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QUANTITATIVE RESULTS ON OPENNESS OF SET-VALUED MAPPINGS AND IMPLICIT MULTIFUNCTION THEOREMS

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Abstract: The aim of this paper is to obtain some quantitative results of openness for usual and parametric set-valued mappings in terms of coderivatives. In particular, we obtain a proof of the Robinson-Ursescu Theorem. Then, implicit multifunction results are obtained by simply specializing the openness results. Moreover, we study a kind of metric regularity and the Lipschitz-like property of the implicit multifunctions. The results of the paper generalize and put in the same framework several results in literature.

Key words: *set-valued mappings, Fréchet coderivative, Mordukhovich coderivative, openness, implicit multifunctions, metric regularity, Lipschitz-like property*

Mathematics Subject Classification: *90C29, 90C26, 49J52*

1 Introduction

In this paper we deal with some quantitative openness results for set-valued maps. We consider separately the non-parametric and the parametric case and then we get several assertions on implicit multifunctions. To this end, we strongly exploit a wide-used coderivative condition, i.e. there exist $c > 0$, $r > 0$, $s > 0$ such that for every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$ and every $y^* \in Y^*$, $x^* \in \widehat{D}^* F(x, y)(y^*)$,

$$c \|y^*\| \leq \|x^*\| , \tag{1.1}$$

where $\widehat{D}^* F$ stands for the Fréchet coderivative of the multifunction F acting between some Asplund spaces X, Y. This condition was firstly developed in [7, Theorem 5.6] and we quote here, without being exhaustive, the references [9], [2], [4]. However, most of the results hold as well for other coderivatives on appropriate spaces (see Section 3).

First, we give an openness result which, in contrast to the existing similar theorems, has the advantage to exactly point out the constants involved in the openness property. The estimation of the radius of the neighborhood where the linear openness holds allows us to pass to the study of the parametric case and to put it in dialog with the non-parametric case. The link we establish here between these two cases is one of the main features of the paper. Besides its own interest, these exact estimations provide the necessary basis to obtain as well some results for implicit set-valued maps.

The study of openness with linear rate started with the Banach open principle and continued with several famous results as Lyusternik-Graves Theorem and Robinson-Ursescu Theorem. In these results the setting was extended from the case of linear bounded operators to the case of (strictly) differentiable functions and convex, closed-graph set-valued maps, respectively. Until now, the Robinson-Ursescu Theorem was included in the framework of the results obtained on the base of condition (1.1) only in the particular case of Asplund spaces (see [6, Theorem 4.21]). At the end of the third section, we observe that our main openness result holds on general Banach spaces if F is a convex-graph multifunction, and on this basis we provide a new proof of the Robinson-Ursescu Theorem.

The fourth section concerns the case of parametric set-valued maps. We show an openness result for this case and we give again exact estimations for the constants. Moreover, we present a discussion on the equivalence between the main results of the previous section and those of the current section. We emphasize the fact that when we want to obtain the parametric result directly from the non-parametric result, we get a weaker estimation of the constant.

The last section and maybe the most important one deals with implicit multifunctions, showing how several properties of the initial parametric set-valued map transfer to them. These results would not be possible without the estimations mentioned before. The properties we envisage are: the lower semicontinuity, several kinds of metric regularity and the Lipschitz-like property. Moreover, on this basis we recover a formula for the coderivative of the implicit multifunction.

2 Preliminaries

Let X and Y be topological spaces. Consider a set-valued mapping F from X into Y. As usual, the domain and the graph of F are denoted respectively by

$$\operatorname{Dom} F := \{x \in X \mid F(x) \neq \emptyset\}$$

and

$$\operatorname{Gr} F = \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

If $A \subset X$ then $F(A) := \bigcup_{x \in A} F(x)$. The set $F(X)$ is denoted by $\operatorname{Im} F$ and is called the image of F. The inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$. The following concepts are standard in the theory of set-valued maps.

One says that F is lower semicontinuous (lsc for short) at $x \in X$ if for every open set $D \subset Y$ with $F(x) \cap D \neq \emptyset$, there exists $U \in \mathcal{V}(x)$ such that for every $x' \in U$, $F(x') \cap D \neq \emptyset$ (where $\mathcal{V}(x)$ stands for the system of the neighborhoods of x).

In what follows, we shall use some weaker continuity assumptions (see, e.g., [6, Definition 1.63]). One says that F is inner semicontinuous at $(x, y) \in \operatorname{Gr} F$ if for every open set $D \subset Y$ with $y \in D$, there exists $U \in \mathcal{V}(x)$ such that for every $x' \in U$, $F(x') \cap D \neq \emptyset$. It is easy to see that this notion is strictly weaker that the lower semicontinuity at x (in fact F is lsc at x iff it is inner semicontinuous at every (x, y) with $y \in F(x)$). For example, the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ given by $F(0) = [-1, 1]$, $F(x) = \{0\}$ for every $x \in \mathbb{R} \setminus \{0\}$ is inner semicontinuous at $(0, 0)$ but it fails to be lsc at 0.

Suppose now that X, Y are normed vector spaces. In this setting, $B(x, r)$ and $D(x, r)$ denote the open and the closed ball with center x and radius r, respectively. Sometimes we write B_X, D_X, S_X for the open and closed unit balls of X and for the unit sphere of X, respectively. If $x \in X$ and $A \subset X$, one defines the distance from x to A as $d(x, A) := \inf\{\|x - a\| \mid a \in A\}$. As usual, we use the convention $d(x, \emptyset) = \infty$. For a non-empty set $A \subset X$ we put $\operatorname{cl} A$, $\operatorname{aint} A$ for the topological closure and the algebraic interior, respectively. When we work on the product space $X \times Y$, we consider the sum norm.

One says that F is open at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ if the image through F of every neighborhood of $\overline{x}$ is a neighborhood of $\overline{y}$. Let us observe that F is inner semicontinuous at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ if and only if F^{-1} is open at $(\overline{y},\overline{x})$.

A stronger openness property is the openness with linear rate. One says that $F : X \rightrightarrows Y$ is open with linear rate $c > 0$ around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ and a positive number $\varepsilon > 0$ such that, for every $(x, y) \in \operatorname{Gr} F \cap (U \times V)$ and every $\rho \in (0, \varepsilon)$,

$$B(y, \rho c) \subset F(B(x, \rho)).$$

It is well known that this property is equivalent to the metric regularity property of F around $(\overline{x},\overline{y})$ which requires to exist $a > 0$ and two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that for every $u \in U$ and every $v \in V$ to have

$$d(u, F^{-1}(v)) \leq ad(v, F(u)).$$

Another property closely related with the previous two is the Lipschitz-like property: one says that F is Lipschitz-like around $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ with modulus $l > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that

$$F(x) \cap V \subset F(u) + ld(x, u)D_X \text{ for all } x, u \in U.$$

It is well-known that F is Lipschitz-like around $(\overline{x},\overline{y})$ iff F^{-1} is metrically regular around $(\overline{x},\overline{y})$ iff F^{-1} is open with linear rate around $(\overline{x},\overline{y})$. For more details see [6, Sections 1.2.2, 1.2.3].

One of the main tools for the proofs of our main results is the well-known Ekeland variational principle (see [1]).

