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TRANSFORMED FUNCTIONS FOR GLOBAL OPTIMIZATION WITH LINEAR CONSTRAINTS*

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Abstract: In this paper, two transformed functions are presented for constrained global optimization with multi-extremum objective function. These transformed functions take a key role in the proceeding for the global optimization. We prove that if the current minimizer is not a global one, there must exist an optimal point of the transformed function in a "lower region" such that the objective value is less than the current minimizer. Thus, the primal problem can get a better minimizer. Moreover, an algorithm is given to show the application of transformed functions.

Key words: *nonlinear programming, global optimization, linear constraints, transformed function*

Mathematics Subject Classification: *90C26, 90C30*

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1 Introduction

It is well know that for many optimization methods such as [8], [13] et al., convexity is a key condition to get its global minimizer. However, many practical applications in engineering, finance and management rely on solving such a global optimization problem with a nonconvex objective function. So, the existence of multiple local minimums is a challenge because they may bring two difficulties: how to judge the current minimizer is a global one, and how to leave a minimizer to another better one if it is not global. In the last few years, many theories and algorithms for global optimizations had been developed. We can see the literature summary from Horst, Pardalos and Thoai [5]. Specifically, some practical methods such as the tunnelling algorithm and the filled function algorithm successively make the movement from current local minimizer to another better one by constructing an auxiliary function (see [3], [6], [9], [12]).

Let us recall some existing auxiliary functions, tunneling functions and filled functions. The tunneling function for unconstrained optimization proposed by Levy and Montalvo [6] is

$$T(x, x^*) = \frac{f(x) - f(x_m^*)}{\prod_{i=1}^{m}[(x - x_i^*)^T (x - x_i^*)]^{\eta_i}},$$

where x_i^* is a local minimizer and η_i is a sufficient large number so that x_i^* becomes a pile of $T(x, x^*)$.

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The first filled function presented by Ge [3] is

$$P(x, r, \rho) = \frac{1}{r + f(x)} exp\left(-\frac{\|x - x^*\|^2}{\rho}\right),$$

where r and ρ are two adjustable parameters. And another filled function with prefixed point x_0,

$$U(x, A, h) = \eta(\|x - x_0\|)\varphi(A[f(x) - f(x^*) + h])$$

was proposed by Ge and Qin [4] and further discussed by Lucidi and Piccialli [7]. In paper [9], two classes of functions which possess both characters of filled functions and tunneling functions are discussed. The usage of the filled function is similar to the tunneling function. Besides the tunneling function and the filled function, other methods such as the dynamic tunneling function [11] and sub-energy method [1], etc. are also efficient to global optimization. Because all of those auxiliary functions are composite of $f(x)$, they are called *the transformed function method*.

The transformed function method for unconstrained global optimization consists of two phases. In the first phase, a classical algorithm such as Newton's method or steepest descent method can be used to find a local minimizer x^* of objective function $f(x)$. In the second phase, we search for either a root or a minimizer of the transformed function. Under the help of the transformed function, move the iterative point out of the current valley. In this paper, we extend the transformed function method for unconstrained global optimization to constrained global optimization.

Consider the problem with nonconvex objective and linear constraints as follows:

$$\min\{f(x) : x \in X\} \tag{1.1}$$

where $f(x) : R^n \to R$ is a nonconvex function, $X = \{x \in R^n : Ax \leq b\}$ is the feasible region, A is an $m \times n$ matrix, and $b = (b_1, b_2, \ldots, b_m)^T \in R^m$. Our objective is to find a global minimizer of problem (1.1).

This paper is organized as follows. In Section two, we introduce some assumptions, notes and two transformed functions. In Section three, we discuss some properties of the transformed functions defined in section two. In Section four, we introduce an algorithm for problem (1.1). At last, in the Section 5, we show two numerical results concerning the algorithm.

2 Notes and Assumptions

Let $J(x) = \{i : a_i^T x = b_i,\ i \in \{1, 2, \ldots, m\}\}$ be an index set of active constraints at point x , where a_i^T is ith row of matrix A. If $J(x) = \emptyset, \forall x \in X$, the problem degenerates to unconstrained. So we always assume that $J(x) \neq \emptyset$, and the number of element in $J(x)$ is $|J(x)| = S$.

