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LEVITIN-POLYAK WELL-POSEDNESS IN GENERALIZED EQUILIBRIUM PROBLEMS WITH FUNCTIONAL CONSTRAINTS

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Abstract: In this paper, we introduce four types of (generalized) Levitin-Polyak well-posedness for generalized equilibrium problems with abstract and functional constraints. Criteria and characterizations for these types of well-posedness are given. Under suitable conditions, we show that any type of well-posedness of generalized equilibrium problems is equivalent to the nonemptiness and compactness of its solution set.

Key words: *generalized equilibrium problems, well-posedness, set-valued map, approximating solution sequence*

Mathematics Subject Classification: *90C48; 46N16, 49K40, 90C31*

1 Introduction and Preliminaries

Well-posedness of unconstrained and constrained scalar optimization problem was first introduced and studied in Tykhonov [14] and Levitin and Polyak [11], respectively. Presently, the concept of well-posedness has also been generalized to variational inequality problems [7], generalized variational inequality problems [8], equilibrium problems [4, 12, 13]. However, to the best of our knowledge, the study of Levitin-Polyak (LP in short) well-posedness of generalized equilibrium problems with functional constraints is still very limited.

In this paper, we investigate LP well-posedness of generalized equilibrium problems with functional constraints. We establish some characterizations of LP well-posedness. Under suitable conditions, we also show that any type of LP well-posedness of generalized equilibrium problems is equivalent to the nonemptiness and compactness of its solution set.

Let X, U be a normed space, (Y, d) be a metric space. Let $X_1 \subseteq X, K \subseteq Y$ be two nonempty and closed sets. Let $T : X_1 \to 2^U$ be a nonempty-compact-valued map (i.e., for each $x \in X_1$, $T(x)$ is a nonempty compact subset of U). Let $h(u, x, y) : U \times X_1 \times X_1 \to R^1 \bigcup \{+\infty\}$ and $g : X_1 \to Y$ be two functions.

Let

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Consider the following generalized equilibrium problems with functional constraints:

$$\text{(GEP)} \qquad \begin{array}{c}\text{Find } \bar{x} \in X_0 \text{ such that there exists } \bar{u} \in T(\bar{x}) \text{ satisfying} \\ h(\bar{u}, \bar{x}, y) \geq 0, \quad \forall y \in X_0.\end{array}$$

Definition 1.1. (i) A trifunction $h(u,x,y) : U \times X_1 \times X_1 \to R^1 \bigcup \{+\infty\}$ is said to be monotone if for any $x, y \in X_1$ and $u \in T(x), v \in T(y)$, the following relation holds:

$$h(u,x,y) + h(v,y,x) \leq 0.$$

(ii) Let Z_1, Z_2 be two topological spaces. A set-valued map F from Z_1 to 2^{Z_2} is said to be upper semicontinuous at $x \in Z_1$ if for any neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(x') \subseteq V$, $\forall x' \in U$. If F is upper semicontinuous at every point of Z_1, we say that F is u.s.c on Z_1.
(iii) $h(.,.,y)$ is is said to be T-upper semicontinuous if for $x_n \to x$ and $u_n \in T(x_n)$, there exists $u \in T(x)$ such that

$$h(u,x,y) \geq \limsup_{n \to +\infty} h(u_n, x_n, y), \forall y \in X_0.$$

Let (P, d_1) be a metric space, $P_1 \subseteq P$ and $p \in P$. In the sequel, we denote by $d_{P_1}(p) = \inf\{d(p,p') : \ p' \in P_1\}$ the distance function from point p to set P_1.

Throughout this paper, we always assume that $X_0 \neq \emptyset$ and $h(.,.,y)$ is T-upper semi continuous on $U \times X_1$ and $h(u,x,.)$ is lower semicontinuous on X_1, $g(x)$ is continuous on X_1. The nonempty-compact-valued map T is upper semicontinuous on X_1 and $h(u,x,x) = 0, \forall x \in X_1, u \in T(x)$.

Denote by $\bar{X}$ the solution set of (GEP).

Definition 1.2. (i) A sequence $\{x_n\} \subseteq X_1$ is called a type I LP approximating solution sequence for (GEP) if there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ and $u_n \in T(x_n)$ such that

$$d_{X_0}(x_n) \leq \epsilon_n, \tag{1.1}$$

$$h(u_n, x_n, y) \geq -\epsilon_n, \quad \forall y \in X_0. \tag{1.2}$$

(ii) A sequence $\{x_n\} \subseteq X_1$ is called a type II LP approximating solution sequence for (GEP) if there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ and $u_n \in T(x_n)$, $\{y_n\} \subset X_0$ satisfying (1.1), (1.2)and

$$h(u_n, x_n, y_n) \leq \epsilon_n. \tag{1.3}$$

(iii) A sequence $\{x_n\} \subseteq X_1$ is called a generalized type I LP approximating solution sequence for (GEP) if there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ and $u_n \in T(x_n)$ satisfying (1.2) and

$$d_K(g(x_n)) \leq \epsilon_n. \tag{1.4}$$

(iv) A sequence $\{x_n\} \subseteq X_1$ is called a generalized type II LP approximating solution sequence for (GEP) if there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ and $u_n \in T(x_n)$, $\{y_n\} \subset X_0$ satisfying (1.2), (1.3) and (1.4).