Theorem 2.1 (Ekeland variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper function (i.e. $\operatorname{dom} f := \{x \in X \mid f(x) \in \mathbb{R}\} \neq \emptyset$) which is lsc and lower bounded on X. Then for every $\overline{x} \in \operatorname{Dom} f$ and every $\varepsilon > 0$ there exists $x_\varepsilon \in X$ such that*

$$f(x_\varepsilon) \leq f(\overline{x}) - \varepsilon d(\overline{x}, x_\varepsilon)$$

and, for every $x \in X \setminus \{x_\varepsilon\}$,

$$f(x_\varepsilon) < f(x) + \varepsilon d(x, x_\varepsilon).$$

Most of the results of this paper work for several types of generalized differentiation objects as we shall made precise later. But, for the clarity of our discussion we mainly use the constructions developed by Mordukhovich and his collaborators (see [6]). We briefly remind these concepts and results. First, recall that X^* denotes the topological dual of the normed vector space X, while the symbols w, w^* are used for the weak and weak-star topologies of the dual system (X, X^*). The symbol $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its topological dual. Given a nonempty set S and a function $f : X \to \overline{\mathbb{R}}$, we use the following notations:

$$x \xrightarrow{S} \overline{x}, \text{ if } x \to \overline{x} \text{ and } x \in S,$$
$$x \xrightarrow{f} \overline{x}, \text{ if } x \to \overline{x} \text{ and } f(x) \to f(\overline{x}).$$

Definition 2.2. Let X be a normed vector space, S be a non-empty subset of X and let $x \in S$, $\varepsilon \geq 0$. The set of $\varepsilon-$normals to S at x is

$$\widehat{N}_\varepsilon(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u-x)}{\|u-x\|} \leq \varepsilon \right\}. \tag{2.1}$$

If $\varepsilon = 0$, the elements in the right-hand side of (2.1) are called Fréchet normals and their collection, denoted by $\widehat{N}(S,x)$, is the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0,\ x_n \xrightarrow{S} \overline{x},\ x_n^* \xrightarrow{w^*} x^*,\ x_n^* \in \widehat{N}_{\varepsilon_n}(S,x_n),\ \forall n \in \mathbb{N}\}.$$

If X is an Asplund space (i.e. a Banach space where every convex continuous function is generically Fréchet differentiable), the formula for the basic normal cone takes a simpler form, namely:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x},\ x_n^* \xrightarrow{w^*} x^*,\ x_n^* \in \widehat{N}(S,x_n),\ \forall n \in \mathbb{N}\}.$$

Let $f : X \to \overline{\mathbb{R}}$ be finite at $\overline{x} \in X$; the Fréchet subdifferential of f at $\overline{x}$ is the set

$$\widehat{\partial} f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in \widehat{N}(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\}$$

and the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is

$$\partial f(\overline{x}) := \{x^* \in X^* \mid (x^*, -1) \in N(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f$ denotes the epigraph of f. On Asplund spaces one has

$$\partial f(\overline{x}) = \limsup_{x \xrightarrow{f} \overline{x}} \widehat{\partial} f(x),$$

and, in particular, $\widehat{\partial} f(\overline{x}) \subset \partial f(\overline{x})$. If f is convex, then both these subdifferential do coincide with the classical Fenchel subdifferential. If δ_Ω denotes the indicator function associated with a nonempty set $\Omega \subset X$ (i.e. $\delta_\Omega(x) = 0$ if $x \in \Omega$, $\delta_\Omega(x) = \infty$ if $x \notin \Omega$), then for any $\overline{x} \in \Omega$, $\widehat{\partial}\delta_\Omega(\overline{x}) = \widehat{N}(\Omega,\overline{x})$ and $\partial\delta_\Omega(\overline{x}) = N(\Omega,\overline{x})$. Let $\Omega \subset X$ be a nonempty set and take $\overline{x} \in \Omega$; then one has:

$$\widehat{\partial} d(\cdot,\Omega)(\overline{x}) = \widehat{N}(\Omega,\overline{x}) \cap D_{X^*}, \widehat{N}(\Omega,\overline{x}) = \bigcup_{\lambda>0} \lambda \widehat{\partial} d(\cdot,\Omega)(\overline{x}).$$

If, in addition, Ω is closed, then $N(\Omega,\overline{x}) = \bigcup_{\lambda>0} \lambda \partial d(\cdot,\Omega)(\overline{x})$.

An element $x^* \in \widehat{\partial} f(\overline{x})$ is called Fréchet subgradient of f at $\overline{x}$ and admits a smooth variational description which will be useful in the sequel (see [6, Theorem 1.88(i)]):

Proposition 2.3 (Smooth variational description of Fréchet subgradients). *Let $f : X \to \overline{\mathbb{R}}$ be finite at $\overline{x}$. Given $x^* \in X^*$, if there are a neighborhood U of $\overline{x}$ and a function $s : U \to \mathbb{R}$ which is Fréchet differentiable at $\overline{x}$ with the derivative $\nabla s(\overline{x}) = x^*$ such that $f - s$ achieves a local minimum at $\overline{x}$, then $x^* \in \widehat{\partial} f(\overline{x})$. Conversely, if $x^* \in \widehat{\partial} f(\overline{x})$ then there are a neighborhood U of $\overline{x}$ and a function $s : U \to \mathbb{R}$ which is Fréchet differentiable at $\overline{x}$ such that*

$$s(\overline{x}) = f(\overline{x}),\ \nabla s(\overline{x}) = x^* \text{ and } s(x) \leq f(x) \text{ for every } x \in U.$$

The next fuzzy sum rule for the Fréchet subdifferential is another main tool for obtaining the desired openness results (see [6, Theorem 2.33]):

Theorem 2.4 (Fuzzy sum rule). *Let X be an Asplund space and $\varphi_1, \varphi_2 : X \to \mathbb{R} \cup \{\infty\}$ be such that φ_1 is Lipschitz continuous around $\overline{x} \in \operatorname{dom} \varphi_1 \cap \operatorname{dom} \varphi_2$ and φ_2 is lower semicontinuous around $\overline{x}$. Then for any $\gamma > 0$ one has*

$$\widehat{\partial}(\varphi_1+\varphi_2)(\overline{x}) \subset \bigcup\{\widehat{\partial}\varphi_1(x_1)+\widehat{\partial}\varphi_2(x_2) \mid x_i \in \overline{x}+\gamma D_X, |\varphi_i(x_i) - \varphi_i(\overline{x})| \le \gamma, i = 1, 2\}+\gamma D_{X^*}.$$

The basic subdifferential satisfies a robust sum rule (see [6, Theorem 3.36]): if X is Asplund, $f_1, f_2, ..., f_{n-1} : X \to \mathbb{R}$ are Lipschitz around $\overline{x}$ and $f_n : X \to \overline{\mathbb{R}}$ is lsc around this point, then

$$\partial(\sum_{i=1}^{n} f_i)(\overline{x}) \subset \sum_{i=1}^{n} \partial f_i(\overline{x}).$$

Definition 2.5. Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative at $(\overline{x}, \overline{y})$ is the set-valued map $\widehat{D}^*F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x}, \overline{y})$ is the set-valued map $D_N^*F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D_N^*F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

Note that, in fact, the concept of normal coderivative, independently of the normal cone used in its definition, was introduced in [5]. If we consider convex-graph multifunctions, we obtain a special form for these two coderivatives, which will be useful to derive a new proof of the Robinson-Ursescu theorem from our openness results (see [6, Proposition 1.37]).

Proposition 2.6. *Let $F : X \rightrightarrows Y$ be convex-graph and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Then one has the following coderivative representation:*

$$\begin{aligned}\widehat{D}^*F(\overline{x}, \overline{y})(y^*) &= D_N^*F(\overline{x}, \overline{y})(y^*)\\ &= \left\{x^* \in X^* \mid \langle x^*, \overline{x}\rangle - \langle y^*, \overline{y}\rangle = \max_{(x,y)\in \operatorname{Gr} F}[\langle x^*, x\rangle - \langle y^*, y\rangle]\right\}.\end{aligned}$$

In this case, we denote by $D^*F(\overline{x}, \overline{y})(y^*)$ any of the preceding two coderivatives.

3 Openness Results

We start with an openness result for set-valued mappings. The conclusion and the technique of proof displayed in the next (main) result are fundamental in the sense that it can be used as well (as we shall see later) for deriving openness results for parametric set-valued maps and implicit multifunctions theorems. This technique as well as the following result can be found in [9, Theorem 2.3] but we obtain here a more precise estimate for the neighborhoods of $(\overline{x}, \overline{y})$ involved in the openness property. This will be essential in the sequel. For these reasons we completely establish and prove here the result.

