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REGULARIZED GAP FUNCTIONS AND ERROR BOUNDS FOR VECTOR VARIATIONAL INEQUALITIES

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Abstract: In this paper we introduce a scalar gap function for a Stampacchia type vector variational inequality problem. We show how properties of the gap function can be improved by regularizing it in the sense of Fukushima. We study in detail the properties of this regularized gap function. An error bound for the vector variational inequality is obtained in terms of the regularized gap function. We also introduce the notion of D-gap function for vector variational inequalities. We study its properties and also present an error bound in terms of the D-gap functions.

Key words: *vector optimization, vector variational inequalities, Lipschitz maps, strongly monotone maps, gap function*

Mathematics Subject Classification: *90C33; 90C29*

1 Introduction and Motivation

The subject of vector variational inequalities has been studied intensively by optimization researchers in the recent past. In fact a whole monograph [5] has been devoted to vector variational inequalities. Further a look into Mathscinet would immediately show the huge number of papers that are being published in this area.

We begin by defining the *weak Stampacchia type vector variational inequality* which we denote by $(SVVI)^w$. This problem consists of finding a vector $\bar{x} \in K$ such that

$$(\langle F_1(\bar{x}), y-\bar{x}\rangle, \langle F_2(\bar{x}), y-\bar{x}\rangle, ..., \langle F_m(\bar{x}), y-\bar{x}\rangle) \notin -int\mathbb{R}^m_+ \text{ for all } y \in K, \tag{1.1}$$

where each $F_i : \mathbb{R}^n \to \mathbb{R}^n$, i=1,2,...m is a vector-valued function and K is a nonempty closed convex set in $\mathbb{R}^n$. The set of solutions of $(SVVI)^w$ is denoted by $sol(SVVI)^w$. Observe that if we set for each $i = 1, \dots, m$, $F_i = \nabla f_i$ where each f_i is a differentiable convex function then a solution $\bar{x}$ of the $(SVVI)^w$ is a weak Pareto minimum for the convex vector optimization problem,

$$\min f(x) = (f_1(x), \dots, f_m(x)) \quad \text{subject to} \quad x \in K.$$

Conversely any weak Pareto minimum of the above mentioned convex vector optimization problem is a also a solution of the problem $(SVVI)^w$ with $F_i = \nabla f_i$ for each $i = 1, \dots, m$. Thus the problem $(SVVI)^w$ arises in a natural way as a generalization of the optimality condition for the existence of a weak Pareto minimum for a convex vector optimization problem. Let us recall that a scalar Stampacchia variational inequality problem in finite dimensions can also be viewed as a generalization of the optimality condition for a single-objective convex optimization problem. In the case of a scalar variational inequality there is a rich theory of gap functions associated with it. See for example Fukushima [4], Wu, Florian and Marcotte [12], Larsson and Patriksson [6], Yamashita, Taji and Fukushima [11] and the references there in. In the scalar case the gap function allows us to reformulate the variational inequality into a constrained or unconstrained optimization problem. Gap functions for vector variational inequalities have received some attention in the recent past. For example in Chen, Goh and Yang [1] and Li, Yan and Chen [8] the gap functions defined for vector variational inequalities are set-valued maps. On the other hand Mastroeni [9] introduces a scalar gap function for a Minty type vector variational inequality. Our aim in this article is to develop scalar gap functions for the weak Stamphaccia type vector variational inequality $(SVVI)^w$. Scalar valued gap functions may become helpful from the computational point of view. We also use the gap functions developed here to devise error bounds which are important from the computational point of view.

We proceed as follows. First we introduce a gap function for $(SVVI)^w$ and then develop a regularized version of the gap function which is directionally differentiable. Our approach to devise regularized version of the gap function is motivated by the approach due to Fukushima [4] for the scalar case. The gap function and its regularized version introduced here provides a constrained optimization reformulation of $(SVVI)^w$. Further we also introduce the notion of a difference gap function or D-gap function for $(SVVI)^w$ which leads to an unconstrained optimization reformulation of $(SVVI)^w$.

The paper is planned as follows. In Section 2 we introduce the scalar gap function for $(SVVI)^w$ and its regularization in the sense of Fukushima [4]. We study in detail the various properties of the regularized gap function. We then present an error bound for $(SVVI)^w$ in terms of the regularized gap function when each of the component function are strongly monotone. In section 3 we introduce the notion of a D-gap function for $(SVVI)^w$. We study in detail its main properties and also present an error bound in terms of the D-gap functions at the end of that section.

2 A Gap Function and its Regularization

We will now introduce a scalar gap function for the vector variational inequality problem. Our gap function here will depend on the scalarization scheme of Lee, et al [7]. We will first state the scalarization scheme of Lee et al [7].

For every $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, consider the following scalar variational inequality:

$$(SVI)_\xi : \qquad \text{Find } \bar{x} \in K \text{ such that } \left\langle \sum_{i=1}^{m} \xi_i F_i(\bar{x}), y - \bar{x} \right\rangle \geq 0 \text{ for all } y \in K. \tag{2.1}$$

The solution set of the above problem is denoted as $sol(SVI)_\xi$. Using the above variational inequality Lee et al [7] provided the following scalarization result for $(SVVI)^w$.

Theorem 2.1. *The following properties hold.*

(i) $\bigcup_{\xi \in int\mathbb{R}^m_+} sol(SVI)_\xi \subset sol(SVVI)^w = \bigcup_{\xi \in \mathbb{R}^m_+ \setminus \{0\}} sol(SVI)_\xi$.

