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FUZZY NONLINEAR MIXED RANDOM VARIATIONAL-LIKE INCLUSIONS

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Abstract: In this paper, we consider a class of randomly h-η-maximal monotone mappings and a class of generalized nonlinear mixed random variational-like inclusions for random fuzzy mappings and define an iterative algorithm for finding approximate solutions for the class of variational inclusions. By using the random resolvent operator of randomly h-η-maximal monotone mappings, we establish the approximate solutions obtained by our algorithm converge to the exact solutions of the generalized nonlinear mixed random variational-like inclusions for random fuzzy mappings.

Key words: *nonlinear mixed random variational-like inclusions, random fuzzy mappings, randomly (h_t, η)-maximal monotone mappings, randomly strongly monotone mappings, randomly relaxed Lipschitz continuous mappings, randomly relaxed monotone mappings, Hausdorff metric*

Mathematics Subject Classification: *49J40, 47H19*

1 Introduction

A variational inclusion is one of the useful and important generations of variational inequalities. It was introduced and considered by Hassouni and Moudafi [20] in 1994, and a perturbed algorithm for finding approximate solutions of the variational inclusions was developed by them. May authors [1, 2, 6, 15, 16, 23, 28, 31, 33, 41] have obtained some important results on variational inclusions with their algorithms to obtain approximate solutions to them in various different assumptions.

A fuzzy set introduced by Zadeh [45] is an extension of a crisp set by enlarging the truth valued set $\{0, 1\}$ to the real unit interval $[0, 1]$. A fuzzy set is characterized by, and identified with a mapping called a membership-grade function from the whole set into $[0, 1]$. Heilpern [21] introduced the concept of fuzzy mappings and showed a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of Nadler's fixed point theorem for multi-valued mappings. In 1989, Chang and Zhu [11] introduced the concept of variational inequalities with fuzzy mappings and extended some of results of Lassonde [30], Shih and Tan [37], Takahashi [38], Yen [43] in the fuzzy setting. Later, they were developed by Agarwal *et al.* [3], Ahmad *et al.* [4], Ding [12, 13], etc..

On the other hand, random variational inequality problems and random quasi-variational inequality problems have been considered by Chang [7], Chang and Huang [9, 10], Huang [24, 25], Husain *et al.* [27], Tan *et al.* [39], Yuan [44], Khan and Salahuddin [29], Salahuddin [36] and Tan [40], etc..

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In 2003, Fang and Huang [17] introduced a class of H-monotone operators and the resolvent operator associated with the operators, with its Lipschitz continuity. They also considered a class of variational inclusions involving H-monotone operators and constructed an algorithm for solving the class of variational inclusions by using their resolvent operator technique.

At the same time, Huang and Fang [26] introduced a class of maximal η-monotone operators and defined an associated resolvent operator. Using their resolvent operator methods, they developed some iterative algorithms to approximate the solution of a class of variational inclusions involving maximal η-monotone operators. Huang and Fang's method extended the resolvent operator method associated with an η-subdifferential operator.

In 2005, Fang et al. [18] introduced a new class of (H, η)-monotone operators which unify a framework for a class of maximal monotone operators, a class of maximal η-monotone operators and a class of H-monotone operators, and studied a system of variational inclusions by using the resolvent operators associated with (H, η)-monotone operators in Hilbert spaces.

Very recently, Peng and Zhu [34] introduced and studied one new system of generalized mixed quasi-variational inclusions with (H, η)-monotone operators. By using the resolvent technique for the (H, η)-monotone operators, they proved the existence of solutions for the system of generalized mixed quasi-variational inclusions and the convergence of a new iterative algorithm approximating the solution for the system.

In [35], they also, very recently, introduced and studied another new system of set-valued variational inclusions with (H, η)-monotone operators. By using the resolvent technique for the (H, η)-monotone operators, they showed the existence of solutions for the system, and proved the convergence of a new three-step iterative algorithm approximating the solution for the system.

Basing on the notion of (H, η)-monotonicity for solving a generalized inclusion problem, Verma [42] also developed a generalized framework for the Eckstein-Bertsekas proximal point algorithm.

Our aim of this paper is to introduce and study generalized nonlinear mixed random variational-like inclusions for random fuzzy mappings. By using random resolvent operator technique of randomly (h_t, η)-maximal monotone mappings, we prove the approximate solutions obtained by the iterative algorithm converge to the exact solution of the generalized nonlinear mixed random variational-like inclusions for random fuzzy mappings.

2 Preliminaries

Throughout this paper, (Ω, Σ) is a measurable space with a set Ω and a σ-algebra Σ of subsets of Ω. H is a real separable Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Notations $\mathcal{B}(H)$, 2^H and $CB(H)$ denote the class of Borel σ-fields in H, the family of all nonempty subsets of H, the family of all nonempty closed bounded subsets of H, respectively.

Let $D(\cdot,\cdot)$ represent the Hausdorff metric on $CB(H)$ defined by

$$D(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\} \text{ for all } A, B \in CB(H),$$

where

$$d(a, B) = d(B, a) = \inf_{b\in B} \|a - b\| \text{ for } a \in A.$$

Definition 2.1. A mapping $u : \Omega \to H$ is said to be measurable if for any $B \in \mathcal{B}(H)$, $u^{-1}(B) = \{t \in \Omega : u(t) \in B\} \in \Sigma$.

Definition 2.2. A mapping $f : \Omega \times H \to H$ is called a random mapping if for each fixed $x \in H$, a mapping $f(\cdot, x) : \Omega \to H$ is measurable. A random mapping f is said to be continuous if for each fixed $t \in \Omega$, a mapping $f(t, \cdot) : H \to H$ is continuous.

Definition 2.3. A multi-valued mapping $T : \Omega \to 2^H$ is said to be measurable if for any $B \in \mathcal{B}(H)$, $T^{-1}(B) = \{t \in \Omega : T(t) \cap B \neq \emptyset\} \in \Sigma$.

Definition 2.4. A mapping $u : \Omega \to H$ is called a measurable selection of a measurable multi-valued mapping $T : \Omega \to 2^H$, if u is measurable and for any $t \in \Omega$, $u(t) \in T(t)$.

Definition 2.5. A mapping $T : \Omega \times H \to 2^H$ is called a random multi-valued mapping if for each fixed $x \in H$, $T(\cdot, x) : \Omega \to 2^H$ is a measurable multi-valued mapping. A random multi-valued mapping $T : \Omega \times H \to CB(H)$ is said to be D-continuous if for each fixed $t \in \Omega$, $T(t, \cdot) : H \to 2^H$ is continuous with respect to the Hausdorff metric D.

Let $\mathcal{F}(H)$ be a collection of fuzzy sets over H. A mapping F from Ω into $\mathcal{F}(H)$ is called a fuzzy mapping on H. If F is a fuzzy mapping on H, then for any $t \in \Omega$, $F(t)$ (denoted by F_t) is a fuzzy set on H and $F_t(x)$ is the membership-grade of x in F_t.

Let $A \in \mathcal{F}(H)$, $\alpha \in [0, 1]$, then the set

$$(A)_\alpha = \{x \in H : A(x) \geq \alpha\}$$

is called an α-cut of A.

Definition 2.6. A fuzzy mapping $F : \Omega \to \mathcal{F}(H)$ is said to be measurable, if for any $\alpha \in (0, 1]$, a multi-valued mapping $(F(\cdot))_\alpha : \Omega \to 2^H$ is measurable.

Definition 2.7. A fuzzy mapping $F : \Omega \times H \to \mathcal{F}(H)$ is called a random fuzzy mapping, if for each fixed $x \in H$, $F(\cdot, x) : \Omega \to \mathcal{F}(H)$ is a measurable fuzzy mapping.