Define hyperplane $H_J = \{x|\bar{A}x = \bar{b}\}$, where $J = J(\bar{x})$, $\bar{A} = \{a_i^T : i \in J(\bar{x})\}$, $\bar{b} = (b_i : i \in J(\bar{x}))^T \in R^S$ for an $\bar{x} \in X$. Let $S_1(\bar{x}) = \{x|f(x) \geq f(\bar{x}), x \neq \bar{x}\}$ be the high-level set and $S_2(\bar{x}) = \{x|f(x) < f(\bar{x})\}$ be the low-level set. Let $L(P)$ be the set of local minimizers and $G(P)$ be the set of global minimizers of problem (1.1) respectively. x_G denotes a global minimizer of (1.1), i.e. $x_G \in G(P)$.

We need the following assumptions:

Assumption 2.1. $f(x)$ has only a finite number of local minima (minimal function values).

Assumption 2.2. $f(x)$ is coercive, namely, $f(x) \to +\infty$ as $||x|| \to +\infty$.

Assumption 2.3. The gradient $\nabla f(x)$ is continuous on Ω, where Ω is a box.

Assumption 2.4. The row vectors of sub-matrix A_x are linearly independent.

Assumption 2.1 means that there is a large enough box Ω such that $L(P) \subset \Omega$, i.e. Ω contains all of local minimizers of $f(x)$ in its interior. According to Assumption 2.1 and 2.2, we only consider the global minimizers of (1.1) in $X \cap \Omega$. So, we suppose that X is a bounded domain.

In the paper, two transformed functions for problem (1.1) at point $x' \in X$ are defined

$$T(x, x') = \frac{f(x) - f(x') + r}{\|x - x'\|^{\alpha}}, \tag{2.1}$$

$$Q(x, x') = \frac{1}{\|x - x'\|^{\alpha}}[f(x) - f(x') + r + \alpha^3 \max_{1 \le j \le m} \{0, a_j^T x - b_j\}] \tag{2.2}$$

where r, α are two parameters satisfying that $r > 0$ is a small number and $\alpha \ge 1$ is a large one. In (2.1) and (2.2), the parameter $r > 0$ is regarded as a user-defined tolerance because it would to be adjusted in the computing process. If the zero point of the transformed function can not be found when $r > 0$ is small enough, then the current local minimizer is a global minimizer. Based on this idea, the parameter r plays a key role in deciding when the method should stop. Theoretically, r can be set as

$$0 < r < min\{f(x^*) - f(x_G)\}. \tag{2.3}$$

where $x^* \in L(P) \setminus G(P)$.

The auxiliary problem of (1.1) at x' is defined

$$\min\{T(x, x') : \bar{A}x = \bar{b}\} \tag{2.4}$$

where $\bar{A}$ is an $S \times n$ matrix, $\bar{b} \in R^S$.

3 Properties of Transformed Functions

From Assumption 2.4, the following lemma holds obviously.

Lemma 3.1. *If $\bar{A}$ is full of row rank , then the projection matrix $P = I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}$ is positive semi-definite and $\bar{A}P = 0$.*

If we define $d_f(x) = -P\nabla f(x)$, then the following result is clear.

Theorem 3.2. (1) *$d_f(x) \neq 0$ is a feasible descent direction of subproblem*

$$\min\{f(x) : \bar{A}x = \bar{b},\ x \in \Omega\}. \tag{3.1}$$

(2) *There exists an $x' \in H_J \cap \Omega$ satisfying $d_f(x') = 0$ if and only if x' is a KKT point of subproblem (3.1).*

If x' is a local minimizer of subproblem (3.1), then the transformed function (2.1) has both characters of the filled functions and the tunneling functions [9]. The descend direction of auxiliary problem (2.4) can be taken as

$$\begin{aligned} d_T &= d_f + \frac{\alpha(f(x)-f(x')+r)}{\|x-x'\|^2}(x-x') \\ &= -P\nabla f(x) + \frac{\alpha(f(x)-f(x')+r)}{\|x-x'\|^2}(x-x'). \end{aligned} \tag{3.2}$$

