th iteration.

(ii) $S' \subseteq S_j$ and $S \cap S_j = \emptyset$ in t-th iteration.

For each $j \in Q$, we define $\rho(j) \subseteq P$ in each iteration of `Job_Allocation` at Step 1 by

$$\rho(j) := \{i \in S_j \mid \nu_{ji}(-p_{ij}) = r_j^{S_j}\}.$$

Lemma 5.10. *In each iteration of* `Job_Allocation`*, if* $|\rho(j)|$ *decreases for some* $j \in Q$ *then* $\tilde{\mu}(j)$ *increases or* $z_Q(i, j)$ *increases for some* $i \in P$*, where* z_Q *is the vector updated at Step 1.*

Proof. If $|\rho(j)|$ decreases for some $j \in Q$ at Step 1 in t-th iteration, $t \geq 2$, then either $\tilde{\mu}(j)$ increases or there exists $S, S' \in 2^P$ with $S \cap S' = \emptyset$ and $|S| = |S'| \neq 0$ such that j replaces S by S'. In the later case, one can easily see that z_Q increases. This completes the proof. □

Lemma 5.11. `Job_Allocation` *terminates in a finite number of iterations.*

Proof. We first mention that in each iteration at Step 4 and Step 6, we use Moore-Bellman-Ford algorithm to find the shortest distances from a single source to all other vertices of the graph T. We show that all executions of `Job_Allocation` for different values of ε are finite.

If [Step 5 (a) $\rightarrow$ Step 7] is executed then $\tilde{\mu}$ increases. By Lemma 5.5 (i), $\tilde{\mu}$ increases or remains the same in each iteration. Therefore this execution is possible at most $\sum_{j\in Q}\mu(j)$ times.

If [Step 5 (b) $\rightarrow$ Step 7] is executed then $\hat{z}_P$ increases by Lemma 5.5 (ii). From Lemma 5.5 (v), we observe that [Step 5 (b) $\rightarrow$ Step 7] can be executed at most $|E|$ times.

By Lemma 5.5 (i), z_P and z_0 decrease or remain the same in each iteration and if [Step 5 (c) $\rightarrow$ Step 7] is executed then z_0 decreases. Therefore [Step 5 (c) $\rightarrow$ Step 7] is executed atmost $|P|$ times. Similarly, if [Step 5 (d) $\rightarrow$ Step 7] is executed then z_P decreases. Therefore [Step 5 (d) $\rightarrow$ Step 7] is executed at most $|E|$ times.

Note that in any iteration, if $\sum_{j\in Q}|\rho(j)| = |S_P|$ at Step 1 then [Step 5 (e) $\rightarrow$ Step 7] cannot be executed. For any $j \in Q$, $|\rho(j)|$ may increase, decrease or remain unchanged in any iteration. Lemma 5.10 implies that if $|\rho(j)|$, for some $j \in Q$, decreases then $\tilde{\mu}$ or z_Q increase. By Lemma 5.5 (i), $\tilde{\mu}$ and z_Q remain the same or increase. Therefore, the total number of possible iterations when $|\rho(j)|$, for some $j \in Q$, decreases are $\sum_{j\in Q}\mu(j) + |E|$. Hence [Step 5 (e) $\rightarrow$ Step 7] can be executed at most $2\sum_{j\in Q}\mu(j) + |E|$ times.

Next, we consider the case when [Step 5 (f) $\rightarrow$ Step 7] is executed. We suppose that the above mentioned cases do not occur. Then the sum in (5.14) remains the same or increases. Since P and Q are finite, this sum can be increased a finite number of times only. If the sum in (5.14) remains the same then the sum in (5.15) increases. Again, since P and Q are finite, the sum in (5.15) can be increased a finite number of times if the sum in (5.14) remains same constantly. Therefore, [Step 5 (f) $\rightarrow$ Step 7] can be executed only a finite number of times.

Finally, we see that [Step 5 (f) $\rightarrow$ Step 6] is executed in a finite number of times if the other cases do not occur. Let us suppose the other cases do not occur. Then in execution of [Step 5 (f) $\rightarrow$ Step 6], $R(i_0)$ enlarges or remains the same. Since $R(i_0)$ can be enlarged at most $|E|$ times, we discuss the case when $R(i_0)$ remains the same. In such a case, distance of some $(i, j) \in R(i_0)$ is decreased. Also, the distance of each element of $R(i_0)$ remains the same or decreases in each execution of [Step 5 (f) $\rightarrow$ Step 6]. Since finite number of paths from i_0 to each $(i, j) \in R(i_0)$ can be found, therefore, [Step 5 (f) $\rightarrow$ Step 6] is executed finite number of times if the other cases do not occur. By Lemma 5.5 (iv), observe that the graph $(\eta, Q; \Gamma)$ remains intact and X is a matching in $(\eta, Q; \Gamma)$ satisfying (5.13)$-$(5.15).