Let $A, T, G : \Omega \times H \to \mathcal{F}(H)$ be random fuzzy mappings satisfying the following condition $(*)$;

$(*)$ there exist functions $\alpha, \beta, \gamma : H \to (0, 1]$ such that $(A_{t,x})_{\alpha(x)}$, $(T_{t,x})_{\beta(x)}$ and $(G_{t,x})_{\gamma(x)} \in CB(H)$ for all $(t, x) \in \Omega \times H$, where $A_{t,x}$ denotes the value of A at (t, x).

Induce random multi-valued mappings $\tilde{A}$, $\tilde{T}$ and $\tilde{G}$ from A, T and G, respectively as follows:

$$\tilde{A} : \Omega \times H \to CB(H), \quad (t, x) \mapsto (A_{t,x})_{\alpha(x)},$$

$$\tilde{T} : \Omega \times H \to CB(H), \quad (t, x) \mapsto (T_{t,x})_{\beta(x)},$$

and

$$\tilde{G} : \Omega \times H \to CB(H), \quad (t, x) \mapsto (G_{t,x})_{\gamma(x)} \text{ for all } (t, x) \in \Omega \times H.$$

Let $N : \Omega \times H \times H \times H \to H$ and $\eta : \Omega \times H \times H \to H$ be random mappings. Let $g : \Omega \times H \to H$ be a random mapping with $g(t, x(t)) \cap \mathrm{Dom}\partial\varphi(\cdot, y(t)) \neq \emptyset$ for $t \in H$, $x(t) \in H$ and fixed $y(t) \in H$, where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semi-continuous functional $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$.

Now we consider the following problem;

Find measurable mappings $x, u, v, w : \Omega \to H$ such that for all $t \in \Omega$ and each fixed $y(t) \in H$, $A_{t,x(t)}(u(t)) \geq \alpha(x(t))$, $T_{t,x(t)}(v(t)) \geq \beta(x(t))$, $G_{t,x(t)}(w(t)) \geq \gamma(x(t))$,

$$g(t, x(t)) \cap \mathrm{Dom}\,\partial\varphi(\cdot, y(t)) \neq \emptyset$$

and

$$\langle N(t,\, u(t),\, v(t), w(t)), \eta(t,\, y(t),\, g(t, x(t)))\rangle \;\geq\; \varphi(g(t, x(t)), x(t)) - \varphi(x(t), y(t)),$$

called a fuzzy nonlinear mixed random variational-like inclusion (**FNMRVLI**). The set of measurable mappings (x, u, v, w) is called a random solution of **FNMRVLI**.

If $\alpha(x) = \beta(x) = \gamma(x) = 1$, for all $x \in H$, then **FNMRVLI** is reduced to finding measurable mappings $x, u, v, w : \Omega \to H$ such that for $t \in \Omega$ and fixed $y(t) \in H$, $u(t) \in \tilde{A}(t, x(t))$, $v(t) \in \tilde{T}(t, x(t))$, $w(t) \in \tilde{G}(t, x(t))$, $g(t, x(t)) \cap \mathrm{Dom}\partial\varphi(\cdot, y(t)) \neq \emptyset$ and

$$\langle N(t,\, u(t),\, v(t), w(t)), \eta(t,\, y(t),\, g(t, x(t)))\rangle \;\geq\; \varphi(g(t, x(t)), x(t)) - \varphi(x(t), y(t)),$$

called a nonlinear mixed random variational-like inclusion. In fact, **FNMRVLI** includes many kind of variational inequalities, quasi-variational inequalities and variational inclusions as well as quasi-variational inclusions in [5, 17, 19, 24, 36] as special cases.

3 Conceptual Background

We recall some useful concepts and results. Throughout this section, $x, y, u, v, w : \Omega \to H$ denote measurable mappings.

Lemma 3.1 ([8]). *Let $G : \Omega \times H \to CB(H)$ be a D-continuous random multi-valued mapping. Then for a measurable mapping $u : \Omega \to H$, a multi-valued mapping $G(\cdot, u(\cdot)) : \Omega \to CB(H)$ is measurable.*

Lemma 3.2 ([8]). *Let $A, T : \Omega \to CB(H)$ be measurable multi-valued mappings and $u : \Omega \to H$ be a measurable selection of A. Then there exists a measurable selection $v : \Omega \to H$ of T such that for all $t \in \Omega$ and $\epsilon > 0$,*

$$\|u(t) - v(t)\| \;\leq\; (1 + \epsilon)\, D(A(t), T(t)).$$

Definition 3.3. Let $x, y : \Omega \to H$ be random mappings and $t \in \Omega$. A random mapping $\eta : \Omega \times H \times H \to H$ is said to be

(i) randomly monotone if

$$\langle x(t) - y(t),\, \eta(t,\, x(t),\, y(t))\rangle \;\geq\; 0, \quad \text{for all} \;\; x(t), y(t) \in H;$$

(ii) randomly strictly monotone if η is randomly monotone and the equality holds if and only if $x(t) = y(t)$ for all $t \in \Omega$;

(iii) randomly α_η-strongly monotone if there exists a function $\alpha_\eta : \Omega \to (0, \infty)$ such that

$$\langle x(t) - y(t),\, \eta(t,\, x(t),\, y(t))\rangle \;\geq\; \alpha_\eta(t)\|x(t) - y(t)\|^2, \quad \text{for all} \;\; x(t), y(t) \in H;$$

(iv) randomly L_η-Lipschitz continuous if there exists a function $L_\eta : \Omega \to (0, \infty)$ such that

$$\|\eta(t,\, x(t),\, y(t))\| \;\leq\; L_\eta(t)\|x(t) - y(t)\|, \quad \text{for all} \;\; x(t), y(t) \in H.$$

Remark 3.4. If $\alpha_\eta(t) = 1$ and $L_\eta(t) = 1$ for all $t \in \Omega$, then η is called randomly strongly monotone and randomly continuous, respectively.

Definition 3.5. Let $x,\, y : \Omega \to H$, $h : \Omega \times H \to H$ and $\eta : \Omega \times H \times H \to H$ be random mappings. If we design $h(t, x(t)) = h_t(x(t))$ for all $t \in \Omega$, h_t is said to be

(i) randomly η-monotone if

$$\langle h_t(x(t)) - h_t(y(t)),\, \eta(t,\, x(t),\, y(t))\rangle \;\geq\; 0, \quad \text{for all} \;\; x(t), y(t) \in H;$$

(ii) randomly η-strictly monotone if h_t is randomly η-monotone and

$$\langle h_t(x(t)) - h_t(y(t)),\, \eta(t,\, x(t),\, y(t))\rangle \;=\; 0 \;\; \text{iff} \;\; x(t) = y(t), \quad \text{for all} \;\; t \in \Omega;$$

(iii) randomly α_{h_t}-η-strongly monotone if there exists a function $\alpha_{h_t} : \Omega \to (0, \infty)$ such that

$$\langle h_t(x(t)) - h_t(y(t)),\, \eta(t,\, x(t),\, y(t))\rangle \;\geq\; \alpha_{h_t}(t)\|x(t) - y(t)\|^2, \quad \text{for all} \;\; x(t), y(t) \in H;$$

(iv) randomly L_{h_t}-Lipschitz continuous if there exists a function $L_{h_t} : \Omega \to (0, \infty)$ such that

$$\|h_t(x(t)) - h_t(y(t))\| \;\leq\; L_{h_t}(t)\|x(t) - y(t)\|, \quad \text{for all} \;\; x(t), y(t) \in H.$$

Remark 3.6. If $h_t = I$, the identity mapping on H, then conditions (i), (ii) and (iii) in Definition 3.5 reduce to (i), (ii) and (iii) in Definition 3.3, respectively.