The gradient of $T(x,x')$ is

$$\nabla T(x,x') = \frac{1}{\|x-x'\|^\alpha}[\nabla f(x) - \frac{\alpha(f(x)-f(x')+r)}{\|x-x'\|^2}(x-x')].$$

Let $\bar{A} = (B\ \ N)$, where matrix B is an invertible matrix of size S. From $x = (x_B,\ x_N)^T$ and $\bar{A}x = \bar{b}$, we get $x_B = B^{-1}\bar{b}–B^{-1}Nx_N$ and $f(x) = f(x_B, x_N) = f(B^{-1}\bar{b}–B^{-1}Nx_N,\ x_N)$. Denote $\bar{f}(x_N) = f(B^{-1}\bar{b} - B^{-1}Nx_N,\ x_N)$, then

$$\nabla \bar{f}(x_N) = -(B^{-1}N)^T \nabla f_B(x) + \nabla f_N(x). \tag{3.3}$$

Theorem 3.3. *Under Assumption* 2.1 − 2.4*, the following two results are true on the intersection set of hyperplane* $H = H_J = \{x|\ \bar{A}x = \bar{b}\}$ *and* Ω*.*

(1) *If* $x \neq x'$ *and* $f(x) \geq f(x')$*, then ,* $(x-x')^T\nabla T(x,x') < 0$ *holds for sufficiently large* α*.*

(2) *If* x' *is not a global minimizer of subproblem* (3.1)*, there must be an* $\bar{x}' \in \{x|\ f(x) < f(x'), x \in H \cap \Omega\}$ *to be the minimizer of function* $T(x,x')$ *on* $H \cap \Omega$ *with some* α *and* r*.*

Proof. (1). Let $x \in H,\ x \neq x'$, and $f(x) \geq f(x')$, then

$$\begin{aligned} (x-x')^T\nabla T(x,x') &\leq \frac{1}{\|x-x'\|^\alpha}\left[(x_N - x'_N)^T \begin{pmatrix} -B^{-1}N \\ I_{n-S} \end{pmatrix}^T \begin{pmatrix} \nabla f_B(x) \\ \nabla f_N(x) \end{pmatrix} - \alpha r\right] \\ &= \frac{1}{\|x-x'\|^\alpha}\left[(x_N - x'_N)^T \nabla \bar{f}(x_N) - \alpha r\right]. \end{aligned}$$

From Assumption 2.1 and Assumption 2.3, there exists an $M > 0$ such that $\|(x_N - x'_N)^T\nabla\bar{f}(x_N)\| \leq M$. So, $(x-x')^T\nabla T(x,x') < 0$ holds for $\alpha > \frac{M}{r}$.

(2). Since x' is not a global minimizer of subproblem (3.1) on $H \cap \Omega$, the set $\{x|\ f(x) < f(x'), x \in H \cap \Omega\}$ is not empty. From the definition of $T(x,x')$, we get

$$\begin{aligned} T(x,x') &= \frac{f(x)-f(x')+r}{\|x-x'\|^\alpha} = \frac{\bar{f}(x_N) - \bar{f}(x'_N) + r}{\left\| \begin{pmatrix} -B^{-1}N \\ I_{n-S} \end{pmatrix} (x_N - x'_N) \right\|^\alpha} \\ &\triangleq \bar{T}(x_N, x'_N). \end{aligned}$$

It implies that auxiliary problem (2.4) is an unconstrained optimization with dimension $n - S$ on hyperplane H.