Theorem 3.1. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that the following assumptions are satisfied:*

(i) $\operatorname{Gr} F$ *is closed;*

(ii) *there exist $c > 0$, $r > 0$, $s > 0$ such that for every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$ and every $y^* \in Y^*$, $x^* \in \widehat{D}^* F(x, y)(y^*)$,*

$$c \|y^*\| \leq \|x^*\| .$$

Then for every $a \in (0, c)$ and for every $\rho \in (0, \varepsilon)$, where $\varepsilon := \min \left(\frac{1}{2} \left(\frac{c}{c+1} - \frac{a}{a+1} \right), \frac{r}{a+1}, \frac{s}{2a} \right)$, it holds

$$B(\overline{y}, \rho a) \subset F(B(\overline{x}, \rho)).$$

Proof. Take $a \in (0, c)$, $b \in \left(\frac{a}{a+1}, \frac{1}{2} \left(\frac{c}{c+1} + \frac{a}{a+1} \right) \right)$ and $\rho \in (0, \varepsilon)$. We have

$$b + \rho < \tfrac{c}{c+1}, \tag{3.1}$$

$$b^{-1} a \rho < b^{-1} a \tfrac{r}{a+1} < r. \tag{3.2}$$

Choose $v \in B(\overline{y}, \rho a)$ and define $f : \operatorname{Gr} F \to \mathbb{R}$, $f(x, y) := \|v - y\|$. Since $\operatorname{Gr} F$ is closed we can apply the Ekeland variational principle for f for obtaining $(u_b, v_b) \in \operatorname{Gr} F$ such that

$$\|v_b - v\| \leq \|\overline{y} - v\| - b(\|\overline{x} - u_b\| + \|\overline{y} - v_b\|) \tag{3.3}$$

and

$$\|v_b - v\| \leq \|y - v\| + b(\|x - u_b\| + \|y - v_b\|), \text{ for every } (x, y) \in \operatorname{Gr} F.$$

From (3.3) and (3.2) we have

$$\|\overline{x} - u_b\| \leq b^{-1} \|\overline{y} - v\| < b^{-1} a \rho < r,$$
$$\|\overline{y} - v_b\| \leq \|\overline{y} - v\| + \|v - v_b\| \leq 2 \|\overline{y} - v\| < 2 \rho a < s.$$

Hence, $(u_b, v_b) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$. If $v_b = v$, then

$$b \|\overline{x} - u_b\| \leq (1 - b) \|\overline{y} - v\| < (1 - b) a \rho < b \rho,$$

hence $u_b \in B(\overline{x}, \rho)$ and $v \in F(B(\overline{x}, \rho))$, which is exactly the conclusion.

We want to prove that $v_b = v$ is the sole possible situation. For this, suppose that $v \neq v_b$ and consider the function

$$h : X \times Y \to \mathbb{R}, \ h(x, y) := \|y - v\| + b(\|x - u_b\| + \|y - v_b\|).$$

From the second relation of the Ekeland variational principle, we have that the pair (u_b, v_b) is a minimum point for h on the set $\operatorname{Gr} F$, or, equivalently, (u_b, v_b) is a global minimum point for the function $h + \delta_{\operatorname{Gr} F}$. Applying the generalized Fermat rule, we have

$$(0, 0) \in \widehat{\partial}(h(\cdot, \cdot) + \delta_{\operatorname{Gr} F}(\cdot, \cdot))(u_b, v_b).$$

Using the fact that h is Lipschitz and $\delta_{\operatorname{Gr} F}$ is lsc, we can apply the fuzzy calculus rule for the Fréchet subdifferential. Choose $\gamma \in (0, \rho)$ such that

$$D(u_b, \gamma) \subset B(\overline{x}, r),$$
$$v \notin D(v_b, \gamma) \subset B(\overline{y}, s)$$

and obtain that there exist

$$(u_\gamma^1, v_\gamma^1) \in D(u_b, \gamma) \times D(v_b, \gamma) \subset B(\overline{x}, r) \times B(\overline{y}, s),$$
$$(u_\gamma^2, v_\gamma^2) \in [D(u_b, \gamma) \times D(v_b, \gamma)] \cap \operatorname{Gr} F \subset [B(\overline{x}, r) \times B(\overline{y}, s)] \cap \operatorname{Gr} F$$

such that

$$(0,0) \in \widehat{\partial} h(u_\gamma^1, v_\gamma^1) + \widehat{\partial}\delta_{\operatorname{Gr} F}(u_\gamma^2, v_\gamma^2) + \rho(D_{X^*} \times D_{Y^*}).$$

Observing that h is the sum of three convex functions, Lipschitz on $X \times Y$, $\widehat{\partial} h$ coincides with the sum of the convex subdifferentials. Remarking also that $v \neq v_\gamma^1 \in D(v_b, \gamma)$, we obtain

$$(0,0) \in \{0\} \times S_{Y^*} + b(D_{X^*} \times \{0\} + \{0\} \times D_{Y^*}) + \widehat{N}(\operatorname{Gr} F, (u_\gamma^2, v_\gamma^2)) + \rho(D_{X^*} \times D_{Y^*}).$$

We find then $y_1^* \in S_{Y^*}$, $y_2^*, y_3^* \in D_{Y^*}$, $x_1^*, x_2^* \in D_{X^*}$ such that

$$(-bx_1^* - \rho x_2^*, -y_1^* - by_2^* - \rho y_3^*) \in \widehat{N}(\operatorname{Gr} F, (u_\gamma^2, v_\gamma^2))$$
$$-bx_1^* - \rho x_2^* \in \widehat{D}^* F(u_\gamma^2, v_\gamma^2)(y_1^* + by_2^* + \rho y_3^*).$$

Using that $(u_\gamma^2, v_\gamma^2) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$, we obtain

$$b + \rho \geq \|-bx_1^* - \rho x_2^*\| \geq c\,\|y_1^* + by_2^* + \rho y_3^*\| \geq c(1 - b - \rho),$$

which is a contradiction with the inequality $(1 - b - \rho)^{-1}(b + \rho) < c$. □

Of course, the conclusion of the above theorem can be stated for a neighborhood of $(\overline{x}, \overline{y})$ making some changes of the constants. We are mainly interested by the situation described in the above result for the reasons which can be seen in the next section. Another remark concerns the fact that Theorem 3.1 can be obtained (as well for a neighborhood of the reference point) under the assumption that $\operatorname{Gr} F$ is only locally closed at that point: it should define the multifunction on an appropriate neighborhood of $(\overline{x}, \overline{y})$ and then make some technical manipulations on the involved constants. We write down such a result; see [6, Theorem 4.1].

Theorem 3.2. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ such that $\operatorname{Gr} F$ is locally closed at $(\overline{x}, \overline{y})$. Then the following assertions are equivalent:*

(i) *There exist $r > 0$, $s > 0$ and $c > 0$ such that for every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$ and every $y^* \in Y^*, x^* \in \widehat{D}^* F(x, y)(y^*)$,*

$$c\,\|y^*\| \leq \|x^*\|.$$

(ii) *There exist $\alpha > 0$, $\beta > 0$, $c > 0$ and $\varepsilon > 0$ such that for every $(x, y) \in \operatorname{Gr} F \cap [B(\overline{x}, \alpha) \times B(\overline{y}, \beta)]$, every $a \in (0, c)$ and every $\rho \in (0, \varepsilon]$,*

$$B(y, \rho a) \subset F(B(x, \rho)).$$

The key condition (ii) is fully discussed in [9, Section 3] for the general case where instead of $\widehat{D}^*$ one has a positively homogeneous map. We just point out that in the particular case when $\operatorname{Gr} F$ is convex then one can speak about a convex duality between the derivative of F at a point $(x, y) \in \operatorname{Gr} F$ (denoted by $DF(x, y) : X \rightrightarrows Y$ and defined as the set-valued map whose graph is the tangent cone to $\operatorname{Gr} F$ at (x, y)) and the coderivative of F at (x, y).

In this case $DF(x,y)$ is a convex process (i.e. its graph is a convex cone) and, taking into account Theorem 1.3.16 in [11], the condition (ii) becomes:

$(ii)'$ There exist $r > 0$, $s > 0$ and $a > 0$ such that for every $(x,y) \in [B(\overline{x}, r) \times B(\overline{y}, s)] \cap \operatorname{Gr} F$

$$B(0,a) \subset \operatorname{cl} DF(x,y)(B(0,1)). \tag{3.4}$$

This condition $(ii)'$ describes a property of uniform "almost" openness of the derivative. The word "almost" is due to the presence of the closure in the right hand side of (3.4). So, in this particular situation, Theorem 3.1 deduces a genuine openness property of a multifunction from the "almost" openness property of its normal derivative (see also [10]). Further, for the case when $F = T$ is a linear bounded operator, denoting by T^* the adjoint of T, condition (ii) becomes $\|T^*y^*\| \geq \gamma \|y^*\|$ for every $y^* \in Y^*$ and for some $\gamma > 0$. This relation is in turn equivalent with the surjectivity of T.