Definition 3.7. Let $h : \Omega \times H \to H$ and $\eta : \Omega \times H \times H \to H$ be random mappings and $M : H \to 2^H$ be a multi-valued mapping, M is said to be

(i) randomly η-monotone if

$$\langle u(t) - v(t),\, \eta(t,\, x(t),\, y(t))\rangle \;\geq\; 0,$$
$$\text{for all} \;\; x(t), y(t) \in H, \quad u(t) \in M(x(t)),\; v(t) \in M(y(t));$$

(ii) randomly η-maximal monotone if M is randomly η-monotone and

$$(I + \rho(t)M)(H) \;=\; H,$$

where $\rho : \Omega \to (0, \infty)$ is a function.

(iii) randomly (h_t, η)-maximal monotone if M is randomly η-monotone and

$$(h_t + \rho(t)M)(H) = H,$$

where $\rho : \Omega \to (0, \infty)$ is a function.

Remark 3.8. For $h_t = I$, the identity mapping, the randomly I-η-maximal monotonicity coincides with the randomly η-maximal monotonicity. If $\eta(t, x(t), y(t)) = x(t) - y(t)$ for all $x(t), y(t) \in H$, the concept of a randomly (h_t, η)-maximal monotone mapping reduces to that of a random mapping, which is called a randomly h_t-monotone mapping.

Definition 3.9. Let $\eta : \Omega \times H \times H \to H$ be a random mapping and $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper functional, φ is said to be η-subdifferentiable at a point $x(t) \in H$ if there exists a point $f^* \in H$ such that

$$\varphi(y(t)) - \varphi(x(t)) \ \geq \ \langle f^*, \, \eta(t, \, y(t), \, x(t)) \rangle \ \text{ for all } \ y(t) \in H,$$

where f^* is called a η-subgradient of φ at $x(t)$. The set of all η-subgradients of φ at $x(t)$ is designed by $\partial_\eta \varphi(x(t))$. A multi-valued mapping $\partial_\eta \varphi : H \to 2^H$ defined by

$$\partial_\eta \varphi(x(t)) \ = \ \{f^* \in H : \varphi(y(t)) - \varphi(x(t)) \ \geq \ \langle f^*, \, \eta(t, \, y(t), \, x(t)) \rangle \ \text{ for all } \ y(t) \in H\}$$

is called a η-subdifferential of φ at $x(t)$.

Proposition 3.10. *Let $\eta : \Omega \times H \times H \to H$ be a randomly continuous and randomly strongly monotone mapping such that $\eta(t, x(t), y(t)) + \eta(t, y(t), x(t)) = 0$ for all $x(t), y(t) \in H$ and for any given $x(t) \in H$, a function*

$$h(y(t), \, u(t)) = \langle x(t) - u(t), \, \eta(t, \, y(t), \, u(t)) \rangle$$

is 0-diagonally quasi-concave in $y(t)$, where $u : \Omega \to (0, \infty)$ is a function. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous η-subdifferentiable proper functional. Then $\partial_\eta \varphi : H \to 2^H$ is randomly η-maximal monotone, hence for any $\rho(t) > 0$, $(I + \rho(t)\partial_\eta \varphi)(H) = H$.

Remark 3.11. It may be considered that Theorem 2.8 of Ding and Luo [15] is a deterministic case of Proposition 3.10.

Proposition 3.12. *Let a mapping $h : \Omega \times H \to H$ be randomly η-strictly monotone and a multi-valued mapping $M : H \to 2^H$ be randomly (h_t, η)-maximal monotone. Then M is randomly η-maximal monotone.*

Proof. Since M is randomly η-monotone. From [14] it is sufficient to prove that

$$\langle u(t) - v(t), \, \eta(t, \, x(t), \, y(t)) \rangle \ \geq \ 0 \ \text{ for all } \ (y(t), \, v(t)) \in \text{Gr}(M) \text{ implies } u(t) \in M(x(t)),$$

where $\text{Gr}(M) = \{(x(t), u(t)) \in H \times H : u(t) \in M(x(t))\}$ denotes the graph of M.

Suppose that M is not randomly η-maximal monotone, then there exists $(x_0(t), u_0(t)) \notin \text{Gr}(M)$ such that

$$\langle u_0(t) - v(t), \, \eta(t, \, x_0(t), \, y(t)) \rangle \ \geq \ 0 \ \text{ for all } \ (y(t), \, v(t)) \in \text{Gr}(M).$$

By assumption, for any $\rho(t) > 0$, $(h_t + \rho(t)M)(H) = H$, there exists $(x_1(t), u_1(t)) \in \text{Gr}(M)$ such that

$$h_t(x_1(t)) + \rho(t)u_1(t) \ = \ h_t(x_0(t)) + \rho(t)u_0(t).$$

It follows that

$$\rho(t)\langle u_0(t) - u_1(t), \, \eta(t, \, x_0(t), \, x_1(t)) \rangle \ = \ - \langle h_t(x_0(t)) - h_t(x_1(t)), \, \eta(t, \, x_0(t), \, x_1(t)) \rangle \ \geq \ 0.$$

Since h_t is randomly η-strictly monotone, we must have $x_0(t) = x_1(t)$ and so $u_0(t) = u_1(t)$. Hence $(x_0(t), u_0(t)) \in \text{Gr}(M)$, which is a contradiction. Therefore M is randomly η-maximal monotone. ∎

Theorem 3.13. *Let $\eta : \Omega \times H \times H \to H$ be a random mapping. Let a random mapping $h : \Omega \times H \to H$ be randomly η-strictly monotone and a multi-valued mapping $M : H \to 2^H$ be randomly (h_t, η)-maximal monotone. Then for a funtion $\rho : \Omega \to (0, \infty)$, the inverse mapping $(h_t + \rho(t)M)^{-1} : H \to H$ is single-valued.*

Proof. For any $u(t) \in H$, let $x(t), y(t) \in (h_t + \rho(t)M)^{-1}(u(t))$. Then we have

$$u(t) - h_t(x(t)) \in \rho(t)M(x(t))$$

and

$$u(t) - h_t(y(t)) \in \rho(t)M(y(t)).$$

Since M is random η-monotone, we have

$$\begin{aligned} 0 &\leq \langle u(t) - h_t(x(t)) - (u(t) - h_t(y(t))),\, \eta(t,\, x(t),\, y(t))\rangle \\ &= -\langle h_t(x(t)) - h_t(y(t)),\, \eta(t,\, x(t),\, y(t))\rangle. \end{aligned}$$

It follows from the randomly η-strict monotonicity of h_t that $x(t) = y(t)$.
Therefore $(h_t + \rho(t)M)^{-1}$ is a single-valued mapping. □

Definition 3.14. Let $\eta : \Omega \times H \times H \to H$ be a random mapping. Let a random mapping $h : \Omega \times H \to H$ be randomly η-strictly monotone and a multi-valued mapping $M : H \to 2^H$ be randomly (h_t, η)-maximal monotone. Then for a function $\rho : \Omega \to (0, \infty)$, the resolvent operator $R^M_{h_t,\rho} : H \to H$ of M is defined by

$$R^M_{h_t,\rho}(x(t)) \;=\; (h_t + \rho(t)M)^{-1}(x(t)), \quad \text{for all} \;\; x(t) \in H.$$

Remark 3.15. If $\eta : \Omega \times H \times H \to H$ is a random mapping, $h : \Omega \times H \to H$ is a randomly η-strictly monotone mapping and $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ is a functional such that

$$\mathrm{Range}(h_t(\cdot) + \rho(t)\partial_\eta \varphi(\cdot, \cdot)) \;=\; H$$

for any measurable function $\rho : \Omega \to (0, \infty)$, then from Proposition 3.12 and Theorem 3.13, we have

$$R^{\partial_\eta \varphi(\cdot,\cdot)}_{h_t,\rho}(x(t)) \;=\; (h_t(\cdot) + \rho(t)\partial_\eta \varphi(\cdot, \cdot))^{-1}(x(t)), \quad \text{for all} \;\; x(t) \in H, \quad t \in \Omega.$$

The single-valued mapping $R^{\partial_\eta \varphi(\cdot,\cdot)}_{h_t,\rho} = (h_t(\cdot) + \rho(t)\partial_\eta \varphi(\cdot, \cdot))^{-1}$ is called a random proximal mapping.