Definition 1.3. (GEP) is said to be type I (resp. type II, generalized type I, generalized type II) LP well-posed if the solution set $\bar{X}$ of (GEP) is nonempty, and for any type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence $\{x_n\}$, there exist a subsequence$\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \bar{X}$ such that $x_{n_j} \to \bar{x}$.

Remark 1.4. (i) It is clear that any (generalized) type II LP approximating solution sequence is a (generalized) type I LP approximating solution sequence. Thus, (generalized) type I LP well-posedness implies (generalized) type II LP well-posedness.

(ii) Each type of LP well-posedness of (GEP) implies that the solution set $\bar{X}$ is compact.

(iii) Suppose that g is uniformly continuous on a set

$$X_1(\delta_0) = \{x \in X_1 : d_{X_0}(x) \leq \delta_0\} \tag{1.5}$$

for some $\delta_0 \geq 0$. Then, generalized type I (type II) LP well-posedness of (GEP) implies its type I (type II) LP well-posedness.

Consider the following statement:

[$\bar{X} \neq \emptyset$ and for any type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence $\{x_n\}$, we have $d_{\bar{X}}(x_n) \to 0$.] (STA)

The next proposition can be trivially established.

Proposition 1.5. *If (GEP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed, then (STA) is true. Conversely, if (STA) holds and $\bar{X}$ is compact, then (GEP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed.*

To see that the various LP well-posednesses of (GEP) are adaptations of the corresponding LP well-posednesses in minimization problems by using a gap function, we consider the following general constrained optimization problem:

$$\text{(P)} \qquad \begin{array}{l} \min f(x) \\ \text{s.t. } x \in X_1, g(x) \in K, \end{array}$$

where $f : X_1 \to R^1 \bigcup\{+\infty\}$ is proper and lower semicontinuous. The feasible set of (P) is X_0, where $X_0 = \{x \in X_1 : g(x) \in K\}$. The optimal set and optimal value of (P) are denoted by $\bar{X}'$ and $\bar{v}$, respectively. Note that if $Dom(f) \bigcap X_0 \neq \emptyset$, where

$$Dom(f) = \{x \in X_1 : f(x) < +\infty\},$$

then $\bar{v} < +\infty$. In this paper, we always assume that $\bar{v} > -\infty$.

LP well-posedness for the special case when f is finite-valued has been studied in [6]. We note here that all the results in [6] are valid when $f : X_1 \to R^1 \bigcup\{+\infty\}$ is proper.

Definition 1.6. (i) A sequence $\{x_n\} \subseteq X_1$ is called a type I LP minimizing sequence for (P) if

$$\limsup_{n\to+\infty} f(x_n) \leq \bar{v}, \tag{1.6}$$

$$d_{X_0}(x_n) \to 0. \tag{1.7}$$

(ii) A sequence $\{x_n\} \subseteq X_1$ is called a type II LP minimizing sequence for (P) if

$$\lim_{n\to+\infty} f(x_n) = \bar{v} \tag{1.8}$$

and (1.7) holds.

(iii) A sequence $\{x_n\} \subseteq X_1$ is called a generalized type I LP minimizing sequence for (P) if

$$d_K(g(x_n)) \to 0. \tag{1.9}$$

and (1.6) holds.

(iv) A sequence $\{x_n\} \subseteq X_1$ is called a generalized type II LP minimizing sequence for (P) if (1.8) and (1.9) hold.

Definition 1.7. (P) is said to be type I (resp. type II, generalized type I, generalized type II) LP well-posed if the solution set $\bar{X}'$ of (P) is nonempty, and for any type I (resp. type II, generalized type I, generalized type II) LP minimizing sequence $\{x_n\}$, there exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in \bar{X}'$ such that $x_{n_j} \to \bar{x}$.

Similar to the Auslender gap function, we can define a gap function for (GEP)

$$f(x) = \inf_{u \in T(x)} \sup_{y \in X_0} -h(u, x, y), \forall x \in X_1. \tag{1.10}$$

First, we have the following lemma concerning some properties of f defined by (1.10).

