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THE FISCHER-BURMEISTER COMPLEMENTARITY FUNCTION ON EUCLIDEAN JORDAN ALGEBRAS*

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Abstract: Recently, Gowda et al. [10] established the Fischer-Burmeister (FB) complementarity function (C-function) on Euclidean Jordan algebras. In this paper, we prove that FB C-function as well as the derivatives of the squared norm of FB C-function are Lipschitz continuous.

Key words: *Fischer-Burmeister function, Euclidean Jordan algebra, Lipschitz continuity*

Mathematics Subject Classification: *65K05, 90C33; 26B05, 15A09*

1 Introduction

It is well-known that the scalar-valued FB function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is specified by

$$\phi(a, b) := a + b - \sqrt{a^2 + b^2}, \ a, b \in \mathbb{R}, \tag{1.1}$$

which is attributed by Fischer to Burmeister (see [5, 6, 7]). It is a *complementarity function for nonlinear complementarity problem* (NCP)(called C-function or NCP function), that is,

$$\phi(a, b) = 0 \ \Leftrightarrow \ a \geq 0, b \geq 0, \ ab = 0. \tag{1.2}$$

FB function has been much studied in the context of NCP, because it has nice properties, such as strong semismoothness. Moreover, the squared norm of FB function has a Lipschitz continuous gradient, which can be effectively employed in the algorithmic development, see, e.g., [3, 8, 13].

Recently, FB function has been generalized to solve the semidefinite complementarity problem (SDCP) and the second-order cone complementarity problem (SOCCP). For instance, Tseng [21] (also see Borwein and Lewis [1]) proved that FB function is a C-function for SDCP, and Fukushima, Luo and Tseng [9] showed that this is true in the setting of SOCCP. It was proved by Sim, Sun and Ralph [17] and Chen, Sun and Sun [2] that the squared norm of FB function has a Lipschitz continuous gradient in the settings of SDCP and SOCCP, respectively.

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Gowda, Sznajder and Tao [10] proposed the following *(vector-valued) FB function* on Euclidean Jordan algebras as

$$\Phi_{FB}(x, y) := x + y - (x^2 + y^2)^{\frac{1}{2}}, \tag{1.3}$$

(detailed description is in the next section) and showed that it is a C-function for *symmetric cone complementarity problem* (SCCP) which is to find a vector $x \in \mathcal{J}$ such that

$$x \in K,\ y \in K,\ \langle x, y\rangle = 0, \quad y = F(x), \tag{1.4}$$

where $\mathcal{J}$ is a space of n-dimensional real column vectors, $(\mathcal{J}, \langle\cdot,\cdot\rangle, \circ)$ is a Euclidean Jordan algebra, K is the symmetric cone in $\mathcal{V}$ (see Section 2), and $F : \mathcal{J} \to \mathcal{J}$ is a given continuously differentiable mapping. SCCP provides a simple, natural, and unified framework for various complementarity problems, such as NCP, SOCCP and SDCP. Because of wide applications in engineering, management science and other fields, it has attracted much attention, see, e.g., [10, 11, 12, 15, 16, 20, 22]. Here, we say $\Phi : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a *C-function (for SCCP)* if it satisfies

$$\Phi(x, y) = 0 \iff x \in K,\ y \in K,\ \langle x, y\rangle = 0. \tag{1.5}$$

Liu, Zhang and Wang [16] showed that the *squared norm of FB function*, $\Psi_{FB} : \mathcal{J} \times \mathcal{J} \to \mathbb{R}$, defined by

$$\Psi_{FB}(x, y) := \frac{1}{2}\|\Phi_{FB}(x, y)\|^2 \tag{1.6}$$

is differentiable. Motivated by all of the cited work above, a natural question arises:

> Are the derivatives of the squared norm of FB function
> Ψ_{FB} (given by (1.6)) Lipschitz continuous?

We answer the above question in the affirmative. To do so, we establish useful inequalities on the Lyapunov operator, employing the norm induced by the underlying inner product.