From Assumption 2.3 and $\bar{T}(x_N, x'_N) \to +\infty$ as $x_N \to x'_N$, we learn that there is a neighborhood of x'_N, $O(x'_N) \subset H$ such that $\bar{T}(x_N, x'_N)$ is continuous in $H \backslash O(x'_N)$. Therefore, function $T(x,x')$ is continuous on bounded closed region $H \cap \{\Omega \setminus O(\bar{x})\}$, and it must have a minimizer. Let

$$\bar{x}' = arg \min_{x \in H \cap \{\Omega \setminus O(\bar{x})\}} T(x,x'). \tag{3.4}$$

Then both x' and $\bar{x}^{'}$ are in hyperplane H.

On the other hand, according to the assumption of the theorem, $f(x')$ is not a global minimum of the subproblem(3.1). But, $f(x)$ certainly obtain its global minimum on the bounded closed region $H \cap \Omega$ from the continuity. Now, we denote this global minimizer of subproblem (3.1) by x_H. If let $0 < r < f(x') - f(x_H)$, then it is true that $T(x_H, x') < 0$ based on the definition of $T(x, x')$ in (2.1).

From (3.4) and the definition of $\bar{x}^{'}$, we have $T(\bar{x}^{'}, x') \leq T(x_H, x') < 0$, which implies $f(\bar{x}^{'}) < f(x')$ from (2.1), namely, $\bar{x}^{'} \in \{x|\ f(x) < f(x'), x \in H \cap \Omega\}$. We finish the proof. □

Theorem 3.3 shows that transformed function $T(x, x')$ has not any stationary point in $H \cap S_1(x')$, since $x - x'$ is a descent direction of $T(x, x')$. Therefore, if x^* is a stationary point of $T(x, x')$, there must be $x^* \in H \cap S_2(x')$. The following theorem holds, provided the descent method is used for $T(x, x')$.

Theorem 3.4. *If $x \in H \cap \Omega$ and $d_T \neq 0$, then d_T is a feasible descent direction of auxiliary problem* (2.4) .

Proof. From (3.2), Lemma 3.1 and $\bar{A}x = \bar{b} = \bar{A}x'$, we have

$$\bar{A}d_T = -\bar{A}P\nabla f(x) + \frac{\alpha(f(x) - f(x') + r)}{\|x - x'\|^2}\bar{A}(x - x') = 0,$$

i.e. d_T is feasible. To prove the descent, we only need to show $\nabla T(x, x')^T d_T < 0$. From $\bar{A}(x - x') = 0$, we have

$$\begin{aligned} & \nabla T(x, x')^T d_T \\ = \quad & -\nabla T(x, x')^T \left\{ P\nabla f(x) - \frac{\alpha(f(x) - f(x') + r)}{\|x - x'\|^2}[(x - x') - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}(x - x')] \right\} \\ = \quad & -\|x - x'\|^\alpha \nabla T(x, x')^T P \nabla T(x, x'). \end{aligned}$$

From $d_T \neq 0$ and $P(x - x') = x - x'$, we have

$$\|x - x'\|^\alpha P\nabla T(x, x') = P\left[\nabla f(x) - \frac{\alpha(f(x) - f(x') + r)}{\|x - x'\|^2}(x - x')\right] = -d_T \neq 0.$$

Finally, from the positive semi-definition of P, we get $\nabla T(x, x')^T d_T < 0$. □

Now we discuss the properties of another transformed function (2.2).

Theorem 3.5. *If $x \in \Omega$ but $x \notin X$, then $\nabla_x Q(x, \bar{x}^*) \neq 0$ for large enough α.*

Proof. Set $V = \{j \mid a_j^T x > b_j, j \in J\}$. Then $V \neq \emptyset$ from $x \notin X$. Let

$$a_{j_0}^T x - b_{j_0} = \max_{j \in V}\{a_j^T x - b_j\}.$$

Suppose $x \notin X$. From (2.2), we have

$$\nabla_x Q(x, \bar{x}^*) = \frac{1}{\|x - \bar{x}^*\|^\alpha}\left[\nabla f(x) + \alpha^3 a_{j_0} - \frac{\alpha(f(x) - f(\bar{x}^*) + r + \alpha^3(a_{j_0}^T x - b_{j_0}))(x - \bar{x}^*)}{\|x - \bar{x}^*\|^2}\right].$$