(ii) $sol(SVVI)^w$ *is a closed set, provided* F_i *is continuous for every* $i = 1, \ldots, m$

The gap function for $(SVVI)^w$ is given as follows

$$\theta(x) = \min_{\xi \in S^m} \max_{y \in K} \left\langle \sum_{i=1}^{m} \xi_i F_i(x), x - y \right\rangle$$

The symbol S^m in the above expression denotes the unit simplex in $\mathbb{R}^m_+$ i.e. it is given as

$$S^m = \left\{ x \in \mathbb{R}^m_+ : \sum_{i=1}^{m} x_i = 1 \right\}$$

The use of S^m in the above expression in the above definition is to stress the fact that the vector $\xi \neq 0$ and we just express the normalized version. Further use of S^m has an advantage since if additionally we take K to be compact then the function θ is finite. It is not difficult to see that $\theta(x) \geq 0$ for all $x \in K$ and $\theta(x) = 0$ if and only if x is a solution of $(SVVI)^w$. Though the proof of this fact is simple we provide it here for the sake of completeness in form of the following theorem

Theorem 2.2. *The function θ satisfies the following properties*

i) $\theta(x) \geq 0$ *for all* $x \in K$.

ii) $\theta(x^*) = 0$, $x^* \in K$, *if and only if* x^* *solves* $(SVVI)^w$.

Proof. The fact that $\theta(x) \geq 0$ for all $x \in K$ follows simply by setting $y = x$ in the right hand side of the expression for $\theta(x)$. Let us now consider a $x^* \in K$ such that $\theta(x^*) = 0$. Let us set

$$\beta(x^*, \xi) = \max_{y \in K} \left\langle \sum_{i=1}^{m} \xi_i F_i(x^*), x^* - y \right\rangle.$$

It is simple to observe that since x^* is fixed $\beta(x, \xi)$ is a lower-semicontinuous convex function of ξ. Further since $\theta(x^*) = 0$ we see that $\beta(x^*, \xi)$ is a proper convex function. This shows that there exists $\xi^* \in S^m$ such that

$$\beta(x^*, \xi^*) = \theta(x^*) = 0$$

This shows that for all $y \in K$

$$\left\langle \sum_{i=1}^{m} \xi_i^* F_i(x^*), y - x^* \right\rangle \geq 0.$$

This shows that x^* solve $(SVI)_{\xi^*}$. Thus using Theorem 2.1 we conclude that x^* is a solution of $(SVVI)^w$. Conversely let x^* solves $(SVVI)^w$. Then again using Theorem 2.1 we conclude that there exists $\xi' \in S^m$ such that x^* solves $(SVI)_{\xi'}$. This shows that $\beta(x, \xi') \leq 0$. This shows that $\theta(x^*) \leq 0$. However we know that $\theta(x^*) \geq 0$ and thus proving that $\theta(x^*) = 0$. □

Thus if we minimize the function θ over the set K and the minimum is attained at $\bar{x}$ and the minimum value is zero then $\bar{x}$ is the solution $(SVVI)^w$.

As we have mentioned above that if we assume that K is compact then we can guarantee that the gap function θ is finite-valued. A natural question is whether we can develop a gap function for $(SVVI)^w$ which is finite-valued irrespective of whether K is compact or not. This is our primary motivation to regularize the gap function θ along the lines of Fukushima [4].

Though the very evaluation of $\theta(x)$ for a given x may look formidable since one has to carry out two optimization procedures to do that. However we would like to point out that $\theta(x)$ is actually evaluated through two convex optimization problems. First of all let us set

$$\beta(x, \xi) = \max_{y \in K} \left\langle \sum_{i=1}^{m} \xi_i F_i(x), x - y \right\rangle.$$

Now observe that for each $(x, \xi) \in \mathbb{R}^n \times S^m$ the function value $\beta(x, \xi)$ is evaluated by maximizing an affine function in y which is the instance of a concave maximization problem. Also observe that for each fixed $x \in \mathbb{R}^n$ the function $\xi \mapsto \beta(x, \xi)$ is a convex function in ξ and hence for each $x \in \mathbb{R}^n$ the function $\theta(x)$ is evaluated through a convex minimization problem. However the function $\theta(x)$ is not in general a convex function in x and hence it is not possible to say whether the directional derivative of θ exists or not. However directional differentiability or some generalized differentiability of θ is useful for minimizing the function θ. Further it is not possible even to guarantee whether θ is locally Lipschitz.

This is another major motivation for us to consider the regularization of θ using the regularization approach of Fukushima [4]. In fact for the regularized gap function we will show that the directional derivative exists and we will estimate it. We are now in a position to define the regularized gap function for $(SVVI)^w$. However let us mention here that in the rest of the paper we will consider each F_i to be a continuously differentiable function and in all the results that follow this will be assumed and we will not explicitly mention it.

Definition 2.3. The regularized gap function for $(SVVI)^w$ is defined as

$$\phi_\alpha(x) = \min_{\xi \in S^m} \max_{y \in K} \left\{ \left\langle \sum_{i=1}^m \xi_i F_i(x), x - y \right\rangle - \frac{\alpha}{2}\|y - x\|^2 \right\}, \quad \alpha > 0. \tag{2.2}$$

For fixed $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m_+ \setminus \{0\}$ consider the following problem,

$$f_\alpha(x, \xi) = \max_{y \in K} \left\{ \left\langle \sum_{i=1}^m \xi_i F_i(x), x - y \right\rangle - \frac{\alpha}{2}\|y - x\|^2 \right\}, \quad \alpha > 0. \tag{2.3}$$

which is equivalently written as

$$f_\alpha(x, \xi) = -\min_{y \in K} \left\{ \left\langle \sum_{i=1}^m \xi_i F_i(x), y - x \right\rangle + \frac{\alpha}{2}\|y - x\|^2 \right\}, \quad \alpha > 0.$$