Remark 3.3. We can see from the proof of Theorem 3.1 that the assumption that X and Y are Asplund spaces is used only to apply the fuzzy calculus rule for the Fréchet subdifferential of the sum. So, the results hold as well if one considers other types of subdifferentials which satisfy similar calculus on the appropriate classes of Banach spaces. For, example, this is the case of the following objects:

- the proximal subdifferential on the class of Hilbert spaces;
- the Fréchet subdifferential of viscosity on the class of Banach spaces which admit a C^1 Lipschitz bump function;
- the β-subdifferential of viscosity on the class of Banach spaces which admit a β-differentiable bump function.

Of course, in the case when the subdifferential satisfies exact calculus rules the results holds as well. We illustrate by another well-known examples (for details, see Subsection 3.2.3 and the commentaries to it from [6], as well as [8]):

- the limiting (or Mordukhovich) subdifferential on the class of Asplund spaces;
- the approximate (or Ioffe) subdifferential on the class of Banach spaces;
- the Clarke subdifferential on Banach spaces.

Remark 3.4. Observe that if we add the assumption that $\operatorname{Gr} F$ is convex, we do not need X and Y to be Asplund spaces, because in this case $\delta_{\operatorname{Gr} F}$ is a convex function and we can use the classical sum rule for the convex subdifferential instead of the fuzzy sum rule for the Fréchet subdifferential on Asplund spaces, so Theorem 3.1 holds in this case on general Banach spaces. In this way, we obtain another proof of the well-known Robinson-Ursescu theorem, as follows.

Theorem 3.5 (Robinson-Ursescu). *Let X and Y be Banach spaces and $F : X \rightrightarrows Y$ be a set-valued map whose graph is convex and closed. Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ such that $\overline{y} \in \operatorname{aint}(\operatorname{Im} F)$. Then for every $V \in \mathcal{V}(\overline{x})$ we have $F(V) \in \mathcal{V}(\overline{y})$.*

Proof. The first part of the proof is classical, showing that there exists $\gamma > 0$ such that $D(\overline{y}, \gamma) \subset \operatorname{cl} F(D(\overline{x}, 1))$ (see, for example, [11, Theorem 1.3.5]) and it is based on the observation that $F(D(\overline{x}, 1)) - \overline{y}$ is absorbing and convex and on the Baire's Theorem.

Take now $(x, y) \in [D(\overline{x}, 1) \times D(\overline{y}, 2^{-1}\gamma)] \cap \operatorname{Gr} F$. We have then

$$D(y, 2^{-1}\gamma) \subset D(\overline{y}, \gamma) \subset \operatorname{cl} F(D(\overline{x}, 1)) \subset \operatorname{cl} F(D(x, 2)). \tag{3.5}$$

Choose $v^* \in Y^*$ arbitrary, $u^* \in D^*F(x, y)(v^*)$ and using the representation of the coderivatives from Proposition 2.6 we have

$$\langle v^*, y - v \rangle \le \langle u^*, x - u \rangle, \text{ for every } (u, v) \in \operatorname{Gr} F.$$

By (3.5), for every $v \in D(y, 2^{-1}\gamma)$ we find $(v_n) \subset F(D(x, 2))$ such that $v_n \to v$ and consequently $(u_n) \subset D(x, 2)$ such that $v_n \in F(u_n)$ for every $n \in \mathbb{N}$. Hence

$$\langle v^*, y - v_n \rangle \le \langle u^*, x - u_n \rangle \le \|u^*\| \|x - u_n\| \le 2 \|u^*\|. \tag{3.6}$$

Passing to the limit in (3.6) we obtain

$$\langle v^*, y - v \rangle \le 2 \|u^*\|, \text{ for every } v \in D(y, 2^{-1}\gamma),$$

showing that $2^{-1}\gamma \|v^*\| \le 2 \|u^*\|$.

We have shown that (ii) from Theorem 3.1 holds with $r := 1, s := 2^{-1}\gamma$ and $c := 4^{-1}\gamma$.

Let $V \in \mathcal{V}(\overline{x})$ and $a \in (0, c)$. Then we can find $\rho > 0$ sufficiently small such that $B(\overline{x}, \rho) \subset V$. In view of the Remark 3.4, we can use Theorem 3.1 to obtain that $B(\overline{y}, \rho a) \subset F(B(\overline{x}, \rho)) \subset F(V)$, whence the conclusion. □

If one fully exploits the conclusion of Theorem 3.1 then one obtains another known thing (see, e.g., [11, Theorem 1.3.11]), namely the fact that under the assumptions of Robinson-Ursescu Theorem one has openness with linear rate. We would like to mention that the proof presented here is not (necessarily) simpler than other proofs in literature, but it has the advantage that it is included in the framework of the results obtained on the base of condition (1.1). Note that the derivation of the Robinson-Ursescu Theorem from the coderivative condition in Asplund spaces was given in [6, Theorem 4.21].

4 Parametric Multifunctions

In the sequel we show that the openness theorem and its proof from the previous section can be used to derive openness of some parametric set-valued maps. Moreover, we show a partial equivalence between the non-parametric and the parametric cases.

Theorem 4.1. *Let X, Y be Asplund spaces, P be a topological space and $F : X \times P \rightrightarrows Y$ be a set-valued map. Denote $F_p(\cdot) := F(\cdot, p)$ and take $(\overline{x}, \overline{y}, \overline{p}) \in X \times Y \times P$ such that $\overline{y} \in F(\overline{x}, \overline{p})$. Suppose that the following hypothesis are satisfied:*

(i) *there exists $U_1 \in \mathcal{V}(\overline{p})$ such that, for every $p \in U_1$,* $\operatorname{Gr} F_p$ *is closed;*

(ii) *$F(\overline{x}, \cdot)$ is inner semicontinuous at $(\overline{p}, \overline{y})$;*

(iii) *there exist $r, s, c > 0$ and $U_2 \in \mathcal{V}(\overline{p})$ such that, for every $p \in U_2$, every $(x, y) \in \operatorname{Gr} F_p \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$ and every $y^* \in Y^*, x^* \in \widehat{D}^* F_p(x, y)(y^*)$,*

$$c \|y^*\| \le \|x^*\|.$$

Then for every $a \in (0, c)$ and $\rho \in (0, \varepsilon)$, where $\varepsilon := \min\left(\frac{1}{2}\left(\frac{c}{c+1} - \frac{a}{a+1}\right), \frac{r}{a+1}, \frac{2s}{3a}\right)$, there exists $U \in \mathcal{V}(\overline{p})$ such that for every $p \in U$,

$$B\left(\overline{y}, \tfrac{a\rho}{2}\right) \subset F_p(B(\overline{x}, \rho)).$$

Proof. Take, as above $a \in (0,c)$, $b \in \left(\frac{a}{a+1}, \frac{1}{2}\left(\frac{c}{c+1}+\frac{a}{a+1}\right)\right)$, $\rho \in (0,\varepsilon)$ and use the inner semicontinuity of $F(\overline{x},\cdot)$ at $(\overline{p},\overline{y})$ to find $U_3 \in \mathcal{V}(\overline{p})$ such that, for every $p \in U_3$, $F(\overline{x},p) \cap B\left(\overline{y}, \frac{a\rho}{2}\right) \neq \emptyset$. Choose $U := U_1 \cap U_2 \cap U_3$ and fix $p \in U$. Then there exists $y' \in F_p(\overline{x})$ such that $\|y'-\overline{y}\| < \frac{a\rho}{2}$.