Therefore,

$$(x-\bar{x}^*)^T\nabla_x Q(x,\bar{x}^*) = \frac{1}{\|x-\bar{x}^*\|^\alpha}[(x-\bar{x}^*)^T\nabla f(x)-\alpha(f(x)-f(\bar{x}^*)+r) + \alpha^3 a_{j_0}^T(x-\bar{x}^*)-\alpha^4(a_{j_0}^T x-b_{j_0})].$$

Since $x\notin X$, we learn $(a_{j_0}^T x-b_{j_0})>0$. And according to Assumption 2.1 and Assumption 2.3, both $(x-\bar{x}^*)^T\nabla f(x)$ and $f(x)-f(\bar{x}^*)+r$ are bounded. Thus,

$$(x-\bar{x}^*)^T\nabla_x Q(x,\bar{x}^*)<0$$

holds for large enough α. This implies that

$$\nabla_x Q(x,\bar{x}^*)\neq 0, \quad \forall x\notin X.$$

We finish the proof. □

Theorem 3.6. *If $x\in X\cap S_1(\bar{x}^*)$, then $\nabla_x Q(x,\bar{x}^*)\neq 0$ for sufficiently large α.*

Proof. For $x\in X$,

$$Q(x,\bar{x}^*)=\frac{1}{\|x-\bar{x}^*\|^\alpha}[f(x)-f(\bar{x}^*)+r].$$

For $x\in S_1(\bar{x}^*)$, namely $f(x)\geq f(\bar{x}^*)$ and $x\neq\bar{x}^*$,

$$(x-\bar{x}^*)^T\nabla_x Q(x,\bar{x}^*)\leq\frac{1}{\|x-\bar{x}^*\|^\alpha}[(x-\bar{x}^*)^T\nabla f(x)-\alpha r].$$

As we know, $|(x-\bar{x}^*)^T\nabla f(x)|$ is bounded and $r>0$. So, there must be $(x-\bar{x}^*)^T\nabla_x Q(x,\bar{x}^*,a_j)<0$ when α is large enough. This means $\nabla_x Q(x,\bar{x}^*)\neq 0$. □

Theorem 3.7. *Suppose that $\bar{x}^*\in X$ is a local minimizer of $f(x)$, $x_1,x_2\in X\cap S_1(\bar{x}^*)$, and $\|x_1-\bar{x}^*\|\leq\|x_2-\bar{x}^*\|-\epsilon$, where $0<\epsilon<1$. If*

$$\alpha>\max\left\{1,\frac{\ln(f(x_1)-f(\bar{x}^*)+r)-\ln(f(x_2)-f(\bar{x}^*)+r)}{\ln(1-\epsilon)}\right\}, \tag{3.5}$$

then

$$Q(x_1,\bar{x}^*)>Q(x_2,\bar{x}^*).$$

Proof. For $x_1,x_2\in X\cap S_1(\bar{x}^*)$, we have $f(x_1)-f(\bar{x}^*)+r>0$, $f(x_2)-f(\bar{x}^*)+r>0$, and

$$Q(x_i,\bar{x}^*)=\frac{1}{\|x_i-\bar{x}^*\|^\alpha}[f(x_i)-f(\bar{x}^*)+r], \quad i=1,2.$$

Consider the following two cases.

(i) $f(x_1)\geq f(x_2)$. Take notice of $\|x_1-\bar{x}^*\|<\|x_2-\bar{x}^*\|$, it is obvious that

$$\frac{f(x_1)-f(\bar{x}^*)+r}{f(x_2)-f(\bar{x}^*)+r}>\Big[\frac{\|x_1-\bar{x}^*\|}{\|x_2-\bar{x}^*\|}\Big]^\alpha, \qquad \alpha\geq 1.$$

This means $Q(x_1,\bar{x}^*)>Q(x_2,\bar{x}^*)$.