Let us consider the following convex minimization problem

$$\min_{y \in K} \left\langle \sum_{i=1}^m \xi_i F_i(x), y - x \right\rangle + \frac{\alpha}{2}\|y - x\|^2 \tag{2.4}$$

It is important to observe that the objective function in the above convex optimization problem is strongly convex and therefore coercive. Thus the above minimization problem has a solution and the solution is unique. There is no requirement of a compactness assumption on K. It is well known that the necessary and sufficient condition for vector $y \in K$ to solve (2.4) is given as follows

$$-\sum_{i=1}^m \xi_i F_i(x) - \alpha(y - x) \in N_K(y), \tag{2.5}$$

where $N_K(y)$ denotes the normal cone to the convex set K at y. This shows that

$$\left(x - \alpha^{-1} \sum_{i=1}^m \xi_i F_i(x)\right) - y \in N_K(y)$$

which implies that y is the projection of the vector $x - \alpha^{-1} \sum_{i=1}^m \xi_i F_i(x)$ onto the set K. Hence for each $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m_+ \setminus \{0\}$ and $\alpha > 0$, the optimal solution of (2.4) can be uniquely determined. Let us denote it by $H_\alpha(x, \xi)$. i.e.,

$$H_\alpha(x,\xi) = \mathrm{Proj}_K\Big(x - \alpha^{-1}\sum_{i=1}^m \xi_i F_i(x)\Big) \tag{2.6}$$

Therefore $f_\alpha(x,\xi)$ can be written as

$$f_\alpha(x,\xi) = -\Big\langle \sum_{i=1}^m \xi_i F_i(x), H_\alpha(x,\xi) - x\Big\rangle - \frac{\alpha}{2}\|H_\alpha(x,\xi) - x\|^2 \tag{2.7}$$

Lemma 2.4. *For any $x \in \mathbb{R}^n$ $\alpha > 0$ and any $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let $H_\alpha(x,\xi)$ be given by (2.6). Then the map $H_\alpha : \mathbb{R}^n \times S^m \to K$ is continuous on $\mathbb{R}^n \times S^m$. Hence $f_\alpha(x,\xi)$ is continuous on $\mathbb{R}^m_+ \times S^m$ and ϕ_α is well-defined.*

Proof. Let (x_n, ξ_n) be a sequence in $\mathbb{R}^n \times S^m$ and let $(x_n, \xi_n) \to (\bar{x}, \bar{\xi})$. It is clear that $(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times S^m$.

Let $z_n = x_n - \alpha^{-1}\sum_{i=1}^m \xi_i^n F_i(x_n)$ where $\xi_n = (\xi_1^n, \xi_2^n, ..., \xi_m^n)$. Then the sequence z_n of vectors converges to $\bar{x} - \alpha^{-1}\sum_{i=1}^m \bar{\xi}_i F_i(\bar{x})$.

From (2.6),

$$\lim_{n\to\infty} H_\alpha(x_n,\xi_n) \quad = \quad \lim_{n\to\infty} \mathrm{Proj}_K(z_n)$$

Since the projection mapping on a closed convex set is a continuous function we have

$$\lim_{n\to\infty} \mathrm{Proj}_K(z_n) = \mathrm{Proj}_K(\bar{x} - \alpha^{-1}\sum_{i=1}^m \bar{\xi}_i F_i(\bar{x})) = H_\alpha(\bar{x},\bar{\xi})$$

This shows that

$$\lim_{n\to\infty} H_\alpha(x_n,\xi_n) = H_\alpha(\bar{x},\bar{\xi}).$$

Thus H_α is continuous on $\mathbb{R}^n \times S^m$. Further from (2.7) it is clear that $f_\alpha(x,\xi)$ is continuous on $\mathbb{R}^n \times S^m$. Since S^m is a compact set it is clear that ϕ_α is finite. Hence ϕ_α is well-defined. □

Remark 2.5. It is important to note that the regularized gap function ϕ_α is finite without the assumption that K is compact while the compactness of K is an important criteria to have the finiteness of the gap function θ.

Theorem 2.6. *For each $x \in \mathbb{R}^n$, $\alpha > 0$ and $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let $H_\alpha(x,\xi)$ be defined by (2.6). Then x solves the variational inequality problem $(SVI)_\xi$ if and only if $H_\alpha(x,\xi) = x$.*

Proof. Let x solve the problem $(SVI)_\xi$. It implies

$$\Big\langle \sum_{i=1}^m \xi_i F_i(x), y - x\Big\rangle \geq 0 \text{ for all } y \in K.$$

Since $H_\alpha(x,\xi)$ solves the problem (2.4), we have

$$\Big\langle \sum_{i=1}^m \xi_i F_i(x), H_\alpha(x,\xi) - x\Big\rangle + \frac{\alpha}{2}\|H_\alpha(x,\xi) - x\|^2 \leq \Big\langle \sum_{i=1}^m \xi_i F_i(x),\ y - x\Big\rangle + \frac{\alpha}{2}\|y - x\|^2, \forall y \in K.$$

This holds in particular for $y = x$. Hence we have

$$\frac{\alpha}{2}\|H_\alpha(x,\xi) - x\|^2 \leq \Big\langle \sum_{i=1}^{m} \xi_i F_i(x),\ x - H_\alpha(x,\xi) \Big\rangle. \tag{2.8}$$

Since x is a solution of $(SVI)_\xi$ we have

$$\Big\langle \sum_{i=1}^{m} \xi_i F_i(x),\ x - H_\alpha(x,\xi) \Big\rangle \leq 0.$$

Hence from (2.8) we conclude that

$$\|H_\alpha(x,\xi) - x\|^2 \leq 0$$

Hence we conclude that $H_\alpha(x,\xi) = x$.