Lemma 1.8. *The function f defined by (1.10) has the following properties.*
(i) $f(x) \geq 0$ *if* $x \in X_0$.
(ii) *Let* $\bar{x} \in X_0$. *Then* $f(\bar{x}) = 0$ *if and only if* $\bar{x} \in \bar{X}$.
(iii) $f : X_1 \to R^1 \bigcup +\infty$. *Further, if* $\bar{X} \neq \emptyset$, *then* $dom(f) \bigcap X_0 \neq \emptyset$.
(iv) $f(x)$ *is lower semicontinuous on* X_1.

Proof. (i) As $h(u, x, x) = 0, \forall x \in X_1, u \in T(x)$, we have

$$\sup_{y \in X_0} -h(u, x, y) \geq -h(u, x, x) = 0, \forall x \in X_0, u \in T(x).$$

Thus,

$$\inf_{u \in T(x)} \sup_{y \in X_0} -h(u, x, y) \geq 0.$$

That is, $f(x) \geq 0$ if $x \in X_0$.

(ii) If $\bar{x} \in \bar{X} \subseteq X_0$, there exists $\bar{u} \in T(\bar{x})$ such that

$$h(\bar{u}, \bar{x}, y) \geq 0, \forall y \in X_0,$$

$$\sup_{y \in X_0} -h(\bar{u}, \bar{x}, y) \leq 0.$$

Further,

$$f(\bar{x}) = \inf_{u \in T(x)} \sup_{y \in X_0} -h(u, \bar{x}, y) \leq 0.$$

This combined with $f(x) \geq 0, \forall x \in X_0$ yields $f(\bar{x}) = 0$.

Conversely, let $f(\bar{x}) = 0$. Since $\sup_{y \in X_0} -h(u, x, y)$ is lower semicontinuous and $T(\bar{x})$ is compact, there exists $\bar{u} \in T(\bar{x})$ satisfying

$$\sup_{y \in X_0} -h(\bar{u}, \bar{x}, y) = 0.$$

Thus, $h(\bar{u}, \bar{x}, y) \geq 0, \forall y \in X_0$. That is, $\bar{x} \in \bar{X}$.

(iii) It is obvious that $f(x) > -\infty$, $\forall x \in X_1$. Moreover, if $\bar{X} \neq \phi$, then there exists $\bar{x} \in \bar{X}$ and $f(\bar{x}) = 0$. So, $\bar{x} \in dom(f) \bigcap X_0 \neq \phi$.

(v) Let $t \in R^1$. Suppose that sequence $x_n \subset X_1$ satisfies

$$f(x_n) \leq t, \tag{1.11}$$

and $x_n \to x^* \in X_1$. We show that $f(x^*) \leq t$. From (11), we have that for any $\epsilon > 0$ and each n, there exists $u_n \in T(x_n)$ such that

$$-h(u_n, x_n, y) \leq t + \epsilon, \forall y \in X_0. \tag{1.12}$$

By the upper semicontinuity of T at x^* and compactness of $T(x^*)$, we obtain a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and some $u^* \in T(x^*)$ such that $u_{n_j} \to u^*$. Since $h(.,.,y)$ is T-upper semicontinuous, taking the limit in (12) (with n replaced by n_j), we have

$$-h(u^*, x^*, y) \leq \liminf_{j\to+\infty} -h(u_{n_j}, x_{n_j}, y) \leq t + \epsilon, \forall y \in X_0. \tag{1.13}$$

By the definition of $f(x^*)$, we have $f(x^*) \leq t + \epsilon$. By the arbitrariness of $\epsilon > 0$, we obtain $f(x^*) \leq t$. Hence, f is lower semicontinuous on X_1. □

The next proposition establishes relationships between the various LP well-posednesses of (GEP) and those of (P) with $f(x)$ defined by (1.10).

Proposition 1.9. *Assume that $\bar{X} \neq \emptyset$. Then, (GEP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed if and only if (P) is type I (resp. type II, generalized type I, generalized type II) LP well-posed with $f(x)$ defined by (1.10).*

Proof. Since $\bar{X} \neq \emptyset$, by Lemma 1.8, $x \in \bar{X}$ is a solution of (GEP) if and only if $\bar{x}$ is an optimal solution of (P) with $\bar{v} = f(\bar{x}) = 0$. It is also routine to check that a sequence $\{x_n\}$ is a type I (resp. type II, generalized type I, generalized type II) LP approximating solution sequence if and only if it is a type I (resp. type II, generalized type I, generalized type II) LP minimizing sequence of (P). It follows that (GEP) is type I (resp. type II, generalized type I, generalized type II) LP well-posed if and only if (P) is type I (resp. type II, generalized type I, generalized type II) LP well-posed with $f(x)$ defined by (1.10). □

2 Criteria and Characterizations for LP Well-Posedness of (GEP)

In this section, we present necessary and /or sufficient conditions for the various types of (generalized) LP well-posednesses defined in Section 1.