In Section 2 we establish the preliminaries and present some useful results about Lyapunov transformation. We show that FB function is Lipschitz continuous in Section 3. Section 4 establishes that the derivatives of squared norm of FB function are Lipschitz continuous. We conclude the paper in Section 5 and raise an open question.

2 Preliminaries

We review some results on Euclidean Jordan algebras (see for instance [4, 14]) and develop some basic inequalities on Euclidean Jordan algebras.

A *Euclidean Jordan algebra* is a triple $(\mathcal{J}, \langle\cdot,\cdot\rangle, \circ)$($\mathcal{V}$ for short), where $(\mathcal{J}, \langle\cdot,\cdot\rangle)$ is a n-dimensional inner product space over real field $\mathbb{R}$ and $(x, y) \mapsto x \circ y : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is a bilinear mapping which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathcal{J}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathcal{J}$ where $x^2 := x \circ x$ and
(iii) $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ for all $x, y, z \in \mathcal{J}$.

We call $x \circ y$ the *Jordan product* of x and y. In general, the Jordan product is not associative, i.e., $(x \circ y) \circ z \neq x \circ (y \circ z)$ for all $x, y, z \in \mathcal{J}$. We assume that there exists an element e (called the *identity* element) such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{J}$. Define *the set of squares* as $K := \{x^2 : x \in \mathcal{J}\}$. It is well-known that K is a *symmetric cone* in $\mathcal{V}$, i.e., K is a closed, convex, homogeneous and self-dual cone. For $x \in \mathcal{J}$, the *degree* of x denoted by $\deg(x)$ is the smallest positive integer m such that the set $\{e, x, x^2, \cdots, x^m\}$ is linearly dependent. The *rank* of $\mathcal{V}$ is defined as $\max\{\deg(x) : x \in \mathcal{J}\}$. In this paper, r will denote the rank of the underlying Euclidean Jordan algebra. Let $\dim(\mathcal{J})$ denote the dimension of $\mathcal{J}$. Obviously, $r \leq \dim(\mathcal{J})$.

Recall that an element $c \in \mathcal{J}$ is *idempotent* if $c^2 = c \neq 0$. It is also *primitive* if it cannot be written as a sum of two idempotents. *A complete system of orthogonal idempotents* is a finite set $\{c_1, c_2, \cdots, c_k\}$ of idempotents with $c_i \circ c_j = 0$ $(i \neq j)$ and $\sum_{i=1}^k c_i = e$. A complete system of orthogonal primitive idempotents is called a *Jordan frame* of $\mathcal{V}$. Thus, for any element $x \in \mathcal{J}$, we have the following important spectral decomposition theorem.

Theorem 2.1 (Theorem III.1.2, [4]). *Let $\mathcal{V}$ be a Euclidean Jordan algebra of rank r. Then for every vector $x \in \mathcal{J}$ there exist a Jordan frame $\{c_1(x), c_2(x), \cdots, c_r(x)\}$ and real numbers $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$, the eigenvalues of x, such that*

$$x = \lambda_1(x)c_1(x) + \lambda_2(x)c_2(x) + \cdots + \lambda_r(x)c_r(x). \tag{2.1}$$

We call (2.1) the spectral decomposition of x.

Let $x = \sum_{j=1}^r \lambda_j(x)c_j(x)$ and $\|\cdot\|$ be *the norm* on $\mathcal{J}$ induced by the inner product, i.e.,

$$\|x\| := \sqrt{\langle x, x\rangle} = \sqrt{\sum_{j=1}^r \lambda_j^2(x)}.$$

We have $\|c_j(x)\| = 1$ for $j \in \{1, 2, \cdots, r\}$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function. We define the vector-valued function $G : \mathcal{J} \to \mathcal{J}$ as

$$G(x) := \sum_{j=1}^r g(\lambda_j(x))c_j(x) = g(\lambda_1(x))c_1(x) + g(\lambda_2(x))c_2(x) + \cdots + g(\lambda_r(x))c_r(x),$$

which is a *Löwner operator*. In particular, taking $t_+ := \max\{0, t\}$, we can define *the projection* of x onto K as

$$x_+ := \sum_{j=1}^r (\lambda_j(x))_+ c_j(x).$$

Note that $x \in K$ if and only if $\lambda_i(x) \geq 0, \forall i \in \{1, 2, \cdots, r\}$. Letting $g(t) := \sqrt{t}$ for $t \in \mathbb{R}_+$, we define

$$x^{\frac{1}{2}} := \sum_{j=1}^r \sqrt{\lambda_j(x)}c_j(x) \quad \text{for} \quad x \in K.$$

Therefore, FB function (1.3) and its squared norm (1.6) are well-defined.