Assumption 3.16. A random mapping $\eta : \Omega \times H \times H \to H$ satisfies the condition

$$\eta(t,\, x(t),\, y(t)) + \eta(t,\, y(t),\, x(t)) \;=\; 0, \quad \text{for all} \;\; x(t),\, y(t) \in H, \; t \in \Omega.$$

Theorem 3.17. *Let $\varphi : H \times H \to \mathbb{R} \cup \{+\infty\}$ be a functional with $Dom\varphi(\cdot, y(t)) \neq \emptyset$ for fixed $y(t) \in H$. Then $t \in \Omega$, $u(t) \in \tilde{A}(t, x(t))$, $v(t) \in \tilde{T}(t, x(t))$, $w(t) \in \tilde{G}(t, x(t))$ is a solution set of* **FNMRVLI** *if and only if $g(t, x(t)) \in \partial_\eta \varphi(\cdot, y(t))$ and*

$$N(t,\, u(t), v(t), w(t)) \in \partial_\eta \varphi(\cdot,\, g(t,\, x(t))).$$

Proof. This directly follows from the definition of η-subdifferential. □

Theorem 3.18. *Let $\eta : \Omega \times H \times H \to H$ be randomly L_η-Lipschitz continuous, $h : \Omega \times H \to H$ be randomly α_{h_t}-η-strongly monotone and $\partial_\eta \varphi : H \times H \to 2^H$ be randomly (h_t, η)-maximal monotone. Then resolvent operator $R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}$ of $\partial_\eta \varphi$ is randomly $\dfrac{L_\eta}{\alpha_{h_t}}$-Lipschitz continuous.*

Proof. By the definition of the resolvent operator $R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}$ of $\partial_\eta \varphi(\cdot,\cdot)$, for any $x(t), y(t) \in H$

$$R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t)) = (h_t + \rho(t)\partial_\eta \varphi(\cdot,\cdot))^{-1}(x(t))$$

and

$$R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t)) = (h_t + \rho(t)\partial_\eta \varphi(\cdot,\cdot))^{-1}(y(t)).$$

It follows that

$$\frac{1}{\rho(t)}(x(t) - h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t)))) \in \partial_\eta \varphi(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t)))$$

and

$$\frac{1}{\rho(t)}(y(t) - h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t)))) \in \partial_\eta \varphi(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))).$$

Since $\partial_\eta \varphi(\cdot,\cdot)$ is randomly η-monotone, we have

$$\begin{aligned}&\langle x(t) - h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))) - (y(t) - h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t)))),\\ &\qquad \eta(t, (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))), (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))))\rangle \ \geq \ 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\langle x(t) - y(t) - (h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))) - h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t)))),\\ &\qquad \eta(t, (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))), (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))))\rangle \ \geq \ 0\end{aligned}$$

or

$$\begin{aligned}&\langle x(t) - y(t), \eta(t, (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))), (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))))\rangle \ \geq\\ &\langle h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))) - h_t(R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))), \eta(t, (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t))), (R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))))\rangle.\end{aligned}$$

Since η is randomly L_η-Lipschitz continuous and h_t is randomly α_{h_t}-η-strongly monotone, we have

$$\begin{aligned}&L_\eta(t)\|x(t) - y(t)\|\,\|R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t)) - R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))\|\\ &\geq \ \alpha_{h_t}(t)\|R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t)) - R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))\|^2.\end{aligned}$$

Hence

$$\|R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(x(t)) - R_{h_t,\rho}^{\partial_\eta \varphi(\cdot,\cdot)}(y(t))\| \ \leq \ \frac{L_\eta(t)}{\alpha_{h_t}(t)}\|x(t) - y(t)\|,$$

for all $x(t), y(t) \in H$, $t \in \Omega$. □

4 Iterative Algorithm

We first give the following lemma.

Lemma 4.1. *The set of measurable mappings $x, u, v, w : \Omega \to H$ is a random solution of* **FNMRVLI** *if and only if for all $t \in \Omega$, $x(t) \in H$, $u(t) \in \tilde{A}(t, x(t))$, $v(t) \in \tilde{T}(t, x(t))$, $w(t) \in \tilde{G}(t, x(t))$ and*

$$g(t, x(t)) \;=\; R^{\partial_\eta \varphi(\cdot, x(t))}_{h_t, \rho}[g(t, x(t)) - \rho(t) N(t, u(t), v(t), w(t))], \tag{4.1}$$

where $\rho : \Omega \to (0, \infty)$ is a measurable function and $R^{\partial_\eta \varphi(\cdot, x(t))}_{h_t, \rho} = (h_t(x(t)) + \rho(t)\partial_\eta \varphi(\cdot, x(t)))^{-1}$.

To obtain an approximate solution of **FNMRVLI**, we can apply a successive approximate method to the problem of solving

$$x(t) \in Q(t, x(t)), \quad \text{for all} \quad t \in \Omega,$$

where

$$\begin{aligned} Q(t, x(t)) \;=\; & \big\{x(t) - g(t, x(t)) + R^{\partial_\eta \varphi(\cdot, x(t))}_{h_t, \rho}[g(t, x(t)) - \rho(t) N(t, u(t), v(t), w(t))] \\ & : u(t) \in \tilde{A}(t, x(t)), v(t) \in \tilde{T}(t, x(t)), w(t) \in \tilde{G}(t, x(t))\big\}. \end{aligned} \tag{4.2}$$

Based on (4.1) and (4.2), we propose the following random iterative algorithm to compute the approximate solution of **FNMRVLI**.

Algoritheorem 4.2. Suppose that $A, T, G : \Omega \times H \to \mathcal{F}(H)$ be random fuzzy mappings satisfying the condition $(*)$. Let $\tilde{A}, \tilde{T}, \tilde{G} : \Omega \times H \to CB(H)$ be D-continuous random multi-valued mappings induced by A, T and G, respectively and $g : \Omega \times H \to H$ be a continuous random mapping. Let $\eta : \Omega \times H \times H \to H$, $N : \Omega \times H \times H \times H \to H$ and $h : \Omega \times H \to H$ be random mappings. For any given measurable mapping $x_0 : \Omega \to H$, multi-valued mappings, $\tilde{A}(\cdot, x_0(\cdot))$, $\tilde{T}(\cdot, x_0(\cdot))$, $\tilde{G}(\cdot, x_0(\cdot)) : \Omega \to CB(H)$ are measurable by Lemma 3.1. Hence there exist selections $u_0 : \Omega \to H$ of $\tilde{A}(\cdot, x_0(\cdot))$, $v_0 : \Omega \to H$ of $\tilde{T}(\cdot, x_0(\cdot))$ and $w_0 : \Omega \to H$ of $\tilde{G}(\cdot, x_0(\cdot))$ by Himmelberg [22]. Let

$$x_1(t) = x_0(t) - g(t, x_0(t)) + R^{\partial_\eta \varphi(\cdot, x_0(t))}_{h_t, \rho}[g(t, x_0(t) - \rho(t) N(t, u_0(t), v_0(t), w_0(t)],$$

then is easy to see that $x_1 : \Omega \to H$ is measurable. By Lemma 3.2 and Nadler [32] there exist measurable selections $u_1 : \Omega \to H$ of $\tilde{A}(\cdot, x_1(\cdot))$, $v_1 : \Omega \to H$ of $\tilde{T}(\cdot, x_1(\cdot))$ and $w_1 : \Omega \to H$ of $\tilde{G}(\cdot, x_1(\cdot))$ such that

$$\begin{aligned} \|u_0(t) - u_1(t)\| \;&\le\; (1+1) D(\tilde{A}(t, x_0(t)), \tilde{A}(t, x_1(t))), \\ \|v_0(t) - v_1(t)\| \;&\le\; (1+1) D(\tilde{T}(t, x_0(t)), \tilde{T}(t, x_1(t))), \\ \|w_0(t) - w_1(t)\| \;&\le\; (1+1) D(\tilde{G}(t, x_0(t)), \tilde{G}(t, x_1(t))). \end{aligned}$$