Consider a real-valued function $c = c(t, s)$ defined for $t, s \geq 0$ sufficiently small, such that

$$c(t, s) \geq 0, \forall t, s \geq 0 \quad c(0, 0) = 0, \tag{2.1}$$

$$s_n \to 0, \, t_n \geq 0, c(t_n, s_n) \to 0 \text{ imply that } t_n \to 0. \tag{2.2}$$

Analogously to ([8], Theorems 2.1 and 2.2), we can prove following two theorems.

Theorem 2.1. *Let $f(x)$ be defined by (1.10). If (GEP) is type II LP well-posed, then there exists a function c satisfying* (14) *and* (15) *such that*

$$|f(x)| \geq c(d_{\bar{X}}(x), d_{X_0}(x)) \quad \forall x \in X_1. \tag{2.3}$$

Conversely, suppose that $\bar{X}$ is nonempty and compact, and (16) *holds for some c satisfying* (14) *and* (15)*. Then, (GEP) is type II LP well-posed.*

Theorem 2.2. *Let $f(x)$ be defined by (1.10). If (GEP) is a generalized type II LP well-posed, then there exists a function c satisfying* (14) *and* (15) *such that*

$$|f(x)| \geq c(d_{\bar{X}}(x), d_K(g(x))) \quad \forall x \in X_1. \tag{2.4}$$

Conversely, suppose that $\bar{X}$ is nonempty and compact, and (17) *holds for some c satisfying* (14) *and* (15)*. Then, (GEP) is generalized type II LP well-posed.*

Let $(X, ||\cdot||)$ be a Banach space. Recall that the Kuratowski measure of noncompactness for a subset H of X is defined as

$$\mu(H) = \inf\{\epsilon > 0 : H \subseteq \bigcup_{i=1}^{n} H_i, diam(H_i) < \epsilon, i = 1, \ldots, n\},$$

where $\mathrm{diam}(H_i)$ is the diameter of H_i defined by

$$\mathrm{diam}(H_i) = \sup\{||x_1 - x_2|| : x_1, x_2 \in H_i\}.$$

The Hausdorff distance between two nonempty bounded subsets A and B of a Banach space $(X, ||\cdot||)$ is

$$\Gamma(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{u \in A} d(u, B)$.

For any $\epsilon \geq 0$, define

$$\Omega^1(\epsilon) = \{x \in X_1 : d_{X_0}(x) \leq \epsilon, \exists u \in T(x), h(u, x, y) + \epsilon \geq 0, \forall y \in X_0\}, \tag{2.5}$$

$$\Omega^2(\epsilon) = \{x \in X_1 : d_K(g(x)) \leq \epsilon, \exists u \in T(x), h(u, x, y) + \epsilon \geq 0, \forall y \in X_0\}. \tag{2.6}$$

Theorem 2.3. *Let $(X, ||\cdot||)$ be a Banach space and $\bar{X} \neq \emptyset$. Let $\Omega_1(\epsilon)$ and $\Omega_2(\epsilon)$ be defined by* (18) *and* (19)*, respectively. Then (GEP) is (generalized) type I LP well-posed if and only if*

$$(\mu(\Omega^2(\epsilon))) \to 0) \quad \mu(\Omega^1(\epsilon)) \to 0; \textit{as } \epsilon \to 0. \tag{2.7}$$

Proof. We first show that $\Omega^1(\epsilon)$ is closed for any $\varepsilon > 0$. Let $x_n \in \Omega^1(\epsilon)$ with $x_n \to \bar{x}$. Then there exists $u_n \in T(x_n)(\forall n \geq 1)$ such that

$$d_{X_0}(x_n) \leq \epsilon, h(u_n, x_n, y) + \epsilon \geq 0, \forall y \in X_0.$$

By the upper semicontinuity of T at $\bar{x}$ and compactness of $T(\bar{x})$, there exist a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and some $\bar{u} \in T(\bar{x})$ such that $u_{n_j} \to \bar{u}$. Since $h(.,.,y)$ is T-upper semicontinuous on $Z \times X_1$, taking the upper limit (with n replaced by n_j), we have

$$h(\bar{u}, \bar{x}, y) + \epsilon \geq \limsup_{j \to +\infty} h(u_{n_j}, x_{n_j}, y) + \epsilon \geq 0, d_{X_0}(\bar{x}) \leq \epsilon, \forall y \in X_0.$$

So $\Omega^1(\epsilon)$ is closed.