Take $v \in B\left(\overline{y}, \frac{a\rho}{2}\right) \subset B(y', a\rho)$ and apply the Ekeland variational principle for the function

$$f : \operatorname{Gr} F_p \to \mathbb{R},\ f(x,y) := \|v-y\|$$

for obtaining $(u_b, v_b) \in \operatorname{Gr} F_p$ such that

$$\|v_b - v\| \leq \|y'-v\| - b(\|\overline{x}-u_b\| + \|y'-v_b\|)$$

and

$$\|v_b - v\| \leq \|y-v\| + b(\|x-u_b\| + \|y-v_b\|),\ \text{for every } (x,y) \in \operatorname{Gr} F_p.$$

Observe that

$$\begin{aligned}
\|\overline{x}-u_b\| &\leq b^{-1}\|y'-v\| < b^{-1}a\rho < (a+1)\rho < r,\\
\|\overline{y}-v_b\| &\leq \|\overline{y}-v\| + \|v-v_b\| \leq 2^{-1}a\rho + \|y'-v\|\\
&\leq 2^{-1}a\rho + \|y'-\overline{y}\| + \|\overline{y}-v\| < 2^{-1}3a\rho < s.
\end{aligned}$$

In the following, the proof is similar with that of Theorem 3.1. □

Let us observe that we can use directly Theorem 3.1 to obtain under the same assumptions as above a slightly weaker estimation. Namely, in this case ε can be obtained as $\min\left(\frac{1}{2}\left(\frac{c}{c+1}-\frac{a}{a+1}\right), \frac{r}{a+1}, \frac{s}{3a}\right)$. To see this, fix $a \in (0,c)$ and $\rho \in (0,\varepsilon)$. Using the inner semicontinuity of $F(\overline{x},\cdot)$ at $(\overline{p},\overline{y})$, we can find as above $U_3 \in \mathcal{V}(\overline{p})$ such that, for every $p \in U_3$, $F(\overline{x},p) \cap B\left(\overline{y}, \frac{a\rho}{2}\right) \neq \emptyset$. Choose again $U := U_1 \cap U_2 \cap U_3$ and fix $p \in U$. Then there exists $y' \in F_p(\overline{x})$ such that $\|y'-\overline{y}\| < \frac{a\rho}{2}$, hence $B\left(\overline{y}, \frac{a\rho}{2}\right) \subset B(y', a\rho)$.

Denote $s' := \frac{2s}{3}$. Then

$$B(y', s') \subset B\left(\overline{y}, \tfrac{2s}{3} + \tfrac{a\rho}{2}\right) \subset B(\overline{y}, s)$$

and $\rho \in \left(0, \min\left(\frac{1}{2}\left(\frac{c}{c+1}-\frac{a}{a+1}\right), \frac{r}{a+1}, \frac{s'}{2a}\right)\right)$. We can apply now Theorem 3.1 for F_p, s' and $(\overline{x}, y')$ instead of F, s and $(\overline{x},\overline{y})$, respectively, to prove that $B(y', a\rho) \subset F_p(B(\overline{x},\rho))$, which completes the proof.

Another interesting fact is that Theorem 4.1 can be used to prove Theorem 3.1.

For this, suppose that all the assumptions of Theorem 3.1 are satisfied, take $P := Y$ and define the set-valued map $\widetilde{F} : X \times Y \rightrightarrows Y$, $\widetilde{F}(x,y) := F(x) - y$. Denote $\widetilde{F}_y(\cdot) := \widetilde{F}(\cdot, y)$ and notice that

$$\operatorname{Gr} \widetilde{F}_y = \operatorname{Gr} F + (0,-y),\ \text{for every } y \in Y.$$

Because $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, we have $(\overline{x},\overline{y},0) \in \operatorname{Gr} \widetilde{F}$. Using the closedness of $\operatorname{Gr} F$, we observe that (i) of Theorem 4.1 is satisfied for $U_1 := Y$.

Also, $\widetilde{F}(\overline{x},\cdot) = F(\overline{x}) - \cdot$ is obviously inner semicontinuous at $(\overline{y}, 0)$.

To prove that (iii) of Theorem 4.1 is satisfied, take $r, c, s > 0$ such that for every $(x,u) \in \operatorname{Gr} F \cap [B(\overline{x},r) \times B(\overline{y},s)]$ and every $y^* \in Y^*$, $x^* \in \widehat{D}^* F(x,u)(y^*)$, the relation $c\|y^*\| \leq \|x^*\|$

holds. Define $U_2 := B(\overline{y}, \frac{s}{4})$ and take $y \in U_2$. Then for every $(x, z) \in \operatorname{Gr} \widetilde{F}_y \cap [B(\overline{x}, r) \times B(0, \frac{3s}{4})]$ we have

$$(x, z+y) \in \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(y, \tfrac{3s}{4})] \subset \operatorname{Gr} F \cap [B(\overline{x}, r) \times B(\overline{y}, s)].$$

Also, for every $y^* \in Y^*$, $x^* \in \widehat{D}^* \widetilde{F}_y(x, z)(y^*)$, we have

$$(x^*, -y^*) \in \widehat{N}(\operatorname{Gr} \widetilde{F}_y, (x, z)) = \widehat{N}(\operatorname{Gr} F, (x, z+y)),$$

hence the relation $c\,\|y^*\| \leq \|x^*\|$ holds.

We can apply now Theorem 4.1 for $\widetilde{F}, (\overline{x}, \overline{y}, 0)$ and $s' := \frac{3s}{4}$ to prove that for every $a \in (0, c)$ and every $\rho \in \left(0, \min\left(\frac{1}{2}\left(\frac{c}{c+1} - \frac{a}{a+1}\right), \frac{r}{a+1}, \frac{2s'}{3a}\right)\right)$, there exists $U \in \mathcal{V}(\overline{y})$ such that, for every $y \in U$,

$$B(0, \tfrac{a\rho}{2}) \subset \widetilde{F}_y(B(\overline{x}, \rho)) = F(B(\overline{x}, \rho)) - y,$$

or, equivalently, for every $a \in (0, c)$ and every $\rho \in \left(0, \min\left(\frac{1}{2}\left(\frac{c}{c+1} - \frac{a}{a+1}\right), \frac{r}{a+1}, \frac{s}{2a}\right)\right)$, there exists $U \in \mathcal{V}(\overline{y})$ such that

$$\bigcup_{y \in U} B(y, \tfrac{a\rho}{2}) \subset F(B(\overline{x}, \rho)).$$

Moreover, we can see from the proof of Theorem 4.1 that $U = U_1 \cap U_2 \cap U_3$, where $U_3 \in \mathcal{V}(\overline{y})$ is chosen such that

$$\widetilde{F}(\overline{x}, y) \cap B(0, \tfrac{a\rho}{2}) \neq \emptyset \text{ for every } y \in U_3. \tag{4.1}$$

Take $U_3 := B(\overline{y}, \frac{a\rho}{2})$ and see that $\overline{y} \in B(y, \frac{a\rho}{2}) \cap F(\overline{x})$ for every $y \in U_3$, which shows (4.1). Also, $\frac{a\rho}{2} < \frac{s}{4}$, hence $U = B(\overline{y}, \frac{a\rho}{2})$ and

$$B(\overline{y}, a\rho) = \bigcup_{y \in U} B(y, \tfrac{a\rho}{2}) \subset F(B(\overline{x}, \rho)).$$

5 Implicit Multifunctions

In the following, we seek for results concerning implicit multifunctions, using the openness theorem above stated for the parametric multifunctions. For a set-valued map $F : X \times P \rightrightarrows Y$, we can define the implicit multifunction $H : P \times Y \rightrightarrows X$ by

$$x \in H(p, y) \Leftrightarrow y \in F(x, p).$$

Denoting $H_p(\cdot) := H(p, \cdot)$, we have $H_p = F_p^{-1}$ for every $p \in P$. Following the classical theory, we can further define $G : P \rightrightarrows X$ as $H(\cdot, 0)$, so for every $p \in P$,

$$G(p) = \{x \in X \mid 0 \in F(x, p)\}.$$

The next result is an implicit multifunction theorem, showing that certain properties of the multifunction F can be transferred to H and G.