We next recall the Peirce decomposition theorem on the space $\mathcal{J}$, where the Jordan frame $\{c_1, c_2, \cdots, c_r\}$ can be fixed beforehand.

Theorem 2.2 (Theorem IV.2.1, [4]). *Let $\{c_1, c_2, \cdots, c_r\}$ be a given Jordan frame in a Euclidean Jordan algebra $\mathcal{V}$ of rank r. Then $\mathcal{J}$ is the orthogonal direct sum of spaces J_{ij} $(i \le j)$, where the subspaces J_{ij} for $i, j \in \{1, 2, \cdots, r\}$ are defined by*

$$J_{ii} := \{x \in \mathcal{J} : x \circ c_i = x\} \quad \text{and} \quad J_{ij} := \left\{x \in \mathcal{J} : x \circ c_i = \frac{1}{2}x = x \circ c_j\right\},\ i \ne j.$$

Furthermore,

(i) $J_{ij} \circ J_{ij} \subseteq J_{ii} + J_{jj}$;

(ii) $J_{ij} \circ J_{jk} \subseteq J_{ik}$, *if* $i \ne k$;

(iii) $J_{ij} \circ J_{kl} = \{0\}$, *if* $\{i, j\} \bigcap \{k, l\} = \emptyset$.

Based on the result above and Lemma IV.2.2 in [4], we have the following connection between $\|x \circ y\|$ and $\|x\|\|y\|$, which is useful in the subsequent analysis.

Lemma 2.3. *Let $x \in J_{ij}, y \in J_{kl}$ with $i < j$ and $k < l$. Then $\|x \circ y\| \le \|x\|\|y\|$. Furthermore,*

$$\|x \circ y\|^2 \begin{cases} = 0 & \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \\ \le \frac{1}{2}\|x\|^2\|y\|^2 & \text{if } i = k,\ j = l, \\ = \frac{1}{8}\|x\|^2\|y\|^2 & \text{if } i < j = k < l \text{ or } k < l = i < j. \end{cases}$$

Proof. Note that if $\{i, j\} \bigcap \{k, l\} = \emptyset$, then $x \circ y = 0$. If $x \in J_{ij}, y \in J_{jl}$ with i, j, l all distinct, by Lemma IV.2.2 in [4], $\|x \circ y\|^2 = \frac{1}{8}\|x\|^2\|y\|^2$. We only need to prove the conclusion in the case of $x, y \in J_{ij}$. By Theorem 2.2, $x \circ y = \delta_1 c_i + \delta_2 c_j$, for some $\delta_1, \delta_2 \in \mathbb{R}$. Thus, $\|x \circ y\|^2 = \delta_1^2 + \delta_2^2$. Meanwhile, by direct computation, we have

$$\begin{aligned} \|x \circ y\|^2 &= \langle x \circ y, x \circ y \rangle \\ &= \langle \delta_1 c_i + \delta_2 c_j, x \circ y \rangle \\ &= \langle (\delta_1 c_i + \delta_2 c_j) \circ x, y \rangle \\ &= \left\langle \frac{1}{2}(\delta_1 + \delta_2)x, y \right\rangle \text{ (by Theorem 2.2)} \\ &\le \frac{\sqrt{\delta_1^2 + \delta_2^2}}{\sqrt{2}}\|x\|\|y\|. \end{aligned}$$

Therefore, we conclude in this case that $\|x \circ y\|^2 \le \frac{1}{2}\|x\|^2\|y\|^2$. □

Below we consider a very fundamental linear operator, Lyapunov transformation, and derive some inequalities on it that will be useful to us.