(ii) $f(x_1)<f(x_2)$. From $\|x_2-\bar{x}^*\|-\|x_1-\bar{x}^*\|\geq\epsilon$, we can get

$$\frac{\|x_1-\bar{x}^*\|}{\|x_2-\bar{x}^*\|}\leq 1-\epsilon. \tag{3.6}$$

From (3.5), we have

$$\frac{f(x_1) - f(\bar{x}^*) + r}{f(x_2) - f(\bar{x}^*) + r} > (1 - \epsilon)^{\alpha}.$$

Combining (3.6) and (3.5), we know $Q(x_1, \bar{x}^*) > Q(x_2, \bar{x}^*)$. The proof is ended. □

Theorem 3.7 shows that as long as α is sufficiently large, the further a point leave the current local minimizer $\bar{x}^*$, the less the value of $Q(x, \bar{x}^*)$ would be in $X \cap S_1(\bar{x}^*)$.

Theorem 3.8. *If $\bar{x}^* \notin G(P)$, then there exists an $x^{'} \in X \cap S_2(\bar{x}^*)$ to be a minimizer of $Q(x, \bar{x}^*)$ for suitable $r > 0$ and $\alpha \geq 1$.*

Proof. Assumption $\bar{x}^* \notin G(P)$ has two meanings: $X \cap S_2(\bar{x}^*) \neq \emptyset$ and $f(x_G) < f(\bar{x}^*)$, where x_G is a global minimizer of problem (1.1) .

From $x_G \in X$ and (2.3), we obtain

$$Q(x_G, \bar{x}^*) = \frac{1}{\|x_G - \bar{x}^*\|}[f(x_G) - f(\bar{x}^*) + r] < 0.$$

We set $O(\bar{x}^*)$ to be a neighborhood of $\bar{x}^*$. Since $Q(x, \bar{x}^*)$ is continuous on bounded and closed region $\Omega \setminus O(\bar{x}^*)$, there is a minimizer $x^{'}$ such that

$$Q(x^{'}, \bar{x}^*) \leq Q(x_G, \bar{x}^*) < 0. \tag{3.7}$$

Now we prove that $x^{'} \in X \cap S_2(\bar{x}^*)$. Firstly, suppose $x^{'} \notin X$. Then, expression (2.2) becomes

$$Q(x^{'}, \bar{x}^*) = \frac{1}{\|x^{'} - \bar{x}^*\|^{\alpha}}[f(x^{'}) - f(\bar{x}^*) + r + \alpha^3(a_{j_0}^T x - b_{j_0})],$$

Because $a_{j_0}^T x - b_{j_0} > 0$ $(j_0 \in V)$, we have $Q(x^{'}, \bar{x}^*) > 0$ for large enough α. This is a contradiction to (3.7). So $x^{'} \in X$.

Secondly, suppose $x^{'} = \bar{x}^*$. Then, from the definition of $Q(x, \bar{x}^*)$, we get $\lim\limits_{x \to \bar{x}^*} Q(x, \bar{x}^*) = +\infty$. This contradicts (3.7) also.

Thirdly, suppose $x \in X \cap S_1(\bar{x}^*)$. Then, from $f(x) \geq f(\bar{x}^*)$, we have

$$Q(x, \bar{x}^*) = \frac{1}{\|x - \bar{x}^*\|^{\alpha}}[f(x) - f(\bar{x}^*) + r] > 0.$$

This still contradicts (3.7).

So, $x^{'} \in X \cap S_2(\bar{x}^*)$ is true. We finish the proof. □

Combining the theorem 3.3, the theorem 3.5, the theorem 3.6 and the theorem 3.8, we conclude that if there is a minimizer of the transformed function, it should be in low-level set $X \cap S_2(\bar{x}^*)$. In other words, the transformed function has no stationary point in high-level set $X \cap S_1(\bar{x}^*)$ or out of feasible X. As a result, we know that if any stationary point of the transformed function can not be found, the current minimizer can be looked as a global minimizer.

4 Transformed Function Method and Numerical Results

In the section, we present the algorithm based on the transformed functions.

Algoritheorem 4.1 (TFM).