Second, we show that

$$\bar{X} = \bigcap_{\epsilon > 0} \Omega^1(\epsilon).$$

It is obvious that $\bar{X} \subset \bigcap_{\epsilon>0} \Omega^1(\epsilon)$. Now suppose that $\epsilon_k \to 0$ and $\bar{x} \in \bigcap_{k=1}^{\infty} \Omega^1(\epsilon_k)$. Then,

$$d_{X_0}(\bar{x}) \leq \epsilon_k, \forall k \tag{2.8}$$

$$\exists \bar{u} \in T(\bar{x}), h(\bar{u}, \bar{x}, y) + \epsilon_k \geq 0, \forall k. \tag{2.9}$$

From (21), we have $\bar{x} \in X_0$. By (22) and the fact that $\bar{x} \in X_0$, we have $\bar{x} \in \bar{X}$.

Now we assume that (20) holds. By [9] (pp. 318), we have

$$\Gamma(\Omega^1(\epsilon), \bar{X}) \to 0, \ as\ \epsilon \to 0,$$

where $\bar{X}$ is is nonempty and compact.

Let $\{x_n\}$ be a type I LP approximating sequence for (GEP). Then, there exist a subsequence $\{x_k\}$ of $\{x_n\}$ and $\epsilon_k > 0$ with $\epsilon_k \to 0$, $u_k \in T(x_k)$ such that

$$d_{X_0}(x_k) \le \epsilon_k, h(u_k, x_k, y) + \epsilon_k \ge 0, \forall y \in X_0.$$

That is, $x_k \in \Omega^1(\epsilon_k)$. It follows from $\Gamma(\Omega^1(\epsilon), \bar{X}) \to 0$ that $d_{\bar{X}}(x_k) \to 0$. By Proposition 1.5, (GEP) is type I LP well-posed.

Conversely, let (GEP) be type I LP well-posed. Taking into account the compactness of $\bar{X}$, we get

$$\Gamma(\Omega^1(\epsilon)) \le h(\Omega^1(\epsilon), \bar{X}) + \mu(\bar{X}) = 2e(\Omega^1(\epsilon), \bar{X}).$$

We show that $e(\Omega^1(\epsilon), \bar{X}) \to 0$ as $\epsilon \to 0$. If not, there exist $\eta > 0, \epsilon_n \to 0, x_n \in \Omega^1(\epsilon_n)$ such that

$$d_{\bar{X}}(x_n) \ge \eta, \forall n,$$

contradicting the type I LP well-posedness of (GEP). So $e(\Omega^1(\epsilon), \bar{X}) \to 0$ as $\epsilon \to 0$. Hence, (20) holds.

For $\Omega^2(\epsilon)$, the conclusion can be analogously established. □

Definition 2.4. (i) Let Z be a topological space and $Z_1 \subseteq Z$ be nonempty. Suppose that $q : Z \to R^1 \bigcup\{+\infty\}$ is an extended real-valued function. q is said to be level-compact on Z_1 if, for any $s \in R^1$, the subset $\{z \in Z_1 : q(z) \le s\}$ is compact.

(ii) Let Z be a finite dimensional normed space and $Z_1 \subseteq Z$ be nonempty. A function $q : Z \to R^1 \bigcup\{+\infty\}$ is said to be level-bounded on Z_1 if Z_1 is bounded or

$$\lim_{z \in Z_1, ||z|| \to +\infty} q(z) = +\infty.$$

(iii) Let $\phi \ne S \subset X_1$. A vector -valued function $u(x)$ from S to U is called a selection of the set-valued map T on S if $u(x) \in T(x), \forall x \in S$.

The following proposition presents some sufficient conditions for type I LP well-posedness of (GEP).

Proposition 2.5. *Assume that one of the following conditions holds:*

(i) *There exists* $0 < \delta_1 < \delta_0$ *such that* $X_1(\delta_1)$ *is compact, where*

$$X_1(\delta_1) = \{x \in X_1 : d_{X_0}(x) \le \delta_1\}; \tag{2.10}$$

(ii) *the function* $f(x)$ *defined by (1.10) is level-compact on* X_1*;*

(iii) X *is finite dimensional and*

$$\lim_{x \in X_1 ||x|| \to +\infty} \max\{f(x), d_{X_0}(x)\} = +\infty; \tag{2.11}$$

(iv) *there exists* $0 < \delta_1 < \delta_0$ *such that* f *is level-compact on* $X_1(\delta_1)$ *defined by* (23). *Then, (GEP) is type I LP well-posed.*

Proof. First, we show that each one of (i), (ii) and (iii) implies (iv). Clearly, either of (i) and (ii) implies (iv). Now we show that (iii) implies (iv). we need only to show that for any $t \in R^1$, the set

$$A = \{x \in X_1(\delta_1) : f(x) \leq t\},$$

where A is bounded and closed since the function f defined by (1.10) is lower semicontinuous on X_1. Suppose to the contrary that there exist $t \in R^1$ and $\{x'_n\} \subseteq X_1(\delta_1)$ such that $||x'_n|| \to +\infty$ and $f(x'_n) \leq t$. From $\{x'_n\} \subseteq X_1(\delta_1)$, we have

$$d_{X_0}(x'_n) \leq \delta_1.$$

Thus,

$$\max\{f(x'_n), d_{X_0}(x'_n)\} \leq \max\{t, \delta_1\},$$

which contradicts condition (24).