Conversely, assume that $H_\alpha(x,\xi) = x$. Since $H_\alpha(x,\xi)$ is a solution to the convex program (2.4) we have

$$\Big\langle \sum_{i=1}^{m} \xi_i F_i(x) + \alpha(H_\alpha(x,\xi) - x),\ y - H_\alpha(x,\xi) \Big\rangle \geq 0, \quad \forall y \in K.$$

Hence

$$\Big\langle \sum_{i=1}^{m} \xi_i F_i(x),\ y - x \Big\rangle \geq 0, \quad \forall y \in K.$$

Therefore x solves $(SVI)_\xi$. □

Theorem 2.7. *Let ϕ_α be the function defined by (2.2). Then $\phi_\alpha(x) \geq 0$ for all $x \in K$. Furthermore, $\phi_\alpha(x^*) = 0,\ x^* \in K$ if and only if x^* solves $(SVVI)^w$.*

Proof. Consider any $x \in K$. Then it follows from (2.3) that for any given $\xi \in S^m$,

$$f_\alpha(x,\xi) \geq \Big\langle \sum_{i=1}^{m} \xi_i F_i(x),\ x - y \Big\rangle - \frac{\alpha}{2}\|y - x\|^2, \quad \text{for all } \ y \in K.$$

This is true in particular when $y = x$. Hence we conclude that for any $\xi \in S^m$ we have

$$f_\alpha(x,\xi) \geq 0$$

This clearly shows that,

$$\phi_\alpha(x) = \min_{\xi \in S^m} f_\alpha(x,\xi) \geq 0.$$

Since $x \in K$ was arbitrarily chosen we have $\phi_\alpha(x) \geq 0$ for all $x \in K$.

For the second part let us begin by assuming that x^* is a solution of $(SVVI)^w$. Hence using using Theorem 2.1 we conclude that there exists $\xi' \in S^m$ such that x^* is also a solution of $(SVI)_{\xi'}$. Further it follows from Theorem 2.6 that $H_\alpha(x^*,\xi') = x^*$. Hence we can conclude from (2.7) that $f_\alpha(x^*,\xi') = 0$. This clearly shows that $\phi_\alpha(x^*) \leq 0$. However from the first part of the proof we know that $\phi_\alpha(x^*) \geq 0$ and this allows us to conclude that $\phi_\alpha(x^*) = 0$.

Conversely, assume that $\phi_\alpha(x^*) = 0$ and $x^* \in K$. Since S^m is compact there exists $\xi \in S^m$ such that $f_\alpha(x^*,\xi) = 0$. Hence using (2.7)we conclude that

$$\Big\langle \sum_{i=1}^{m} \xi_i F_i(x^*),\ x^* - H_\alpha(x^*,\xi) \Big\rangle = \frac{\alpha}{2}\|H_\alpha(x^*,\xi) - x^*\|^2 \tag{2.9}$$

Moreover since $H_\alpha(x^*, \xi)$ solves the convex minimization problem (2.4), we have

$$\Big\langle -\sum_{i=1}^{m} \xi_i F_i(x^*) - \alpha(H_\alpha(x^*, \xi) - x^*),\ z - H_\alpha(x^*, \xi)\Big\rangle \leq 0, \quad \forall z \in K.$$

Now setting $z = x^*$ in the above inequality we have

$$\Big\langle -\sum_{i=1}^{m} \xi_i F_i(x^*),\ x^* - H_\alpha(x^*, \xi)\Big\rangle - \alpha\langle H_\alpha(x^*, \xi) - x^*,\ x^* - H_\alpha(x^*, \xi)\rangle \leq 0.$$

Now using (2.9) the above inequality reduces to

$$-\frac{\alpha}{2}\|H_\alpha(x^*, \xi) - x^*\|^2 - \alpha\Big\langle H_\alpha(x^*, \xi) - x^*,\ x^* - H_\alpha(x^*, \xi)\Big\rangle \leq 0.$$

This shows that

$$(\alpha - \frac{\alpha}{2})\|H_\alpha(x^*, \xi) - x^*\|^2 \leq 0.$$

Therefore $H_\alpha(x^*, \xi) = x^*$. Hence from Theorem 2.6, it follows that x^* is a solution for $(SVI)_\xi$. Therefore we conclude from Theorem 2.1 that the vector x^* solves $(SVVI)^w$. Hence the proof. □

The above theorem shows that $(SVVI)^w$ is equivalent to the constrained minimization problem

$$\min_{x \in K} \phi_\alpha(x) \tag{2.10}$$

in the sense that if x^* is a minimum of the above problem and $\phi_\alpha(x^*) = 0$, then x^* is solution of $(SVVI)^w$ and vice versa.

Theorem 2.8. *The function ϕ_α is directionally differentiable in any direction $d \in \mathbb{R}^n$ and its directional derivative is given by*

$$\phi'_\alpha(x; d) = \min_{\xi \in \Lambda(x)} \left\{ \Big\langle \sum \xi_i F_i(x) - \sum_{i=1}^{m} \xi_i \nabla F_i(x)^T (H_\alpha(x, \xi) - x),\ d\Big\rangle + \alpha\langle H_\alpha(x, \xi) - x,\ d\rangle \right\}$$

where

$$\Lambda(x) = \{\xi \in S^m : \phi_\alpha(x) = f_\alpha(x, \xi)\}$$

If $\Lambda(x)$ is a singleton set,say $\Lambda(x) = \{\xi(x)\}$ the ϕ_α is Gateaux differentiable at x and

$$\nabla \phi_\alpha(x) = \sum_{i=1}^{m} \xi(x)_i F_i(x) - \sum_{i=1}^{m} \xi(x)_i \nabla F_i(x)^T [H_\alpha(x, \xi(x)) - x] + \alpha[H_\alpha(x, \xi(x)) - x] \tag{2.11}$$

Proof. To begin with let us observe from Lemma 2.4 that $f_\alpha(x, \xi)$ is a continuous function over $\mathbb{R}^n \times S^m$. This implies that for each fixed $x \in \mathbb{R}^n$ the functions $\xi \mapsto f_\alpha(x, \xi)$ is continuous over S^m. Using this fact it is simple to show that for each $x \in \mathbb{R}^n$ the set $\Lambda(x)$ is a non-empty closed set. Further $\Lambda(x)$ is a subset of S^m, and hence it is also a compact set.