Theorem 5.1. *Let X, Y be Asplund spaces, P be a topological space and $F : X \times P \rightrightarrows Y$ be a set-valued map such that $\overline{y} \in F(\overline{x}, \overline{p})$. Suppose that all the assumptions of Theorem 4.1 are satisfied. Then there exist $U \in \mathcal{V}(\overline{p}), \delta > 0$ and $\rho > 0$ such that, for every $p \in U$ and $y \in B(\overline{y}, \delta)$, the multifunction $(p, y) \rightrightarrows H(p, y) \cap B(\overline{x}, \rho)$ takes nonempty values. In particular, if $\overline{y} := 0$, the multifunction $p \rightrightarrows G(p) \cap B(\overline{x}, \rho)$ takes nonempty values for every $p \in U$.*

If, in addition, the next hypothesis is satisfied:

(iv) *There exist $U_3 \in \mathcal{V}(\overline{p})$ and $\alpha > 0$ such that $F(x, \cdot)$ is lsc on U_3 for every $x \in B(\overline{x}, \alpha)$, then there exist $U_0 \in \mathcal{V}(\overline{p}), \delta_0 > 0$ and $\rho_0 > 0$ such that the multifunction $(p, y) \rightrightarrows H(p, y) \cap B(\overline{x}, \rho_0)$ is lsc on $U_0 \times B(\overline{y}, \delta_0)$. Again, if $\overline{y} = 0$, the multifunction $p \rightrightarrows G(p) \cap B(\overline{x}, \rho_0)$ is lsc on U_0.*

Proof. Choose $a \in (0, c)$ and $\rho \in (0, \min(\varepsilon, r))$ that provide U from the conclusion of the Theorem 4.1. Then for every $p \in U$, $B(\overline{y}, 2^{-1}a\rho) \subset F_p(B(\overline{x}, \rho))$. Take $\delta := 2^{-1}a\rho$ and $(p, y) \in U \times B(\overline{y}, \delta)$. There exists $x \in B(\overline{x}, \rho)$ such that $y \in F_p(x)$, hence $x \in H(p, y) \cap B(\overline{x}, \rho)$.

For the second part, take again $a \in (0, c)$ and $\rho \in (0, \min(\varepsilon, r, \alpha))$ to obtain U. Take $U_0 := U \cap U_3$, $\delta_0 := 2^{-1}a\rho$ and $\rho_0 := \rho$ and choose $(p, y) \in U_0 \times B(\overline{y}, \delta_0)$. There exists $x \in H(p, y) \cap B(\overline{x}, \rho_0)$, which is nonempty from the first part. We have to prove that for every $\theta > 0$ arbitrarily small, there exist $U' \in \mathcal{V}(\overline{p})$ and $\delta' > 0$ such that, for every $(p', y') \in U' \times B(y, \delta')$, $H(p', y') \cap B(\overline{x}, \rho_0) \cap B(x, \theta) \neq \emptyset$.

For this, choose $\theta > 0$ sufficiently small, $\beta \in (0, \theta)$ such that $B(x, \beta) \subset B(x, \theta) \subset B(\overline{x}, \rho)$ and $\gamma > 0$ such that $B(y, \gamma) \subset B(\overline{y}, s)$. With an analogous argument as in the proof of the first part of the theorem, applied for (x, p, y) instead of $(\overline{x}, \overline{p}, \overline{y})$, we can find $U' \in \mathcal{V}(\overline{p})$, $\rho' \in (0, \beta)$ and $\delta' > 0$ such that, for every $(p', y') \in U' \times B(y, \delta')$, $H(p', y') \cap B(x, \rho') \neq \emptyset$. Because $H(p', y') \cap B(x, \rho') \subset H(p', y') \cap B(x, \beta) \subset H(p', y') \cap B(\overline{x}, \rho) \cap B(x, \theta)$, we have the conclusion. □

Theorem 5.1 generalizes in several directions Theorem 3.1 from [3]. To be explicit, the assumption $(A2)$ is not needed, while $(A3)$ can be meaningfully relaxed and, moreover, the main result can be written to illustrate an openness property and to give some more details about the neighborhoods of the reference point $(\overline{x}, \overline{p}, \overline{y})$ where that property holds.

The next theorem shows a sort of metric regularity, as well as a graphical regularity for the implicit multifunctions.

Theorem 5.2. *Let X, Y be Asplund spaces, P be a topological space and $F : X \times P \rightrightarrows Y$ be a set-valued map such that $\overline{y} \in F(\overline{x}, \overline{p})$. Suppose that the following assumptions are satisfied:*

(i) *there exists $U_1 \in \mathcal{V}(\overline{p})$ such that, for every $p \in U_1$, $\operatorname{Gr} F_p$ is closed;*

(ii) *F is inner semicontinuous at $(\overline{x}, \overline{p})$;*

(iii) *there exist $r, s, c > 0$ and $U_2 \in \mathcal{V}(\overline{p})$ such that, for every $p \in U_2$, every $(x, y) \in \operatorname{Gr} F_p \cap [B(\overline{x}, r) \times B(\overline{y}, s)]$ and every $y^* \in Y^*$, $x^* \in \widehat{D}^* F_p(x, y)(y^*)$,*

$$c \|y^*\| \leq \|x^*\| .$$

Then the following are true:

(a) *For every $a \in (0, c)$, there exist $U \in \mathcal{V}(\overline{p})$, $\delta > 0$ and $\tau > 0$ such that, for every $p \in U$, $y \in B(\overline{y}, \delta)$ and $x \in B(\overline{x}, \tau)$,*

$$d(x, H(p, y)) \leq \tfrac{1}{a} d(y, F(x, p)), \tag{5.1}$$

hence if $\overline{y} = 0$, for every $p \in U$ and $x \in B(\overline{x}, \tau)$,

$$d(x, G(p)) \leq \tfrac{1}{a} d(0, F(x, p)). \tag{5.2}$$

(b) *If P is a metric space, denoting $H_y(\cdot) := H(\cdot, y)$, for every $a \in (0, c)$, there exist $\gamma_0 > 0$, $\delta_0 > 0$, $\tau_0 > 0$ and $l := 1 + \frac{1}{a}$ such that, for every $p \in B(\overline{p}, \gamma_0)$, $y \in B(\overline{y}, \delta_0)$ and $x \in B(\overline{x}, \tau_0)$, one has*

$$d((p, x), \operatorname{Gr} H_y) \leq l d((x, p, y), \operatorname{Gr} F), \tag{5.3}$$

hence if $\overline{y} = 0$, for every $p \in B(\overline{p}, \gamma_0)$ and $x \in B(\overline{x}, \tau_0)$,

$$d((p, x), \operatorname{Gr} G) \leq l d((x, p, 0), \operatorname{Gr} F). \tag{5.4}$$

Proof. (a) Let $a \in (0, c)$ and $\rho \in \left(0, \min\left(\frac{1}{2}\left(\frac{c}{c+1} - \frac{a}{a+1}\right), \frac{r}{2(a+1)}, \frac{s}{4a}\right)\right)$. Using the inner semicontinuity of F at $(\overline{x}, \overline{p})$, we can find $U_0 \in \mathcal{V}(\overline{p})$ and $\nu > 0$ such that for every $(x, p) \in B(\overline{x}, \nu) \times U_0$,

$$F(x, p) \cap B(\overline{y}, \tfrac{a\rho}{2}) \neq \emptyset. \tag{5.5}$$

Denote $U := U_0 \cap U_1 \cap U_2$, $\tau := \min(\nu, \frac{r}{2})$, $\delta := \frac{a\rho}{2}$ and take $(x, p, y) \in B(\overline{x}, \tau) \times U \times B(\overline{y}, \delta)$.