Now we show that if (iv) holds, then (GEP) is type I LP well-posed. Let $\{x_n\}$ be a type I LP approximating solution sequence. Then, there exist $\{\epsilon_n\} > 0$ with $\epsilon_n \to 0$ and $u_n \in T(x_n)$ such that

$$h(u_n, x_n, y) + \epsilon_n \geq 0, \forall y \in X_0, \tag{2.12}$$

$$d_{X_0}(x_n) \leq \epsilon_n. \tag{2.13}$$

From (26), we can assume without loss of generality that $\{x_n\} \subseteq X_1(\delta_1)$. Furthermore, from (25), we can assume without loss of generality that

$$\{x_n\} \subseteq \{x \in X_1(\delta_1) : f(x) \leq 1\},$$

where $f(x)$ is defined by (1.10). By the level-compactness of $f(x)$ on $X_1(\delta_1)$, there exist a subsequence of $\{x_{n_j}\}$ of $\{x_n\}$ and $\bar{x} \in X_1(\delta_1)$ such that $x_{n_j} \to \bar{x}$. From this fact and (26), we have $\bar{x} \in X_0$. Furthermore, by the upper semicontinuity of T at $\bar{x}$ and the compactness of $T(\bar{x})$, there exist a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and some $\bar{u} \in T(\bar{x})$ such that $u_{n_j} \to \bar{u}$. This fact combined with (25) yields

$$f(\bar{x}) \leq \liminf_{j\to+\infty} f(x_{n_j}) \leq \limsup_{j\to+\infty} f(x_{n_j}) = \limsup_{j\to+\infty} -h(u_{n_j}, x_{n_j}, y) \leq 0, \forall y \in X_0.$$

We know that $f(\bar{x}) \geq 0$, so $f(\bar{x}) = 0$ and $\bar{x} \in X_0$. Thus, $\bar{x} \in \bar{X}$. □

Similarly, we can prove the next proposition.

Proposition 2.6. *Assume that one of the following conditions holds:*

(i) *There exists* $0 < \delta_1 < \delta_0$ *such that* $X_1(\delta_1)$ *is compact, where*

$$X_2(\delta_1) = \{x \in X_1 : d_K(g(x)) \leq \delta_1\}; \tag{2.14}$$

(ii) *the function* $f(x)$ *defined by (1.10) is level-compact on* X_1*;*

(iii) X *is finite dimensional and*

$$\lim_{x\in X_1 ||x||\to+\infty} \max\{f(x), d_K(g(x))\} = +\infty; \tag{2.15}$$

(iv) *there exists* $0 < \delta_1 < \delta_0$ *such that* f *is level-compact on* $X_2(\delta_1)$ *defined by (27). Then, (GEP) is generalized type I LP well-posed.*

Proposition 2.7. *Let X be finite dimensional. Let $h(.,.,y)$ be T-continuous on $Z \times X_1$ and the solution set $\bar{X}$ be nonempty. Assume that there exist $\delta_1 > 0$ and $y_0 \in X_0$ such that the function $-h(u(x), x, y_0)$ is level-bounded for any selection $u(x)$ of T on $X_1(\delta_1)$ defined by (23). Then, (GEP) is type I LP well-posed.*

Proof. Let $\{x_n\}$ be a type I LP approximating solution sequence. Then, there exist $\epsilon_n \geq 0$ with $\epsilon_n \to 0$ and $u_n \in T(x_n)$ such that

$$h(u_n, x_n, y) + \epsilon_n \geq 0, d_{X_0}(x_n) \leq \epsilon_n, \forall y \in X_0.$$

From $d_{X_0}(x_n) \leq \epsilon_n$, we can assume $\{x_n\} \subset X_1(\delta_1)$. Let us show by contradiction that $\{x_n\}$ is bounded. Otherwise, we assume without loss of generality that $||x_n|| \to +\infty$. By the level-boundedness condition, we have

$$\lim_{n \to +\infty} h(u_n, x_n, y_0) = -\infty$$

contradicting the fact that $h(u_n, x_n, y)+\epsilon_n \geq 0$ when n is sufficiently large. Consequently, we can assume without loss of generality that $x_n \to \bar{x} \in X_1$. This together with $d_{X_0}(x_n) \leq \epsilon_n$ yields $\bar{x} \in X_0$. By the upper semicontinuity of T at $\bar{x}$ and compactness of $T(\bar{x})$, there exist a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and some $\bar{u} \in T(\bar{x})$ such that $u_{n_j} \to \bar{u}$. This shows $x \in \bar{X}$. □

Similarly, we can prove the next result.