Let

$$x_2(t) \;=\; x_1(t) - g(t, x_1(t)) + R^{\partial_\eta \varphi(\cdot, x_1(t))}_{h_t, \rho}[g(t, x_1(t)) - \rho(t) N(t, u_1(t), v_1(t), w_1(t))].$$

Then $x_2 : \Omega \to H$ is measurable. Continuing the above process inductively we can obtain the following random iterative sequences $\{x_n(t)\}$ of measurable mappings and three sequences $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ of measurable selections for solving **FNMRVLI** as follows:

$$x_{n+1}(t) = x_n(t) - g(t, x_n(t)) + R_{h_t,\rho}^{\partial_\eta \varphi(\cdot, x_n(t))}[g(t, x_n(t)) - \rho(t) N(t,\, u_n(t),\, v_n(t),\, w_n(t))]. \quad (4.3)$$

$$u_n(t) \in \tilde{A}(t, x_n(t)),\ \|u_n(t) - u_{n+1}(t)\| \ \le\ (1 + (1+n)^{-1}) D(\tilde{A}(t, x_n(t)),\, \tilde{A}(t, x_{n+1}(t))),$$

$$v_n(t) \in \tilde{T}(t, x_n(t)),\ \|v_n(t) - v_{n+1}(t)\| \ \le\ (1 + (1+n)^{-1}) D(\tilde{T}(t, x_n(t)),\, \tilde{T}(t, x_{n+1}(t))),$$

$$w_n(t) \in \tilde{G}(t, x_n(t)),\ \|w_n(t) - w_{n+1}(t)\| \ \le\ (1 + (1+n)^{-1}) D(\tilde{G}(t, x_n(t)),\, \tilde{G}(t, x_{n+1}(t)))$$

for any $t \in \Omega$ and $n = 0, 1, 2, \ldots$.

Definition 4.3. A random mapping $g : \Omega \times H \to H$ is said to be

(i) randomly r_g-strongly monotone, if there exists a measurable function $r_g : \Omega \to (0, \infty)$ such that

$$\langle g(t, x(t)) - g(t, y(t)), x(t) - y(t) \rangle \ \ge\ r_g(t) \|x(t) - y(t)\|^2, \quad \text{for} \ \ x(t),\ y(t) \in H;$$

(ii) randomly s_g-Lipschitz continuous, if there exists a measurable function $s_g : \Omega \to (0, \infty)$ such that

$$\|g(t, x(t)) - g(t, y(t))\| \ \le\ s_g(t) \|x(t) - y(t)\|^2, \quad \text{for} \ \ x(t),\ y(t) \in H.$$

Definition 4.4. Let $N : \Omega \times H \times H \times H \to H$ be a random mapping and $\tilde{A}, \tilde{T}, \tilde{G} : \Omega \times H \to CB(H)$ random multi-valued mappings:

(i) N is said to be randomly $\lambda_{\tilde{T}}$-relaxed monotone with respect to the second argument for the mapping $\tilde{T}$, if there exists a measurable function $\lambda_{\tilde{T}} : \Omega \to (0, \infty)$ such that

$$\langle N(t, \cdot, v_1(t), \cdot) - N(t, \cdot, v_2(t), \cdot), x_1(t) - x_2(t) \rangle \ \ge\ -\lambda_{\tilde{T}}(t) \|x_1(t) - x_2(t)\|^2,$$

for all $x_i(t) \in H$, $v_i(t) \in \tilde{T}(t, x_i(t))$, $t \in \Omega$, $i = 1,\, 2$.

(ii) N is said to be randomly $\zeta_{\tilde{G}}$-relaxed Lipschitz continuous with respect to the third argument for the mapping $\tilde{G}$, if there exists a measurable function $\zeta_{\tilde{G}} : \Omega \to (0, \infty)$ such that

$$\langle N(t, \cdot, \cdot, w_1(t)) - N(t, \cdot, \cdot, w_2(t)), x_1(t) - x_2(t) \rangle \ \le\ -\zeta_{\tilde{G}}(t) \|x_1(t) - x_2(t)\|^2,$$

for all $x_i(t) \in H$, $w_i(t) \in \tilde{G}(t, x_i(t))$, $t \in \Omega$, $i = 1,\, 2$.

(iii) N is said to be randomly Lipschitz continuous with respect to the first, the second and the third arguments, if there exist measurable functions d_N, ϵ_N, $p_N : \Omega \to (0, \infty)$ such that

$$\|N(t, x_1(t), x_2(t), x_3(t)) - N(t, y_1(t), y_2(t), y_3(t))\|$$
$$\le\ d_N(t) \|x_1(t) - y_1(t)\| + \epsilon_N(t) \|x_2(t) - y_2(t)\| + p_N(t) \|x_3(t) - y_3(t)\|$$

for $x_i(t)$, $y_i(t) \in H$, $i = 1, 2, 3$ and $t \in \Omega$.

(iv) N is said to be randomly $\xi_{\tilde{A},g}$-strongly monotone with respect to the first argument for the mapping $\tilde{A}$ if there exists a measurable function $\xi_{\tilde{A},g}:\Omega\to\infty$ such that

$$\langle N(t,u_1(t),\cdot,\cdot)-N(t,u_2(t),\cdot,\cdot),\, g(t,x_1(t))-g(t,x_2(t))\rangle \;\geq\; \xi_{\tilde{A},g}(t)\|x_1(t)-x_2(t)\|^2$$

for $x_i(t)\in H$, $u_i(t)\in\tilde{A}(t,x_i(t))$, $i=1,2$, where $g:\Omega\times H\to H$ is a random mapping.

(v) Mappings $\tilde{A},\tilde{T},\tilde{G}$ are said to be randomly D-Lipschitz continuous, if there exist measurable functions $\beta_{\tilde{A}}$, $\gamma_{\tilde{T}}$, $\sigma_{\tilde{G}}:\Omega\to(0,\infty)$ such that

$$\begin{aligned}\|u_1(t)-u_2(t)\| &\leq D(\tilde{A}(t,x_1(t)),\,\tilde{A}(t,x_2(t)))\leq \beta_{\tilde{A}}(t)\|x_1(t)-x_2(t)\|,\\ \|v_1(t)-v_2(t)\| &\leq D(\tilde{T}(t,x_1(t)),\,\tilde{T}(t,x_2(t)))\leq \gamma_{\tilde{T}}(t)\|x_1(t)-x_2(t)\|,\\ \|w_1(t)-w_2(t)\| &\leq D(\tilde{G}(t,x_1(t)),\,\tilde{G}(t,x_2(t)))\leq \sigma_{\tilde{G}}(t)\|x_1(t)-x_2(t)\|\end{aligned}$$

for $x_i(t)\in H$, $u_i(t)\in\tilde{A}(t,x_i(t))$, $v_i(t)\in\tilde{T}(t,x_i(t))$, $w_i(t)\in\tilde{G}(t,x_i(t))$, $t\in\Omega$, $i=1$, 2.