If $y \in F(x, p)$, then (5.1) trivially holds. Suppose that $y \notin F(x, p)$ and then, for every $\varepsilon > 0$, we can find $y_\varepsilon \in F(x, p)$ such that

$$\|y_\varepsilon - y\| < d(y, F(x, p)) + \varepsilon. \tag{5.6}$$

Because from (5.5) and the choice of y we have $d(y, F(x, p)) < a\rho$, we can take $\varepsilon > 0$ sufficiently small such that $d(y, F(x, p)) + \varepsilon < a\rho$. Using (5.6), we have

$$y \in B(y_\varepsilon, d(y, F(x, p)) + \varepsilon) \subset B(y_\varepsilon, a\rho).$$

Moreover,

$$\begin{aligned} B(x, 2^{-1}r) &\subset B(\overline{x}, r), \\ B(y_\varepsilon, 2^{-1}s) &\subset B(y, 2^{-1}s + a\rho) \\ &\subset B(\overline{y}, 2^{-1}s + a\rho + 2^{-1}a\rho) \\ &\subset B(\overline{y}, s). \end{aligned}$$

Hence we can apply Theorem 3.1 for $(x, y_\varepsilon) \in \operatorname{Gr} F_p$, $r_0 := 2^{-1}r$, $s_0 := 2^{-1}s$ and $\rho_0 := \frac{1}{a}(d(y, F(x, p)) + \varepsilon) < \rho$, showing that

$$B(y_\varepsilon, a\rho_0) \subset F_p(B(x, \rho_0)).$$

We can find then $\widetilde{x} \in B(x, \rho_0)$ such that $y \in F_p(\widetilde{x})$, or $\widetilde{x} \in H(p, y)$. Hence

$$d(x, H(p, y)) \leq \|x - \widetilde{x}\| < \tfrac{1}{a}(d(y, F(x, p)) + \varepsilon).$$

Making $\varepsilon \to 0$, we obtain (5.1) with $k := \frac{1}{a}$.

(b) The proof is similar with the proof of (a), but has some important different points. Take as above $a \in (0, c)$ and $\rho \in \left(0, \min\left(\frac{1}{2}\left(\frac{c}{c+1} - \frac{a}{a+1}\right), \frac{r}{4(a+1)}, \frac{s}{4a}\right)\right)$, use again the inner semicontinuity of F at $(\overline{x}, \overline{p})$ and find the neighborhood U_0 of $\overline{p}$ and $\nu > 0$ such that for every $(x, p) \in B(\overline{x}, \nu) \times U_0$, (5.5) holds. If P is a metric space, we can find $\gamma > 0$ such that $B(\overline{p}, \gamma) \subset U_0 \cap U_1 \cap U_2$. Take $\delta_0 := \min(\frac{a\rho}{2}, \frac{\gamma}{6})$, $\tau_0 := \min(\frac{\gamma}{6}, \nu, \frac{r}{4})$, $\gamma_0 := \frac{\gamma}{3}$ and choose $(x, p, y) \in B(\overline{x}, \tau_0) \times B(\overline{p}, \gamma_0) \times B(\overline{y}, \delta_0)$.

We have

$$\begin{aligned} &d((x, p, y), \operatorname{Gr} F) \leq \|x - \overline{x}\| + d(p, \overline{p}) + \|y - \overline{y}\| < \tfrac{\gamma}{6} + \tfrac{\gamma}{3} + \tfrac{\gamma}{6} = \tfrac{2\gamma}{3}, \\ &d((x, p, y), \operatorname{Gr} F) < d(y, F(x, p)) < a\rho. \end{aligned}$$

Without loss of generality suppose that $y \notin F(x,p)$, hence for every $\varepsilon > 0$ sufficiently small such that $d((x,p,y),\operatorname{Gr} F)+\varepsilon < \min(a\rho, \frac{2\gamma}{3})$ we can find $(x_\varepsilon, p_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F$ satisfying

$$\begin{aligned}\max(\|y_\varepsilon - y\|, d(p_\varepsilon, p)) &\le \|y_\varepsilon - y\| + \|x_\varepsilon - x\| + d(p_\varepsilon, p)\\ &< d((x,p,y),\operatorname{Gr} F)+\varepsilon.\end{aligned} \tag{5.7}$$

Hence,

$$\begin{aligned}&p_\varepsilon \in B(p, \tfrac{2\gamma}{3}) \subset B(\overline{p}, \gamma),\\ &y \in B(y_\varepsilon, d((x,p,y),\operatorname{Gr} F)+\varepsilon) \subset B(y_\varepsilon, a\rho)\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}&B(x_\varepsilon, 4^{-1}r) \subset B(x, 4^{-1}r + a\rho) \subset B(x, 2^{-1}r) \subset B(\overline{x}, r),\\ &B(y_\varepsilon, 2^{-1}s) \subset B(\overline{y}, s).\end{aligned}$$

Then we can apply Theorem 3.1 for $(x_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F_{p_\varepsilon}$, $r' := 4^{-1}r$, $s' := 2^{-1}s$ and $\rho' := \frac{1}{a}(d((x,p,y),\operatorname{Gr} F)+\varepsilon) < \rho$ and obtain

$$B(y_\varepsilon, a\rho') \subset F_{p_\varepsilon}(B(x_\varepsilon, \rho')).$$

Using (5.8), we have that there exists $\widetilde{x} \in B(x_\varepsilon, \rho')$ such that $y \in F_{p_\varepsilon}(\widetilde{x})$, or $(\widetilde{x}, p_\varepsilon) \in \operatorname{Gr} H_y$. Hence, using (5.7),

$$\begin{aligned}d((p,x),\operatorname{Gr} H_y) &\le \|\widetilde{x} - x\| + d(p_\varepsilon, p)\\ &\le \|\widetilde{x} - x_\varepsilon\| + \|x_\varepsilon - x\| + d(p_\varepsilon, p)\\ &< \tfrac{1}{a}(d((x,p,y),\operatorname{Gr} F)+\varepsilon) + d((x,p,y),\operatorname{Gr} F)+\varepsilon.\end{aligned}$$

Making again $\varepsilon \to 0$, we obtain (5.3) with $l = \frac{1}{a} + 1$. □

Theorem 5.2 covers Theorem 3.2 in [3], again with the relaxation of some assumptions.

The next result presents a formula for the coderivative of G, following the similarity with the classical implicit function theorem, where a formula for the derivative of the implicit function is present. This formula was shown in [2, Proposition 3.8], but here is obtained as a consequence of (5.4).

Theorem 5.3. *Suppose that all the hypotheses of Theorem 5.2 are satisfied for $\overline{y} = 0$ and that P is an Asplund space. Then there exist $\gamma > 0$ and $\tau > 0$ such that, for every $(x,p) \in B(\overline{p}, \gamma) \times B(\overline{x}, \rho)$ such that $x \in G(p)$ and for every $x^* \in X^*$, the following inclusion for the Fréchet coderivative of G holds:*

$$\widehat{D}^*G(p,x)(x^*) \supset \bigcup_{y^* \in Y^*} \{p^* \in P^* \mid (-x^*, p^*) \in \widehat{D}^*F(x,p,0)(y^*)\}. \tag{5.9}$$

Moreover, if $\operatorname{Gr} F$ is closed, for every $\varepsilon > 0$ and every $x^* \in X^*$, $p^* \in \widehat{D}^*G(p,x)(x^*)$, there exist $(x_\varepsilon, p_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F$ and $(x_\varepsilon^*, p_\varepsilon^*, y_\varepsilon^*) \in X^* \times P^* \times Y^*$ such that

$$\begin{aligned}&\|x_\varepsilon - x\| < \varepsilon, \|p_\varepsilon - p\| < \varepsilon, \|y_\varepsilon\| < \varepsilon,\\ &(-x_\varepsilon^*, p_\varepsilon^*) \in \widehat{D}^*F(x_\varepsilon, p_\varepsilon, y_\varepsilon)(y_\varepsilon^*),\\ &\|x_\varepsilon^* - x^*\| < \varepsilon, \|p_\varepsilon^* - p^*\| < \varepsilon.\end{aligned}$$

Hence, if the multifunction F is $N-$regular at $(x,p,0)$ (i.e. $\widehat{D}^*F(x,p,0) = D_N^*F(x,p,0)$), we have equality in (5.9).