Proposition 2.8. *Let X be finite dimensional. Let $h(.,.,y)$ be T-continuous on $Z \times X_1$ and the solution set $\bar{X}$ be nonempty. Assume that there exist $\delta_1 > 0$ and $y_0 \in X_0$ such that the function $-h(u(x), x, y_0)$ is level-bounded for any selection $u(x)$ of T on $X_2(\delta_1)$ defined by (27). Then, (GEP) is generalized type I LP well-posed.*

Now we consider the case when Y is a normed space, K is a closed and convex cone with nonempty interior $intK$.

Let $e \in intK$ and $t \geq 0$. Denote

$$X_3(t) = \{x \in X_1 : g(x) \in K - te\}.$$

The next proposition can be established similarly to ([8], Proposition 2.3).

Proposition 2.9. *Let Y be a normed space, K be a closed and convex cone with nonempty interior $intK$ and $e \in intK$. The solution set $\bar{X}$ is nonempty. Further assume that there exists $t_1 > 0$ such that the function $f(x)$ defined by (1.10) is level-compact on $X_3(t_1)$. Then, (GEP) is generalized type I LP well-posed.*

Under suitable conditions, we show that any type of well-posedness of generalized equilibrium problems is equivalent to the nonemptiness and compactness of its solution set.

Let

$$\phi(x) = \sup_{y \in X_0} \sup_{v \in T(y)} h(v, y, x), \forall x \in X_1. \tag{2.16}$$

Lemma 2.10. *Assume that $X_1 \subset X$ and X_0 are convex. Assume that for any $x \in X_1, u \in T(x)$, $h(u, x, .)$ is convex and $h(u, x, y)$ is monotone. Then, for $\bar{x} \in X_0$, there exist $\bar{u} \in T(\bar{x})$ such that $h(\bar{u}, \bar{x}, y) \geq 0, \forall y \in X_0$ if and only if $h(v, y, \bar{x}) \leq 0, \forall y \in X_0, v \in T(y)$.*

Proof. Suppose for $\bar{x} \in X_0$, there exist $\bar{u} \in T(\bar{x})$ such that

$$h(\bar{u}, \bar{x}, y) \geq 0, \forall y \in X_0.$$

Since $h(u, x, y)$ is monotone, one has

$$h(\bar{u}, \bar{x}, y) + h(v, y, \bar{x}) \leq 0, \forall y \in X_0, v \in T(y).$$

Consequently,

$$h(v, y, \bar{x}) \leq 0, \forall y \in X_0, v \in T(y).$$

Conversely, for all $y \in X_0$, and $\bar{x} \in \bar{X}$, we define $y_n = \frac{1}{n}y + (1 - \frac{1}{n})\bar{x} \in X_0, n \in N$. From $h(u, x, x) = 0, \forall x \in X_0, u \in T(x)$, we have

$$h(v_n, y_n, y_n) = 0, \forall v_n \in T(\frac{1}{n}y + (1 - \frac{1}{n})\bar{x}).$$

Since $h(u, x, .)$ is convex on X_1 and $h(v_n, y_n, \bar{x}) \leq 0$, we can see that

$$\begin{aligned} 0 = h(v_n, y_n, y_n) = h(v_n, y_n, \tfrac{1}{n}y + (1 - \tfrac{1}{n})\bar{x}) &\leq \tfrac{1}{n}h(v_n, y_n, y) + (1 - \tfrac{1}{n})h(v_n, y_n, \bar{x}) \\ &\leq \tfrac{1}{n}h(v_n, y_n, y) \end{aligned}$$

evn By the upper semicontinuity of T at $\bar{x}$ and compactness of $T(\bar{x})$, there exist $y_n \to \bar{x}$ and a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ and some $\bar{u} \in T(\bar{x})$ such that $v_{n_j} \to \bar{u}$. Since $h(., ., y)$ is T-upper semicontinuous on $Z \times X_1$, one has

$$h(\bar{u}, \bar{x}, y) \geq \limsup_{j \to +\infty} h(v_{n_j}, y_{n_j}, y) \geq 0, \forall y \in X_0.$$

□

Lemma 2.11. *Let $\phi(x)$ be defined by (29). Assume that $X_1 \subset X$ and X_0 are convex. Assume that for any $x \in X_1, u \in T(x)$, $h(u, x, .)$ is convex and $h(u, x, y)$ is monotone. Then, the following assertions are true.*

(i) *The function $\phi(x)$ is convex and lower semicontinuous on X_1;*

(ii) *$\phi(x) \geq 0, \forall x \in X_0$. Let $\bar{x} \in X_0$. Then $\phi(\bar{x}) = 0$ if and only if $\bar{x} \in \bar{X}$.*