Theorem 4.5. *Let a random mapping $\eta:\Omega\times H\times H\to H$ be randomly L_η-Lipschitz continuous and a random mapping $h:\Omega\times H\to H$ be randomly α_{h_t}-η-strongly monotone. Let $N:\Omega\times H\times H\times H\to H$ be a random mapping which is randomly Lipschitz continuous with respect to the first, the second and the third arguments with random coefficients $d_N(t)$, $\epsilon_N(t)$ and $p_N(t)$, respectively. Let $A,T,G:\Omega\times H\to\mathcal{F}(H)$ be random fuzzy mappings satisfying the condition $(*)$. Let $\tilde{A},\tilde{T},\tilde{G}:\Omega\times H\to CB(H)$ be random multi-valued mappings induced by A,T and G, respectively, which are randomly D-Lipschitz continuous with random coefficients $\beta_{\tilde{A}}(t)$, $\gamma_{\tilde{T}}(t)$ and $\sigma_{\tilde{G}}(t)$, respectively. Let $g:\Omega\times H\to H$ be randomly r_g-strongly monotone and randomly s_g-Lipschitz continuous. Let N be randomly $\lambda_{\tilde{T}}$-relaxed monotone with respect to the second argument for the mapping $\tilde{T}$ and randomly $\zeta_{\tilde{G}}$-relaxed Lipschitz continuous with respect to the third argument for the mapping $\tilde{G}$. Let N be randomly $\xi_{\tilde{A},g}$-strongly monotone with respect to the first argument for the mapping $\tilde{A}$. Let $\varphi:H\times H\to\mathbb{R}\cup\{+\infty)$ be a functional such that for fixed $x(t)\in H$, $Ran(h_t(x(t))+\rho(t)\partial_\eta\varphi(\cdot,x(t)))=H$, where $\rho:\Omega\to(0,\infty)$ is a measurable function.*

Assume that

$$\|R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x(t))}(z(t))-R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,y(t))}(z(t))\|\;\leq\;\delta(t)\|x(t)-y(t)\|\quad\text{for }\; x(t),y(t)\text{ and }z(t)\in H,$$

where $\delta:\Omega\to(0,\infty)$ is a function and the following conditions hold;

$$\left|\rho(t)-\frac{(\xi_{\tilde{A},g}(t)s_g^2(t)L_\eta(t)-\chi(t)\alpha_{h_t}(t)(1-\kappa(t)))}{L_\eta(t)(d_N^2(t)\beta_{\tilde{A}}^2(t)-\chi^2(t))}\right|$$
$$<\frac{\sqrt{(\xi_{\tilde{A},g}(t)s_g^2(t)L_\eta(t)-\chi(t)\alpha_{h_t}(t)(1-\kappa(t)))^2-(d_N^2(t)\beta_{\tilde{A}}^2(t)-\chi^2(t))(L_\eta^2(t)s_g^2(t)-\alpha_{h_t}^2(t)(1-\kappa(t))^2)}}{L_\eta^2(t)(d_N^2(t)\beta_{\tilde{A}}^2(t)-\chi^2(t))}$$

$$\xi_{\tilde{A},g}(t)s_g^2(t)L_\eta(t)>\chi(t)\alpha_{h_t}(t)(1-\kappa(t))^2+\sqrt{(d_N^2(t)\beta_{\tilde{A}}^2(t)-\chi^2(t))(L_\eta^2(t)s_g^2(t)-\alpha_{h_t}^2(t)(1-\kappa(t))^2)}\tag{4.4}$$

$$\xi_{\tilde{A},g}(t)s_g^2(t)L_\eta(t)>\chi(t)\alpha_{h_t}(t)(1-\kappa(t))^2,\;\;\kappa(t)<1,\;\;\chi(t)<d_N^2(t)\beta_{\tilde{A}}^2(t),$$

$$\alpha_{h_t}(t)(1-\kappa(t)) < L_\eta(t)s_g^2(t) \;\text{ and }\; \chi(t) \;=\; \Delta+\ell,$$

where $\kappa(t)=\sqrt{1-2r_g(t)+s_g^2(t)}+\delta(t)$,

$$\Delta \;=\; \sqrt{1-2\zeta_{\tilde{G}}(t)+\sigma_{\tilde{G}}^2(t)p_N^2(t)} \;\text{ and }\; \ell \;=\; \sqrt{1-2\lambda_{\tilde{T}}(t)+\epsilon_N^2(t)\gamma_{\tilde{T}}^2(t)}.$$

Then there exist measurable mappings $x,u,v,w:\Omega\to H$ such that **FNMRVLI** holds. Moreover $x_n(t)\to x(t)$, $u_n(t)\to u(t)$, $v_n(t)\to v(t)$ and $w_n(t)\to w(t)$ in H, where $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are random sequences obtained by Algorithm 4.2.

Proof. From (4.3), for any $t\in\Omega$, we have

$$\begin{aligned}
\|x_{n+1}(t)-x_n(t)\| \;=&\; \|x_n(t)-x_{n-1}(t)-(g(t,x_n(t))-g(t,x_{n-1}(t)))\\
&+R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_n(t))}(z_n(t))-R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_{n-1}(t))}(z_{n-1}(t))\|\\
\le&\; \|x_n(t)-x_{n-1}(t)-(g(t,x_n(t))-g(t,x_{n-1}(t)))\|\\
&+\|R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_n(t))}(z_n(t))-R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_{n-1}(t))}(z_{n-1}(t))\|\\
\le&\; \|x_n(t)-x_{n-1}(t)-(g(t,x_n(t))-g(t,x_{n-1}(t)))\|\\
&+\|R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_n(t))}(z_n(t))-R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_n(t))}(z_{n-1}(t))\|\\
&+\|R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_n(t))}(z_{n-1}(t))-R_{h_t,\rho}^{\partial_\eta\varphi(\cdot,x_{n-1}(t))}(z_{n-1}(t))\|\\
\le&\; \|x_n(t)-x_{n-1}(t)-(g(t,x_n(t))-g(t,x_{n-1}(t)))\|\\
&+\frac{L_\eta(t)}{\alpha_{h_t}(t)}\|z_n(t)-z_{n-1}(t)\|+\delta(t)\|x_n(t)-x_{n-1}(t)\|,
\end{aligned} \tag{4.5}$$

where

$$z_n(t) \;=\; g(t,x_n(t))-\rho(t)N(t,u_n(t),\, v_n(t),w_n(t)).$$

Now

$$\begin{aligned}
&\|z_n(t)-z_{n-1}(t)\|\\
=&\|g(t,x_n(t))-\rho(t)N(t,u_n(t),v_n(t),w_n(t))\\
&-g(t,x_{n-1}(t))+\rho(t)N(t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t))\|\\
\le&\|g(t,x_n(t))-g(t,x_{n-1}(t))-\rho(t)(N(t,u_n(t),v_n(t),w_n(t))-N(t,u_{n-1}(t),v_n(t),w_n(t)))\|\\
&+\rho(t)\|x_n(t)-x_{n-1}(t)+(N(t,u_{n-1}(t),v_n(t),w_n(t))-N(t,u_{n-1}(t),v_{n-1}(t),w_n(t)))\|\\
&+\rho(t)\|x_n(t)-x_{n-1}(t)-(N(t,u_{n-1}(t),v_{n-1}(t),w_n(t))-N(t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)))\|.
\end{aligned} \tag{4.6}$$