Now for any $\xi \in S^m$, $\sum_{i=1}^{m} \xi_i F_i$ is continuously differentiable. Hence it follows from Theorem 3.2, [4] that $f_\alpha(x, \xi)$ is also continuously differentiable and the gradient of $f_\alpha(x, \xi)$ is given by

$$\nabla_x f_\alpha(x, \xi) = \sum_{i=1}^{m} \xi_i F_i(x) - \sum_{i=1}^{m} \xi_i \nabla F_i(x)^T [H_\alpha(x, \xi) - x] + \alpha[H_\alpha(x, \xi) - x] \tag{2.12}$$

From (2.2) and (2.7), we have

$$\phi_\alpha(x) = -\max_{\xi\in S^m}\{-f_\alpha(x,\xi)\}$$

Set

$$\hat{\phi}_\alpha(x) = \max_{\xi\in S^m}\{-f_\alpha(x,\xi)\}$$

Since H_α is continuous on $\mathbb{R}^n \times S^m$ by Lemma 2.4, the map $f_\alpha : \mathbb{R}^n \times S^m \to \mathbb{R}$ is continuous on $\mathbb{R}^n \times S^m$. We also conclude from the continuity of H_α and ∇F_i, $i = 1, \ldots, m$, that $\nabla_x f_\alpha(x,\xi)$ is continuous on $\mathbb{R}^n \times S^m$.

Hence using Theorem 10.2.1, [3], $\hat{\phi}_\alpha$ is directionally differentiable at x and

$$\hat{\phi}'_\alpha(x;d) = \max_{\xi\in\Lambda(x)} -\langle \nabla_x f_\alpha(x,\xi), d\rangle$$

Thus the regularized gap function ϕ_α is directionally differentiable at x and the directional derivative is given by

$$\phi'_\alpha(x;d) = -\max_{\xi\in\Lambda(x)} -\langle \nabla_x f_\alpha(x,\xi), d\rangle$$

Therefore

$$\phi'_\alpha(x;d) = \min_{\xi\in\Lambda(x)} \langle \nabla_x f_\alpha(x,\xi), d\rangle. \tag{2.13}$$

Substituting (2.12) for $\nabla_x f_\alpha(x,\xi)$, we have

$$\phi'_\alpha(x;d) = \min_{\xi\in\Lambda(x)} \left\{ \left\langle \sum \xi_i F_i(x) - \sum_{i=1}^{m} \xi_i \nabla F_i(x)^T [H_\alpha(x,\xi) - x] + \alpha[H_\alpha(x,\xi) - x], \quad d \right\rangle \right\}.$$

Hence the proof.

If $\Lambda(x) = \{\xi(x)\}$, a singleton set, then (2.13) reduces to the following

$$\phi'_\alpha(x;d) = \langle \nabla_x f_\alpha(x,\xi(x)), d\rangle.$$

Hence ϕ_α is Gateaux differentiable at x and the gradient $\nabla\phi_\alpha(x)$ is given by (2.11). □

We shall now present an error bound for $(SVVI)^w$ when each of the functions F_i are strongly monotone. In the study of scalar-valued variational inequalities the gap function and its regularization play a very important role in devising error bounds for the variational inequality problem. Error bounds are fundamental since they allow us to estimate how far a feasible element is from the solution set without even having computed a single solution of the associated variational inequality. The gap function plays a pivotal role in allowing us to devise the error bound. We will now show that this pivotal role of the gap function can be transferred in a very natural way from the scalar to the vector case. In our setting we will devise an error bound in terms of the regularized gap function $\phi_\alpha(x)$.

It is important to note that for a scalar variational inequality problem if the operator is strongly monotone then there exists a unique solution for the problem. For details see for example [2], Theorem 2.3.3. However for the vector variational inequality problem $(SVVI)^w$ such a thing need not be true in general even if all functions F_i, $i = 1, \ldots, m$ are strongly monotone. This fact can be gauged from Theorem 2.1. In fact if each F_i is strongly monotone then so is $\sum_{i=1}^m \xi_i F_i$ for each $\xi \in \mathbb{R}^m_+ \setminus \{0\}$. Hence for each $\xi \in \mathbb{R}^m_+ \setminus \{0\}$ the problem $(SVI)_\xi$ has a unique solution. However as we change the vector ξ the solution vector changes as well and all these solutions are in fact a solution of $(SVVI)^w$. Thus $sol(SVVI)^w$ need not be a singleton set in general. In what follows by the notation $d(x,C)$ we mean the distance between the point x and the set C.