Proof. (i) Let us show that $\phi(x)$ is convex. For any $t \in (0, 1)$, since $h(u, x, .)$ is convex, we have

$$\begin{aligned} \phi(tx_1 + (1-t)x_2) &= \sup_{y \in X_0} \sup_{v \in T(y)} h(v, y, tx_1 + (1-t)x_2) \\ &\leq \sup_{y \in X_0} \sup_{v \in T(y)} (th(v, y, x_1) + (1-t)h(v, y, x_2)) \\ &\leq t \sup_{y \in X_0} \sup_{v \in T(y)} h(v, y, x_1) + (1-t) \sup_{y \in X_0} \sup_{v \in T(y)} h(v, y, x_2) \\ &= t\phi(x_1) + (1-t)\phi(x_2). \end{aligned}$$

So, $\phi(x)$ is convex. Similar to the the proof of Lemma 1.2, we obtain $\phi(x)$ is lower semicontinuous on X_1.

(ii) It is obvious from the definition of $\phi(x)$ that $\phi(x) \geq 0, \forall x \in X_0$. Now assume $\bar{x} \in \bar{X}$, there exist $\bar{u} \in T(\bar{x})$ such that

$$h(\bar{u}, \bar{x}, y) \geq 0, \forall y \in X_0.$$

Since $h(u, x, .)$ is convex on X_1 and $h(u, x, y)$ is monotone. By Lemma 2.10, we can show that

$$h(v, y, \bar{x}) \leq 0, \forall y \in X_0, v \in T(y).$$

Then $\phi(\bar{x}) \leq 0$. This combined with $\phi(x) \geq 0$ yields $h(\bar{x}) = 0$.
Conversely, suppose that $\bar{x} \in X_0$ and $\phi(\bar{x}) = 0$. From the definition of $\phi(x)$, one has

$$h(v, y, x) \leq 0, \forall y \in X_0, v \in T(y).$$

By Lemma 2.10, there exists $\bar{u} \in T(\bar{x})$ such that $h(\bar{u}, \bar{x}, y) \geq 0, \forall y \in X_0$. This is, $\bar{x} \in \bar{X}$. □

Assumption 2.12. Assume that X is a finite dimensional, $X_1 \subset X$ is a nonempty closed and convex set and Y is a normed space, $K \subset Y$ is a closed and convex cone with nonempty interior int K. Assume that $g(x)$ is K-concave on X_1 (i.e.for any x_1, x_2 and any $\theta \in (0,1)$ there holds that $g(\theta x_1 + (1-\theta)x_2) - \theta g(x_1) - (1-\theta)g(x_2) \in K$). Assume that for any $x \in X_1, u \in T(x)$, $h(u, x, .)$ is convex on X_1 and $h(u, x, y)$ is monotone. Further assume that the solution set $\bar{X}$ is nonempty.

It is obvious that under Assumption 2.12, the optimization problem (P) (with f replaced by ϕ) is a convex program.

Lemma 2.13. *Let Assumption 2.12 hold. If $\{x_n\}$ is a (generalized) type I LP approximating solution sequence of (GEP), then it is a (generalized) type I LP minimizing solution sequence of (P) (with $f(x)$ replaced by $\phi(x)$).*

Proof. We prove only the type I case, the generalized type I case can be similarly proved. Let $\{x_n\}$ be a type I LP approximating solution sequence of (GEP). Then, there exist $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n \to 0$ and $z_n \in T(x_n)$ satisfying

$$d_{X_0}(x_n) \leq \epsilon_n, h(u_n, x_n, y) \geq -\epsilon_n, \quad \forall y \in X_0.$$

Since $h(u, x, y)$ is monotone

$$h(v, y, x_n) \leq -h(u_n, x_n, y) \leq \epsilon_n$$

we can easily verify that

$$d_{X_0}(x_n) \leq \epsilon_n, \phi(x_n) \leq \epsilon_n.$$

It follows that (1.7) holds with $\bar{v} = 0$. □

The following result can be established by using Lemmas 2.11, 2.3, ([6], Theorem 2.4) and Remark 1.4 (ii).

Lemma 2.14. *Let Assumption 2.12 hold. Then, (GEP) is (generalized) type I LP well-posed if and only if its solution set $\bar{X}$ is nonempty and compact.*

The following theorem is a direct consequence of Lemmas 2.13 and 2.14.

Theorem 2.15. *Let Assumption 2.12 hold. Then, any type of LP well-posedness of (GEP) is equivalent to the fact that the solution set $\bar{X}$ of (GEP) is nonempty and compact.*

Remark 2.16. Necessary and sufficient conditions for the nonemptiness and compactness of the solution set of a monotone (GEP) were established in [5]. By Theorem 2.15, these conditions can be used to verify LP well-posedness of (GEP).

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