Adding (4.5) and (4.6), we get

$$\begin{aligned}
&\|x_{n+1}(t)-x_n(t)\|\\
\le&\|x_n(t)-x_{n-1}(t)-(g(t,x_n(t))-g(t,x_{n-1}(t)))\|\\
&+\frac{L_\eta(t)}{\alpha_{h_t}(t)}[\|g(t,x_n(t))-g(t,x_{n-1}(t))-\rho(t)(N(t,u_n(t),v_n(t),w_n(t))\\
&-N(t,u_{n-1}(t),v_n(t),w_n(t)))\|+\rho(t)\|x_n(t)-x_{n-1}(t)+(N(t,u_{n-1}(t),v_n(t),w_n(t))\\
&-N(t,u_{n-1}(t),v_{n-1}(t),w_n(t)))\|+\rho(t)\|x_n(t)-x_{n-1}(t)-(N(t,u_{n-1}(t),v_{n-1}(t),w_n(t))\\
&-N(t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)))\|+\delta(t)\|x_n(t)-x_{n-1}(t)\|.
\end{aligned} \tag{4.7}$$

By the random s_g-Lipschitz continuity and the random r_g-strong monotonicity of g, we have

$$\begin{aligned}
&\|x_n(t) - x_{n-1}(t) - (g(t, x_n(t)) - g(t, x_{n-1}(t)))\|^2 \\
=&\|x_n(t) - x_{n-1}(t)\|^2 - 2\langle g(t, x_n(t)) - g(t, x_{n-1}(t)), x_n(t) - x_{n-1}(t)\rangle \\
&+ \|g(t, x_n(t)) - g(t, x_{n-1}(t))\|^2 \\
\le&\|x_n(t) - x_{n-1}(t)\|^2 - 2r_g(t)\|x_n(t) - x_{n-1}(t)\|^2 + s_g^2(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
\le&(1 - 2r_g(t) + s_g^2(t))\|x_n(t) - x_{n-1}(t)\|^2.
\end{aligned} \tag{4.8}$$

Since N is randomly Lipschitz continuous with measurable mappings $d_N(\cdot)$, $\epsilon_N(\cdot)$, $p_N(\cdot) : \Omega \to (0, \infty)$ and $\tilde{A}$, $\tilde{T}$, $\tilde{G}$ are randomly D-Lipschitz continuous with mappings $\beta_{\tilde{A}}(\cdot)$, $\gamma_{\tilde{T}}(\cdot)$, $\sigma_{\tilde{G}}(\cdot) : \Omega \to (0, \infty)$, respectively, we have

$$\begin{aligned}
&\|N(t, u_n(t), v_n(t), w_n(t)) - N(t, u_{n-1}(t), v_n(t), w_n(t))\| \\
\le\ & d_N(t)\|u_n(t) - u_{n-1}(t)\| \\
\le\ & d_N(t)D(\tilde{A}(t, x_n(t)),\ \tilde{A}(t, x_{n-1}(t))) \\
\le\ & d_N(t)\beta_{\tilde{A}}(t)(1 + n^{-1})\|x_n(t) - x_{n-1}(t)\|,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\|N(t, u_{n-1}(t), v_n(t), w_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t), w_n(t))\| \\
\le\ & \epsilon_N(t)\|v_n(t) - v_{n-1}(t)\| \\
\le\ & \epsilon_N(t)D(\tilde{T}(t, x_n(t)),\ \tilde{T}(t, x_{n-1}(t))) \\
\le\ & \epsilon_N(t)\gamma_{\tilde{T}}(t)(1 + n^{-1})\|x_n(t) - x_{n-1}(t)\|,
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\|N(t, u_{n-1}(t), v_{n-1}(t), w_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t), w_{n-1}(t))\| \\
\le\ & p_N(t)\|w_n(t) - w_{n-1}(t)\| \\
\le\ & p_N(t)D(\tilde{G}(t, x_n(t)),\ \tilde{G}(t, x_{n-1}(t))) \\
\le\ & p_N(t)\sigma_{\tilde{G}}(t)(1 + n^{-1})\|x_n(t) - x_{n-1}(t)\|.
\end{aligned} \tag{4.11}$$

Since N is randomly $\lambda_{\tilde{T}}$-relaxed monotone with respect to the second argument, from (4.10), we have

$$\begin{aligned}
&\|x_n(t) - x_{n-1}(t) + (N(t, u_{n-1}(t), v_n(t), w_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t), w_n(t)))\|^2 \\
\le\ & \|x_n(t) - x_{n-1}(t)\|^2 \\
&+ 2\langle N(t, u_{n-1}(t), v_n(t), w_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t), w_n(t)), x_n(t) - x_{n-1}(t)\rangle \\
&+ \|N(t, u_{n-1}(t), v_n(t), w_n(t)) - N(t, u_{n-1}(t), v_{n-1}(t), w_n(t))\|^2 \\
\le\ & \|x_n(t) - x_{n-1}(t)\|^2 - 2\lambda_{\tilde{T}}(t)\|x_n(t) - x_{n-1}(t)\|^2 \\
&+ \epsilon_N^2(t)\gamma_{\tilde{T}}^2(t)(1 + n^{-1})^2\|x_n(t) - x_{n-1}(t)\|^2 \\
=\ & (1 - 2\lambda_{\tilde{T}}(t) + \epsilon_N^2(t)\gamma_{\tilde{T}}^2(t)(1 + n^{-1})^2)\|x_n(t) - x_{n-1}(t)\|^2\ .
\end{aligned} \tag{4.12}$$

Since N is randomly $\zeta_{\tilde{G}}$-relaxed Lipschitz continuous with respect to the third argument, from (4.11), we obtain

$$\begin{aligned}
& \|x_n(t) - x_{n-1}(t) - (N(t,u_{n-1}(t),v_{n-1}(t),w_n(t)) - N(t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)))\|^2 \\
\leq\ & \|x_n(t) - x_{n-1}(t)\|^2 - 2\langle N(t,u_{n-1}(t),v_{n-1}(t),w_n(t)) - N(t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t)), \\
& x_n(t) - x_{n-1}(t)\rangle + \|N(t,u_{n-1}(t),v_{n-1}(t),w_n(t)) - N(t,u_{n-1}(t),v_{n-1}(t),w_{n-1}(t))\|^2 \\
\leq\ & \|x_n(t) - x_{n-1}(t)\|^2 + 2\zeta_{\tilde{G}}(t)\|x_n(t) - x_{n-1}(t)\|^2 + \sigma^2_{\tilde{G}}(t)p^2_N(t)(1+n^{-1})^2\|x_n(t) - x_{n-1}(t)\|^2 \\
=\ & (1 + 2\zeta_{\tilde{G}}(t) + \sigma^2_{\tilde{G}}(t)p^2_N(t))(1+n^{-1})^2\|x_n(t) - x_{n-1}(t)\|^2 \ . \qquad (4.13)
\end{aligned}$$