1 Take an initial point $x^0 \in X$.

2 Solve the problem (1.1) and obtain a local minimizer $\bar{x}$.

3.1 Set $P = I - \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}$, $\bar{x}^1 = \bar{x}$, and $t = 1$.

3.2 Solve the auxiliary problem :

$$\min\{T(x, \bar{x}^t) : \bar{A}x = \bar{b}\}. \tag{4.1}$$

If auxiliary problem (4.1) has no solution when α is large enough and $r > 0$ is small enough, then $\bar{x}^* = \bar{x}^t$, goto step 4.1.

3.3 Let $\bar{x}^{'}$ be a solution of (4.1). If $A\bar{x}^{'} \leq b$, goto step 3.4; otherwise, let $\bar{x}^* = \bar{x}^t$ and goto step 4.1.

3.4 Let $\bar{x}^{'}$ be an initial point and $d_f(x) = -P\nabla f(x)$ be a descent direction at point x. Solve subproblem (3.1) and get a local minimizer $\bar{x}^{t+1}$. If $A\bar{x}^{t+1} \leq b$, let $t = t+1$ and goto step 3.2; otherwise, let $x^0 = \bar{x}^{'}$ and goto step 2.

4.1 Set

$$\lambda_J = -(\bar{A}\bar{A}^T)^{-1}\bar{A}\nabla f(\bar{x}^*) = (\lambda_j, j \in J(\bar{x}^1))^T. \tag{4.2}$$

4.2 If there exist $j \in J(\bar{x}^1)$ such that $\lambda_j < 0$, let $x^0 = \bar{x}^*$ and goto step 2.

4.3 If $\lambda_J \geq 0$ and there is a $j \in \{1, 2, \ldots .m\} \setminus J(\bar{x}^1)$ such that $a_j^T\bar{x}^* = b_j$, then let $J(\bar{x}^1) = J(\bar{x}^1) \cup \{j\}$ and goto step 4.1.

4.4 If $\lambda_J \geq 0$ and $a_j^T\bar{x}^* < b_j$ for all $j \in \{1, 2, \ldots .m\} \setminus J(\bar{x}^1)$, goto step 5.

5 At point $\bar{x}^*$, solve the problem

$$\min\{Q(x, \bar{x}^*) : x \in \Omega\}. \tag{4.3}$$

If problem (4.3) has no solution when α is large enough and $r > 0$ is small enough, $\bar{x}^*$ is a global minimizer of (1.1) and stop; If (4.3) has a solution $\hat{x}$, then let $x^0 = \hat{x}$ and goto step 2.

From our discussion in section 3, we have the following convergent theorem.

Theorem 4.2. *When algorithm (TFM) stops, $(\bar{x}^*, \lambda)$ is a KKT-pair of problem* (1.1)*, where λ_j for $j \in J(\bar{x}^*)$ is defined by* (4.2) *and $\lambda_j = 0$ for $j \notin J(\bar{x}^*)$.*

5 Numerical Test

In the section we show some results of numerical test for the algorithm. We choose two problems which are from [10] and [2]. The problems are computed by MATLAB and the adjustable parameters are selected as $3 \leq \alpha \leq 15$, $0.001 \leq r \leq 0.1$. We use the gradient projection method to find a local minimizer. In the following Table 1 and Table 2, $x^k, \bar{x}^k$

and $\hat{x}^k$ represent the local minimizer of (1.1) , the solution of (4.1) and the solution of (4.3) concerning with the k-th iterative for searching local minimizer of the primal problem respectively.

Example 5.1 (Consider the problem [10]).

$$\begin{aligned} \min \quad & f(x) = -\frac{1}{3}x_1^3 + x_2^2 + x_3^2 + x_2x_3 + \sin x_4 + 2x_1 - 4x_2 + 3x_3 - 4x_4 \\ s.t. \quad & 2x_1 + 2x_2 + x_3 + x_4 \leq 0 \\ & 3x_1 - x_2 + 2x_3 - 4x_4 + 2 \leq 0 \\ & -1 \leq x_i \leq 1, \ i = 1, 2, 3, 4. \end{aligned}$$

In this problem, $x^* = (-1, 1, -1, 1)$ is a global minimizer with an objective value $f^* = -10.8249$. We choose the initial point $x^0 = (-1, 0, 0, 0)$, then $f(x^0) = -1.6667$. The numerical results are showed in Table 1.