Thus, `Job_Allocation` terminates after a finite number of iterations. $\square$

Remark 5.12. We have shown the existence of a stable outcome in the job market with linear valuations and possibly bounded salaries. In our model, each worker can work for at most one firm and each firm can employ as many workers as it wishes. One can see that the complexity of the `Job_Allocation` may not be polynomial, specially when $\mu \neq (1, \ldots, 1)$. We leave it as an open problem to design an algorithm which finds a stable outcome in our model having a polynomial complexity.

Acknowledgements

The authors are indebted to Akihisa Tamura (Keio University) for his careful reading of an earlier version of this paper and pointing out a fatal error. Authors are also thankful to Muneer A. Rashid (NUST-CAMP) and Azad A. Siddiqui (NUST-EME) for the discussions and very useful comments. The second author is grateful to the financial support by Global

COE (Kyoto University) which made it possible to visit Keio University where a part of this work was completed.

References

[1] V.P. Crawford and E.M. Knoer, Job matching with heterogeneous firms and workers, *Econometrica* 49 (1981) 437–450.

[2] G. Demange and D. Gale, The strategy structure of two sided matching markets, *Econometrica* 53 (1985) 873–888.

[3] K. Eriksson and J. Karlander, Stable matching in a common generalization of the marriage and assignment models, *Discrete Math.* 217 (2000) 135–156.

[4] R. Farooq, A polynomial-time algorithm for a stable matching problem with linear valuations and bounded side payments, *Japan J. Indust. Appl. Math.* 25 (2008),83–98.

[5] R. Farooq, Y.T. Ikebe and A. Tamura, On labor allocation model with possibly bounded salaries, *J. Oper. Res. Soc. Japan* 51 (2008) 136–154.

[6] S. Fujishige and A. Tamura, A general two-sided matching market with discrete concave utility functions, *Discrete Appl. Math.* 154 (2006) 950–970.

[7] S. Fujishige and A. Tamura, A two-sided discrete-concave market with possibly bounded side payments: An approach by discrete convex analysis, *Math. Oper. Res.* 32 (2007) 136–155.

[8] D. Gale and L.S. Shapley, College admissions and the stability of marriage, *Amer. Math. Monthly* 69 (1962) 9–15.

[9] M. Kaneko, The central assignment game and the assignment markets, *J. Math. Econom.* 10 (1982) 205–232.

[10] J.A.S. Kelso and V.P. Crawford, Job matching coalition formation, and gross substitution, *Econometrica* 50 (1982) 1483–1504.

[11] K. Murota, Convexity and Steinitz's exchange property, *Adv. Math.* 124 (1996) 272–311.

[12] K. Murota, Discrete convex analysis, *Math. Program.* 83 (1998) 313–371.

[13] K. Murota, *Discrete Convex Analysis*, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[14] A.E. Roth and M.A.O. Sotomayor, Stable outcomes in discrete and continuous models of two-sided matching: A unified treatment, *Rev. Econom.* 16 (1996) 1–24.

[15] L.S. Shapley and M. Shubik, The assignment game I: The core, *Internat. J. Game Theory* 1 (1972) 111–130.

[16] M. Sotomayor, Existence of stable outcomes and the lattice property for a unified matching market, *Math. Social Sci.* 39 (2000) 119–132.

[17] M. Sotomayor, Core structure and comparative statics in a hybrid matching market, *Games Econom. Behav.* 60 (2007) 357–380.

Manuscript received 12 November 2009
accepted for publication 6 May 2010

Yasir Ali
Centre for Advanced Mathematics and Physics,
National University of Sciences and Technology, H-12 Islamabad, Pakistan
E-mail address: ali@camp.nust.edu.pk,

Rashid Farooq
Centre for Advanced Mathematics and Physics,
National University of Sciences and Technology, H-12 Islamabad, Pakistan
E-mail address: farooq@kurims.kyoto-u.ac.jp, farooq@camp.nust.edu.pk