Again, since N is randomly $\xi_{\tilde{A},g}$-strongly monotone with respect to the first argument of the mapping $\tilde{A}$, from (4.9), we obtain

$$\begin{aligned}
& \|g(t,x_n(t)) - g(t,x_{n-1}(t)) - \rho(t)(N(t,u_n(t),v_n(t),w_n(t)) - N(t,u_{n-1}(t),v_n(t),w_n(t)))\|^2 \\
& \leq\ \|g(t,x_n(t)) - g(t,x_{n-1}(t))\|^2 \\
& - 2\rho(t)\langle N(t,u_n(t),v_n(t),w_n(t)) - N(t,u_{n-1}(t),v_n(t),w_n(t)), g(t,x_n(t)) - g(t,x_{n-1}(t))\rangle \\
& + \rho^2(t)\|N(t,u_n(t),v_n(t),w_n(t)) - N(t,u_{n-1}(t),v_n(t),w_n(t))\|^2 \\
& \leq\ s^2_g(t)\|x_n(t) - x_{n-1}(t)\|^2 - 2\rho(t)\xi_{\tilde{A},g}(t)\|g(t,x_n(t)) - g(t,x_{n-1}(t))\|^2 \\
& + \rho^2(t)d^2_N(t)\beta^2_{\tilde{A}}(t)(1+n^{-1})^2\|x_n(t) - x_{n-1}(t)\|^2 \\
& \leq\ (s^2_g(t) - 2\rho(t)\xi_{\tilde{A},g}(t)s^2_g(t) + \rho^2(t)d^2_N(t)\beta^2_{\tilde{A}}(t)(1+n^{-1})^2)\|x_n(t) - x_{n-1}(t)\|^2 \ . \qquad (4.14)
\end{aligned}$$

Combining (4.7)-(4.14), we obtain

$$\begin{aligned}
& \|x_{n+1}(t) - x_n(t)\| \\
\leq & \Bigg[\sqrt{1 - 2r_g(t) + s^2_g(t)} + \frac{L_\eta(t)}{\alpha_{h_t}(t)}\Bigg\{\sqrt{(s^2_g(t) - 2\rho(t)\xi_{\tilde{A},g}(t)s^2_g(t) + \rho^2(t)d^2_N(t)\beta^2_{\tilde{A}}(t)(1+n^{-1})^2} \\
& + \rho(t)\sqrt{1 - 2\lambda_{\tilde{T}}(t) + \epsilon^2_N(t)\gamma^2_{\tilde{T}}(t)(1+n^{-1})^2} \\
& + \rho(t)\sqrt{1 + 2\zeta_{\tilde{G}}(t) + \sigma^2_{\tilde{G}}(t)p^2_N(t)(1+n^{-1})^2}\Bigg\}\Bigg]\|x_n(t) - x_{n-1}(t)\|,
\end{aligned}$$

$$\begin{aligned}
& \|x_n(t) - x_{n-1}(t)\| + \delta(t)\|x_n(t) - x_{n-1}(t)\| \\
\leq\ & \Bigg[\sqrt{1 - 2r_g(t) + s^2_g(t)} + \frac{L_\eta(t)}{\alpha_{h_t}(t)}\Bigg\{\sqrt{(s^2_g(t) - 2\rho(t)\xi_{\tilde{A},g}(t)s^2_g(t) + \rho^2(t)d^2_N(t)\beta^2_{\tilde{A}}(t)(1+n^{-1})^2} \\
& + \rho(t)\sqrt{1 - 2\lambda_{\tilde{T}}(t) + \epsilon^2_N(t)\gamma^2_{\tilde{T}}(t)(1+n^{-1})^2} \\
& + \rho(t)\sqrt{1 + 2\zeta_{\tilde{G}}(t) + \sigma^2_{\tilde{G}}(t)p^2_N(t)(1+n^{-1})^2}\Bigg\} + \delta(t)\Bigg]\ \|x_n(t) - x_{n-1}(t)\| \\
\leq\ & \Bigg[\kappa(t) + \frac{L_\eta(t)}{\alpha_{h_t}(t)}\Bigg(\sqrt{(s^2_g(t) - 2\rho(t)\xi_{\tilde{A},g}(t)s^2_g(t) + \rho^2(t)d^2_N(t)\beta^2_{\tilde{A}}(t)(1+n^{-1})^2} \\
& + \rho(t)\chi_n(t)\Bigg)\Bigg]\ \|x_n(t) - x_{n-1}(t)\| \\
\leq\ & \theta_n(t)\|x_n(t) - x_{n-1}(t)\|, \qquad (4.15)
\end{aligned}$$

where $\kappa(t) = \sqrt{1 - 2r_g(t) + s^2_g(t)} + \delta(t)$, $\Delta_n = \sqrt{1 + 2\zeta_{\tilde{G}}(t) + \sigma^2_{\tilde{G}}(t)p^2_N(t)(1+n^{-1})^2}$, and $\ell_n = \sqrt{1 - 2\lambda_{\tilde{T}}(t) + \epsilon^2_N(t)\gamma^2_{\tilde{T}}(t)(1+n^{-1})^2}$, and

$$\chi_n(t) \ = \ \Delta_n + \ell_n$$

$$\theta_n(t) = \kappa(t) + \frac{L_\eta(t)}{\alpha_{h_t}(t)}\left(\sqrt{s_g^2(t) - 2\rho(t)\xi_{\tilde{A},g}(t)s_g^2(t) + \rho^2(t)d_N^2(t)\beta_{\tilde{A}}^2(t)(1+n^{-1})^2} + \chi_n(t)\rho(t)\right).$$

Letting

$$\theta(t) = \kappa(t) + \frac{L_\eta(t)}{\alpha_{h_t}(t)}\left(\sqrt{s_g^2(t) - 2\rho(t)\xi_{\tilde{A},g}(t)s_g^2(t) + \rho^2(t)d_N^2(t)\beta_{\tilde{A}}^2(t)} + \chi(t)\rho(t)\right),$$

we know that $\theta_n(t) \to \theta(t)$ for all $t \in \Omega$. It follows from (4.5) that $\theta(t) < 1$ for all $t \in \Omega$. Hence for any $t \in \Omega$, $\theta_n(t) < 1$ for n sufficiently large.

Therefore $\{x_n(t)\}$ is a Cauchy sequence in H. Since H is complete, there exists a measurable mapping $x : \Omega \to H$ such that $x_n(t) \to x(t) \in H$, for all $t \in \Omega$.

From Algorithm 4.2, we have

$$\begin{aligned}
\|u_n(t) - u_{n-1}(t)\| &\le \beta_{\tilde{A}}(t)(1+n^{-1})\|x_n(t) - x_{n-1}(t)\|,\\
\|v_n(t) - v_{n-1}(t)\| &\le \gamma_{\tilde{T}}(t)(1+n^{-1})\|x_n(t) - x_{n-1}(t)\|,\\
\|w_n(t) - w_{n-1}(t)\| &\le \sigma_{\tilde{G}}(t)(1+n^{-1})\|x_n(t) - x_{n-1}(t)\|,
\end{aligned}$$

which implies that $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are also Cauchy sequences in H. Let $u_n(t) \to u(t)$, $v_n(t) \to v(t)$ and $w_n(t) \to w(t)$. Since $\{u_n(t)\}$, $\{v_n(t)\}$ and $\{w_n(t)\}$ are sequences of measurable mappings. We know that $x, u, v, w : \Omega \to H$ are measurable.

Now we will prove that $u(t) \in \tilde{A}(t, x(t))$, $v(t) \in \tilde{T}(t, x(t))$ and $w(t) \in \tilde{G}(t, x(t))$. For any $t \in \Omega$, we have

$$\begin{aligned}
d(u(t),\, \tilde{A}(t, x(t))) &= \inf\{\|u(t) - z\| : z \in \tilde{A}(t, x(t))\}\\
&\le \|u(t) - u_n(t)\| + d(u_n(t),\, \tilde{A}(t, x(t)))\\
&\le \|u(t) - u_n(t)\| + D(\tilde{A}(t, x_n(t)),\, \tilde{A}(t, x(t)))\\
&\le \|u(t) - u_n(t)\| + \beta_{\tilde{A}}(t)\|x_n(t) - x(t)\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Hence, $u(t) \in \tilde{A}(t, x(t))$, for all $t \in \Omega$. Similarly, we can prove that $v(t) \in \tilde{T}(t, x(t))$ and $w(t) \in \tilde{G}(t, x(t))$ for all $t \in \Omega$. This completes the proof. □

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