Theorem 2.9. *Let us consider that each function $F_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$ are strongly monotone with the modulus of strong monotonicity $\mu_i > 0$. Let $\mu = \min_{1\leq i\leq m} \mu_i$ and let $\alpha > 0$ be chosen so that $\alpha < 2\mu$. Then for any $x \in K$ we have*

$$d(x, sol(SVVI)^w) \leq \frac{1}{\sqrt{\mu - \frac{\alpha}{2}}}\sqrt{\phi_\alpha(x)}.$$

Proof. From our notations we can write the function $\phi_\alpha(x)$ in the following way

$$\phi_\alpha(x) = \min_{\xi\in S^m} f_\alpha(x, \xi)$$

From Lemma 2.4 we know that f_α is a continuous function on $\mathbb{R}^n \times S^m$ it is clear that function $f_\alpha(x, .)$ is a continuous function on S^m. Hence there exists $\xi^* \in S^m$ (ξ^* will depend on the chosen x) such that

$$\phi_\alpha(x) = f_\alpha(x, \xi^*). \tag{2.14}$$

Now as each F_i is strongly monotone with modulus of monotonicity $\mu_i > 0$ then it is simple to observe that $\sum_{i=1}^m \xi_i^* F_i$ is also strongly monotone with $\mu > 0$. Thus $(SVI)_{\xi^*}$ has a unique solution and let us denote this as x^*. Further from Theorem 2.1 we know that x^* also solves $(SVVI)^w$. Now using the definition of $f_\alpha(x, \xi^*)$ and (2.14) we have for all $y \in K$,

$$\phi_\alpha(x) \geq \Big\langle \sum_{i=1}^m \xi_i^* F_i(x), x - y\Big\rangle - \frac{\alpha}{2}||y - x||^2.$$

Now setting $y = x^*$ we have

$$\phi_\alpha(x) \geq \Big\langle \sum_{i=1}^m \xi_i^* F_i(x), x - x^*\Big\rangle - \frac{\alpha}{2}||x - x^*||^2.$$

Now using strong monotonicity of $\sum_{i=1}^m \xi_i^* F_i$ we have

$$\phi_\alpha(x) \geq \Big\langle \sum_{i=1}^m \xi_i^* F_i(x^*), x - x^*\Big\rangle + (\mu - \frac{\alpha}{2})||x - x^*||^2. \tag{2.15}$$

Since x^* solves $(SVI)_{\xi^*}$ we have

$$\Big\langle \sum_{i=1}^m \xi_i^* F_i(x^*), x - x^*\Big\rangle \geq 0.$$

Hence from (2.15) we have noting that $2\mu > \alpha$

$$||x - x^*|| \leq \frac{1}{\sqrt{\mu - \frac{\alpha}{2}}}\sqrt{\phi_\alpha(x)}.$$

Hence we have

$$d(x, sol(SVVI)^w) \leq \frac{1}{\sqrt{\mu - \frac{\alpha}{2}}}\sqrt{\phi_\alpha(x)}.$$

Hence the result. □

3 D-gap Functions

We define the D-gap function for $(SVVI)^w$ which provides an unconstrained minimization formulation of the $(SVVI)^w$.

Definition 3.1. Consider the problem $(SVVI)^w$. For any $x \in \mathbb{R}^n$, $\alpha > 0$ and any $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, let $f_\alpha(x,\xi)$ be defined by (2.3). Then The D-gap function for $(SVVI)^w$ is defined by,

$$\phi_{\alpha\beta}(x) = \min_{\xi \in S^m} \{f_\alpha(x,\xi)) - f_\beta(x,\xi)\}, \ 0 < \alpha < \beta. \tag{3.1}$$

If we set $g_{\alpha\beta}(x,\xi) = f_\alpha(x,\xi) - f_\beta(x,\xi)$, then

$$\phi_{\alpha\beta}(x) = \min_{\xi \in S^m} g_{\alpha\beta}(x,\xi), \ 0 < \alpha < \beta. \tag{3.2}$$

Theorem 3.2. *$\phi_{\alpha\beta}$ is non-negative on $\mathbb{R}^n$.i.e., $\phi_{\alpha\beta}(x) \geq 0$ for all $x \in \mathbb{R}^n$. Further, x^* solves $(SVVI)^w$ if and only if $\phi_{\alpha\beta}(x^*) = 0$*

Proof. Let us take any $x \in \mathbb{R}^n$ and let us consider the function

$$g_{\alpha\beta}(x,\xi) = f_\alpha(x,\xi) - f_\beta(x,\xi), \quad 0 < \alpha < \beta.$$

It follows from Proposition 3.1, [11] that for any fixed but arbitrary $\xi \in S^m$,

$$\frac{1}{2}(\beta - \alpha)\|x - H_\beta(x,\xi)\|^2 \leq g_{\alpha\beta}(x,\xi) \leq \frac{1}{2}(\beta - \alpha)\|x - H_\alpha(x,\xi)\|^2 \tag{3.3}$$

This shows that for every $\xi \in S^m$

$$g_{\alpha\beta}(x,\xi) \geq 0$$

Hence we conclude that $\phi_{\alpha\beta}(x) \geq 0$. Since $x \in \mathbb{R}^n$ was chosen arbitrarily we have $\phi_{\alpha\beta}(x) \geq 0$ for all $x \in \mathbb{R}^n$

To prove the second part we begin by assuming that x^* is a solution of $(SVVI)^w$. Then we conclude from Theorem 2.1, there exists a $\xi' \in S^m$ such that x^* solves $(SVI)_{\xi'}$. Further it follows from Theorem 2.6 that $H_\alpha(x^*,\xi') = x^*$, which combined with (3.3) shows that $g_{\alpha\beta}(x^*,\xi') \leq 0$. Hence

$$\phi_{\alpha\beta}(x^*) = \min_{\xi \in S^m} g_{\alpha\beta}(x^*,\xi') \leq 0$$

Moreover from the first part of the theorem we know that $\phi_{\alpha\beta}(x^*) \geq 0$ this leads to the fact that $\phi_{\alpha\beta}(x^*) = 0$.