Table 1

k	x^k	$f(x^k)$	$\bar{x}^k$	$\hat{x}^k$	$f(\bar{x})$ or $f(\hat{x})$
1	$\begin{pmatrix} -1.0000 \\ 0.9999 \\ -0.7499 \\ 0.7499 \end{pmatrix}$	-9.4217	$\begin{pmatrix} -1.0000 \\ 0.9996 \\ -0.7505 \\ 0.7506 \end{pmatrix}$		-9.4246
2	$\begin{pmatrix} -1.0000 \\ 0.9999 \\ -0.7507 \\ 0.7509 \end{pmatrix}$	-9.4269	no solution	$\begin{pmatrix} -1.0000 \\ 0.9969 \\ -0.7726 \\ 0.7596 \end{pmatrix}$	-9.5014
3	$\begin{pmatrix} -1.0000 \\ 0.9753 \\ -0.9158 \\ 0.9650 \end{pmatrix}$	-10.4565	$\begin{pmatrix} -1.0000 \\ 0.9978 \\ -0.9890 \\ 0.9911 \end{pmatrix}$		-10.7657
4	$\begin{pmatrix} -1.0000 \\ 0.9986 \\ -0.9895 \\ 0.9921 \end{pmatrix}$	-10.7726	no solution	$\begin{pmatrix} -1.0000 \\ 1.0000 \\ -1.0000 \\ 0.9967 \end{pmatrix}$	-10.8138
5	$\begin{pmatrix} -1.0000 \\ 1.0000 \\ -1.0000 \\ 0.9999 \end{pmatrix}$	-10.8248	no solution	no solution	

Example 5.2 (Consider the problem).

$$\begin{aligned} \min \quad & f(x) = -50(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) - 10.5x_1 - 7.5x_2 - 3.5x_3 - 2.5x_4 - 1.5x_5 - 10x_6 \\ s.t. \quad & 6x_1 + 3x_2 + 3x_3 + 2x_4 + x_5 \leq 6.5 \\ & 10x_1 + 10x_3 + x_6 \leq 20 \\ & 0 \leq x_i \leq 1, \ i = 1, 2, 3, 4, 5; \quad x_6 \geq 0 \end{aligned}$$

This example is taken from Test Problem 2 of Ref. [2]. $x^* = (0, 1, 0, 1, 1, 20)$ is a global minimizer with an objective value $f^* = -361.5$. We choose the initial point $x^0 = (0, 0.5, 0, 0, 0, 1)$. The numerical results are showed in Table 2.

Table 2

k	x^k	$f(x^k)$	$\bar{x}^k$	$\hat{x}^k$	$f(\bar{x})$ or $f(\hat{x})$
1	$\begin{pmatrix} 0.0000 \\ 0.9999 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 1.0867 \end{pmatrix}$	-68.3565	$\begin{pmatrix} 0.0000 \\ 0.9951 \\ 0.0000 \\ 0.0000 \\ 0.0977 \\ 1.1022 \end{pmatrix}$		-68.6097
2	$\begin{pmatrix} 0.0000 \\ 1.0000 \\ 0.0000 \\ 0.0000 \\ 0.9999 \\ 1.9075 \end{pmatrix}$	-128.0649	$\begin{pmatrix} 0.0000 \\ 1.0000 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 1.9995 \end{pmatrix}$		-128.9950
3	$\begin{pmatrix} 0.0000 \\ 1.0000 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 19.9999 \end{pmatrix}$	-308.9990		$\begin{pmatrix} 0.0000 \\ 1.0000 \\ 0.0000 \\ 0.0457 \\ 1.0000 \\ 20.0000 \end{pmatrix}$	
4	$\begin{pmatrix} 0.0000 \\ 1.0000 \\ 0.0000 \\ 0.9999 \\ 1.0000 \\ 20.0000 \end{pmatrix}$	-361.4898	no solution	no solution	

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