Proof. Take $\gamma > 0$ and $\rho > 0$ such that for every $p \in B(\overline{p}, \gamma)$ and $x \in B(\overline{x}, \tau)$, (5.4) holds. Choose $x \in G(p) \cap B(\overline{x}, \rho)$ and $x^* \in X^*$, $p^* \in P^*$ such that there exists $y^* \in Y^*$ for which $(-x^*, p^*) \in \widehat{D}^*F(x, p, 0)(y^*)$. Then we have

$$(-x^*, p^*, -y^*) \in \widehat{N}(\operatorname{Gr} F, (x, p, 0)) = \widehat{\partial}\delta_{\operatorname{Gr} F}(x, p, 0).$$

Using the smooth variational description of Fréchet subgradients, we can find $\alpha > 0$, $\beta > 0$, $\theta > 0$ and a Fréchet differentiable function

$$s : B(x, \alpha) \times B(p, \beta) \times B(0, \theta) \to \mathbb{R}$$

such that

$$\begin{aligned} &\nabla s(x, p, 0) = (-x^*, p^*, -y^*),\\ &s(x, p, 0) = \delta_{\operatorname{Gr} F}(x, p, 0) = 0,\\ &s(\widetilde{x}, \widetilde{p}, \widetilde{y}) \leq \delta_{\operatorname{Gr} F}(\widetilde{x}, \widetilde{p}, \widetilde{y}), \text{ for every } (\widetilde{x}, \widetilde{p}, \widetilde{y}) \in B(x, \alpha) \times B(p, \beta) \times B(0, \theta). \end{aligned} \tag{5.10}$$

Defining $\widetilde{s} : B(p, \beta) \times B(x, \alpha) \to \mathbb{R}$ by $\widetilde{s}(\widetilde{p}, \widetilde{x}) := s(\widetilde{x}, \widetilde{p}, 0)$, we have from (5.10)

$$\begin{aligned} \nabla\widetilde{s}(p, x) &= \nabla_{(\widetilde{p},\widetilde{x})} s(x, p, 0) = (p^*, -x^*),\\ \widetilde{s}(p, x) &= s(x, p, 0) = 0 = \delta_{\operatorname{Gr} G}(p, x),\\ \widetilde{s}(\widetilde{p}, \widetilde{x}) &= s(\widetilde{x}, \widetilde{p}, 0)\\ &\leq \delta_{\operatorname{Gr} F}(\widetilde{x}, \widetilde{p}, 0) = \delta_{\operatorname{Gr} G}(\widetilde{p}, \widetilde{x}), \text{ for every } (\widetilde{p}, \widetilde{x}) \in B(p, \beta) \times B(x, \alpha). \end{aligned}$$

Hence, $(p^*, -x^*) \in \widehat{\partial}\delta_{\operatorname{Gr} G}(p, x) = \widehat{N}(\operatorname{Gr} G, (p, x))$, or $p^* \in \widehat{D}^*G(p, x)(x^*)$, showing (5.9).

For the second part, take $\varepsilon > 0$ and $x^* \in X^*, p^* \in \widehat{D}^*G(p, x)(x^*)$. Then there exists $\lambda > 0$ such that $(p^*, -x^*) \in \widehat{\partial}[\lambda d((\cdot, \cdot), \operatorname{Gr} G)](p, x)$. Using (5.4), we can find $\tau', \gamma' > 0$ such that

$$\lambda d((\widetilde{x}, \widetilde{p}), \operatorname{Gr} G) \leq \lambda l(d((\widetilde{x}, \widetilde{p}, \widetilde{y}), \operatorname{Gr} F) + \|\widetilde{y}\|) \leq \delta_{\operatorname{Gr} F}(\widetilde{x}, \widetilde{p}, \widetilde{y}) + \lambda l \|\widetilde{y}\|,$$

for every $(\widetilde{x}, \widetilde{p}, \widetilde{y}) \in B(x, \tau') \times B(p, \gamma') \times Y \subset B(\overline{x}, \tau) \times B(\overline{p}, \gamma) \times Y$.

Now we define the functions $f, g : B(x, \tau') \times B(p, \gamma') \times Y \to \overline{\mathbb{R}}$ by

$$\begin{aligned} f(\widetilde{x}, \widetilde{p}, \widetilde{y}) &:= \delta_{\operatorname{Gr} F}(\widetilde{x}, \widetilde{p}, \widetilde{y}) + \lambda l \|\widetilde{y}\|,\\ g(\widetilde{x}, \widetilde{p}, \widetilde{y}) &:= \lambda d((\widetilde{x}, \widetilde{p}), \operatorname{Gr} G). \end{aligned}$$

Using that $f(x, p, 0) = g(x, p, 0) = 0$ and $f \leq g$, we obtain

$$(-x^*, p^*, 0) \in \widehat{\partial}g(x, p, 0) \subset \widehat{\partial}f(x, p, 0).$$

Because $\operatorname{Gr} F$ is closed, we can apply the fuzzy calculus rule for the Fréchet subdifferential of f and ε chosen before to obtain that there exists $(x_\varepsilon, p_\varepsilon, y_\varepsilon) \in X \times P \times Y$ such that

$$\begin{aligned} \|(x_\varepsilon, p_\varepsilon, y_\varepsilon) - (x, p, 0)\| &< \varepsilon,\\ \|\delta_{\operatorname{Gr} F}(x_\varepsilon, p_\varepsilon, y_\varepsilon) - \delta_{\operatorname{Gr} F}(x, p, 0)\| &< \varepsilon \end{aligned} \tag{5.11}$$

and

$$\widehat{\partial}f(x, p, 0) \subset \widehat{\partial}\delta_{\operatorname{Gr} F}(x_\varepsilon, p_\varepsilon, y_\varepsilon) + \{0\} \times \{0\} \times \lambda l D_{Y^*} + \varepsilon(D_{X^*} \times D_{P^*} \times D_{Y^*}).$$

Using (5.11) we have $(x_\varepsilon, p_\varepsilon, y_\varepsilon) \in \operatorname{Gr} F$ and consequently there exist $(-x^*_\varepsilon, p^*_\varepsilon, y^*_\varepsilon) \in \widehat{N}(\operatorname{Gr} F, (x_\varepsilon, p_\varepsilon, y_\varepsilon))$ and $z^* \in D_{Y^*}$ such that

$$(-x^* + x^*_\varepsilon, p^* - p^*_\varepsilon, -y^*_\varepsilon - \lambda l z^*) \in \varepsilon(D_{X^*} \times D_{P^*} \times D_{Y^*}),$$

which completes the proof. □

Finally, we present a theorem that shows how the Lipschitz-like property transfers from F to the implicit multifunctions H and G.

Theorem 5.4. *Let all the assumptions of the Theorem 5.2 be satisfied. In addition, suppose that F is Lipschitz-like with respect to p uniformly in x around $(\overline{x},\overline{p},\overline{y})$, which means that there exist $\delta_3,\tau_3,L>0$ and $U_3\in\mathcal{V}(\overline{p})$ such that for every $x\in B(\overline{x},\tau_3)$ and every $p_1,p_2\in U_3$*

$$B(\overline{y},\delta_3)\cap F(x,p_2)\subset F(x,p_1)+Ld(p_1,p_2)D_Y. \tag{5.12}$$

Then for every $a\in(0,c)$ there exist $\bar{\delta},\bar{\tau}>0$ and $\bar{U}\in\mathcal{V}(\overline{p})$ such that for every $y\in B(\overline{y},\bar{\delta})$ and every $p_1,p_2\in\bar{U}$

$$B(\overline{x},\bar{\tau})\cap H(p_2,y)\subset H(p_1,y)+\frac{L}{a}d(p_1,p_2)D_X,$$

which means that H is Lipschitz-like with respect to p uniformly for $y\in B(\overline{y},\bar{\delta})$ with modulus $\frac{L}{a}$.

In particular, if $\overline{y}=0$, the implicit multifunction G is Lipschitz-like with the same modulus.

Proof. Fix $a\in(0,c)$ and take $\bar{\delta}:=\min(\delta,\delta_3),\bar{\tau}:=\min(\tau,\tau_3),\bar{U}:=U\cap U_3$, with δ,τ and U provided by Theorem 5.2, (a). Choose now $y\in B(\overline{y},\bar{\delta}),p_1,p_2\in\bar{U}$ and $x\in B(\overline{x},\bar{\tau})\cap H(p_2,y)$. Then, using (5.12), we obtain that

$$y\in F(x,p_2)\cap B(\overline{y},\bar{\delta})\subset F(x,p_1)+Ld(p_1,p_2)D_Y$$

and using (5.1), i.e. $d(x,H(p_1,y))\leq\frac{1}{a}d(y,F(x,p_1))$, we have

$$d(x,H(p_1,y))\leq\frac{1}{a}d(y,F(x,p_1))\leq\frac{L}{a}d(p_1,p_2).$$

The proof is complete. □

We mention that the above result is again a generalization of the corresponding result in [3].

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