Conversely, assume that $\phi_{\alpha\beta}(x^*) = 0$. Since S^m is compact there exists $\xi \in S^m$ such that

$$g_{\alpha\beta}(x^*,\xi) = 0.$$

It follows from (3.3) that $H_\beta(x^*,\xi) = x^*$. Hence we conclude from Theorem 2.6 that x^* solves $(SVI)_\xi$. Therefore from Theorem 2.1, x^* is also a solution for $(SVVI)^w$. Hence the result. $\square$

We will now show that the D-gap function is also directionally differentiable and we will provide an estimate for its directional derivative.

Theorem 3.3. *The D-gap function $\phi_{\alpha\beta}$ is directionally differentiable and its directional derivative at a point $x \in \mathbb{R}^n$ and in the direction $d \in \mathbb{R}^n$ is given by*

$$\phi'_{\alpha\beta}(x;d) = \min_{\xi\in\Lambda(x)} \left\{ \Big\langle \sum_{i=1}^{m} \xi_i \nabla F_i(x)^T [H_\beta(x,\xi) - H_\alpha(x,\xi)], d\rangle + \alpha\langle H_\alpha(x,\xi) - x, d\rangle - \beta\langle H_\beta(x,\xi) - x, d\Big\rangle \right\}$$

where

$$\Lambda(x) = \{\xi \in S^m : \phi_{\alpha\beta}(x) = g_{\alpha\beta}(x,\xi)\}$$

If $\Lambda(x)$ is singleton, say $\Lambda(x) = \{\xi(x)\}$ then $\phi_{\alpha\beta}$ is Gateaux differentiable at x and

$$\begin{aligned}\nabla\phi_{\alpha\beta}(x) \;\; &= \;\; \sum_{i=1}^{m} \xi(x)_i \nabla F_i(x)^T [H_\beta(x,\xi(x)) - H_\alpha(x,\xi(x))] \\ &\quad + \alpha[H_\alpha(x,\xi(x)) - x] - \beta[H_\beta(x,\xi(x)) - x] \end{aligned} \tag{3.4}$$

Proof. Since $\sum_{i=1}^{m} \xi_i F_i$ is continuously differentiable for every $\xi \in \mathbb{R}^m_+ \setminus \{0\}$, it follows from Theorem 3.2, [4] that f_α and f_β are continuously differentiable in x. Hence $g_{\alpha\beta}$ is continuously differentiable and and the gradient of the function $g_{\alpha\beta}$ is given by

$$\nabla_x g_{\alpha\beta}(x,\xi) = \sum_{i=1}^{m} \xi_i \nabla F_i(x)^T [H_\beta(x,\xi) - H_\alpha(x,\xi)] + \alpha[H_\alpha(x,\xi) - x] - \beta[H_\beta(x,\xi) - x] \tag{3.5}$$

From Lemma 2.4, the maps $f_\alpha, f_\beta : \mathbb{R}^n \times S^m \to \mathbb{R}$ are continuous on $\mathbb{R}^n \times S^m$. Hence the map $g_{\alpha\beta} : \mathbb{R}^n \times S^m \to \mathbb{R}$ is continuous on $\mathbb{R}^n \times S^m$. Since S^m is compact it shows that $\Lambda(x)$ is a non-empty compact set. Further since $\nabla_x g_{\alpha\beta} = \nabla_x f_\alpha - \nabla_x f_\beta$, the continuity of the map $\nabla_x g_{\alpha\beta} : \mathbb{R}^n \times S^m \to \mathbb{R}$ on $\mathbb{R}^n \times S^m$ follows form the continuity of the maps $\nabla_x f_\alpha$ and $\nabla_x f_\beta$. Now using Theorem 10.2.1, [3], we conclude that $\phi_{\alpha\beta}$ is differentiable at x and

$$\phi'_{\alpha\beta}(x;d) = \min_{\xi\in\Lambda(x)} \langle \nabla_x g_{\alpha\beta}(x,\xi), \quad d\rangle. \tag{3.6}$$

Now substituting (3.5) for $\nabla_x g_{\alpha\beta}(x,\xi)$, we have

$$\phi'_{\alpha\beta}(x;d) = \min_{\xi\in\Lambda(x)} \left\{ \langle \Sigma_{i=1}^{m} \xi_i \nabla F_i(x)^T [H_\beta(x,\xi) - H_\alpha(x,\xi)] + \alpha[H_\alpha(x,\xi) - x] - \beta[H_\beta(x,\xi) - x], \; d\rangle \right\}.$$

This proves the first part of the theorem.

If $\Lambda(x) = \{\xi(x)\}$, a singleton set, then (3.6) reduces to the following

$$\phi'_{\alpha\beta}(x;d) = \langle \nabla_x g_{\alpha\beta}(x,\xi), d\rangle.$$

Hence $\phi_{\alpha\beta}$ is Gateaux differentiable at x and the gradient $\nabla\phi_{\alpha\beta}(x)$ is given by (3.4). □

We will end this section by providing an error bound for $(SVVI)^w$ in terms of the D-gap function. Before stating the result on error bound we would like to present the definition a vector-valued Lipschitz function. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be Lipschitz over $K \subseteq \mathbb{R}^n$ with Lipschitz rank $L > 0$, if for any $x, y \in K$,

$$||F(y) - F(x)|| \leq L||y - x||.$$

Theorem 3.4. *Let us consider that each of the functions F_i, $i = 1, \dots, m$ are Lipschitz on K with Lipschitz rank $L_i > 0$, $i = 1, \dots, m$ and also assume that each F_i is strongly monotone on K with $\mu_i > 0$, $i = 1, \dots, m$ as the modulus of strong monotonicity. Let $L^* = \max\{L_1, \dots, L_m\}$ and $\mu^* = \min\{\mu_1, \dots, \mu_m\}$. Then for any $x \in K$*

$$d(x, sol(SVVI)^w) \leq \frac{L^* + \beta}{\mu^*} \frac{\sqrt{2}}{\sqrt{(\beta - \alpha)}} \sqrt{\phi_{\alpha\beta}(x)}.$$

Proof. Let us note first of all that the function $g_{\alpha\beta}(x, .)$ is a continuous function over S^m. This fact essentially follows from Lemma 2.4 and noting that $g_{\alpha\beta}(x, \xi) = f_\alpha(x, \xi) - f_\beta(x, \xi)$ with $0 < \alpha < \beta$. Further as S^m is compact there exists $\xi^* \in S^m$ such that

$$\phi_{\alpha\beta}(x) = g_{\alpha\beta}(x, \xi^*). \tag{3.7}$$

Note that ξ^* depends on the choice of x. It is natural that as we change x the vector $\xi^* \in S^m$ will also change. However once we have fixed x the corresponding ξ^* is also fixed. Now since each F_i is Lipschitz on K with Lipschitz rank $L_i > 0$, and $\xi^* \in S^m$, the function $\sum_{i=1}^m \xi_i^* F_i$ is also Lipschitz on K with Lipschitz rank L^* Further since each F_i is strongly monotone with $\mu_i > 0$ as its modulus of strong monotonicity it is simple to observe that $\sum_{i=1}^m \xi_i^* F_i$ is strongly monotone with μ^* as the modulus of strong monotonicity. Hence $(SVI)_{\xi^*}$ has a unique solution which we shall denote as x^*. Given $x \in K$ we know that $H_\beta(x, \xi^*)$ is the unique solution of the strongly convex minimization problem

$$\min_{y \in K} \left\{ \left\langle \sum_{i=1}^m \xi^* F_i(x), y - x \right\rangle + \frac{\beta}{2} ||y - x||^2 \right\} \tag{3.8}$$

We will first show that there exist a constant $c > 0$ such that

$$||x - x^*|| \leq c||H_\beta(x, \xi^*) - x||$$

We will show that the constant c independent of x and ξ^* and will be given by the data of the problem. Now since $H_\beta(x, \xi^*)$ is a unique solution of (3.8) we have for all $y \in K$

$$\left\langle \sum_{i=1}^m \xi^* F_i(x) + \beta(H_\beta(x, \xi^*) - x), y - H_\beta(x, \xi^*) \right\rangle \geq 0.$$

The above inequality is true in particular when $y = x^*$ and hence from the above inequality we have

$$\left\langle \sum_{i=1}^m \xi^* F_i(x) + \beta(H_\beta(x, \xi^*) - x), H_\beta(x, \xi^*) - x^* \right\rangle \leq 0. \tag{3.9}$$

Further since x^* is a solution of $(SVI)_{\xi^*}$ we have

$$\left\langle \sum_{i=1}^m \xi_i^* F_i(x^*), x^* - H_\beta(x, \xi^*) \right\rangle \leq 0. \tag{3.10}$$

Adding (3.9) with (3.10) we get

$$\left\langle \sum_{i=1}^m \xi_i^* F(x) - \sum_{i=1}^m \xi_i^* F_i(x^*) + \beta(H_\beta(x, \xi^*) - x), H_\beta(x, \xi^*) - x^* \right\rangle \leq 0.$$

This can be rearranged as

$$\begin{aligned}\Big\langle \sum_{i=1}^m \xi_i^* F(x) - \sum_{i=1}^m \xi_i^* F_i(x^*), x - x^* \Big\rangle \quad \leq \quad & -\beta||H_\beta(x,\xi^*) - x||^2 \\ & -\Big\langle \sum_{i=1}^m \xi_i^* F(x) - \sum_{i=1}^m \xi_i^* F_i(x^*), H_\beta(x,\xi^*) - x \Big\rangle \\ & +\beta\langle H_\beta(x,\xi^*) - x, x^* - x\rangle.\end{aligned}$$

Noting that $\beta > 0$ and also applying Cauchy-Schwarz inequality we have

$$\begin{aligned}\Big\langle \sum_{i=1}^m \xi_i^* F(x) - \sum_{i=1}^m \xi_i^* F_i(x^*), x - x^* \Big\rangle \quad \leq \quad & \Big|\Big| \sum_{i=1}^m \xi_i^* F(x) - \sum_{i=1}^m \xi_i^* F_i(x^*) \Big|\Big| ||H_\beta(x,\xi^*) - x|| \\ & +\beta||H_\beta(x,\xi^*) - x|| ||x - x^*||.\end{aligned}$$

Noting that $\sum_{i=1}^m \xi_i^* F_i$ is Lipschitz on K with Lipschitz rank L^* and also strongly monotone with μ^* as the modulus of strong monotonicity we have from the above inequality

$$\begin{aligned}\mu^*||x - x^*||^2 \leq L^*||x - x^*|| ||H_\beta(x,\xi^*) - x|| \\ +\beta||x - x^*|| ||H_\beta(x,\xi^*) - x||.\end{aligned}$$

This shows that

$$||x - x^*|| \leq \frac{L^* + \beta}{\mu^*}||H_\beta(x,\xi^*) - x||. \tag{3.11}$$

Thus the required $c > 0$ is given as

$$c = \frac{L^* + \beta}{\mu^*}$$

which as we see is independent of x and ξ^*. Now using (3.3) we have

$$||H_\beta(x,\xi^*) - x|| \leq \frac{\sqrt{2}}{\sqrt{(\beta - \alpha)}}\sqrt{g_{\alpha\beta}(x,\xi^*)}. \tag{3.12}$$

Now combining (3.12) and (3.11) and also noting that $\phi_{\alpha\beta}(x) = g_{\alpha\beta}(x,\xi^*)$ we arrive at the desired conclusion. □

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