For each $x \in \mathcal{J}$, we define the *Lyapunov transformation (operator)* $\mathcal{L}(x) : \mathcal{J} \to \mathcal{J}$ by

$$\mathcal{L}(x)y = x \circ y, \quad \text{for all } y \in \mathcal{J},$$

which is a symmetric operator in the sense that $\langle \mathcal{L}(x)y, z \rangle = \langle y, \mathcal{L}(x)z \rangle$ for all $y, z \in \mathcal{J}$. Given $0 \ne a = \sum_{i=1}^r \lambda_i(a)c_i(a)$ with $\lambda_1(a) \ge \cdots \ge \lambda_{|\wp(a)|} > 0 = \lambda_{|\wp(a)|+1} = \cdots = \lambda_r(a)$, where $\wp(a) := \{i : \lambda_i(a) > 0\}$, we define a subspace

$$J_a := J(e_{\wp(a)}, 1) := \{x \in \mathcal{J} : x \circ e_{\wp(a)} = x\} \quad \text{with} \quad e_{\wp(a)} := \sum_{i=1}^{|\wp(a)|} c_i(a). \tag{2.2}$$

It is well-known that $\mathcal{L}_a := \mathcal{L}(a)$ is a one-to-one mapping from J_a to J_a and therefore it has an inverse $\mathcal{L}_a^{-1}$ on J_a, i.e., for any $x \in J_a$, $\mathcal{L}_a^{-1}(x)$ is the unique $d \in J_a$ such that $a \circ d = x$. Using Lemma 20 in [10], any $x \in J_a$ can be expressed as

$$x = \sum_{i=1}^{|\wp(a)|} x_i c_i(a) + \sum_{1 \le i < j \le |\wp(a)|} x_{ij}, \tag{2.3}$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in J_{ij}$ with the given Jordan frame $\{c_1(a), c_2(a), \cdots, c_r(a)\}$. The following proposition gives a formula for $\mathcal{L}_a^{-1}(x)$.

Proposition 2.4. *Let $0 \neq a = \sum_{i=1}^r \lambda_i(a) c_i(a)$ with $\lambda_1(a) \ge \cdots \ge \lambda_{|\wp(a)|} > 0 = \lambda_{|\wp(a)|+1} = \cdots = \lambda_r(a)$. Let J_a and $e_{\wp(a)}$ be given by (2.2). Then every $x \in J_a$ can be written as in (2.3) and*

$$\mathcal{L}_a^{-1}(x) = \sum_{i=1}^{|\wp(a)|} \frac{x_i}{\lambda_i(a)} c_i(a) + \sum_{1 \le j < l \le |\wp(a)|} \frac{2}{\lambda_j(a) + \lambda_l(a)} x_{jl}. \tag{2.4}$$

In particular, $\mathcal{L}_a^{-1}(a) = e_{\wp(a)}$ is the identity element in J_a, and $\mathcal{L}_a^{-1}(a^k) = a^{k-1}$ for $k > 1$.

Proof. As we noted before, the fact that every $x \in J_a$ can be written in the form (2.3) is given by Lemma 20 in [10]. Let $d := \mathcal{L}_a^{-1}(x)$. Then

$$d = \sum_{i=1}^{|\wp(a)|} d_i c_i(a) + \sum_{1 \le j < l \le |\wp(a)|} d_{jl},$$

for some $d_i \in \mathbb{R}$ and $d_{jl} \in J_{jl}$. By Theorem 2.2, direct calculation yields that

$$\begin{aligned} a \circ d &= \sum_{i=1}^{|\wp(a)|} \lambda_i(a) d_i c_i(a) + \left(\sum_{i=1}^{|\wp(a)|} \lambda_i(a) c_i(a) \right) \circ \left(\sum_{1 \le j < l \le |\wp(a)|} d_{jl} \right) \\ &= \sum_{i=1}^{|\wp(a)|} \lambda_i(a) d_i c_i(a) + \sum_{1 \le j < l \le |\wp(a)|} \left(\sum_{i=1}^{|\wp(a)|} \lambda_i(a) c_i(a) \right) \circ d_{jl} \\ &= \sum_{i=1}^{|\wp(a)|} \lambda_i(a) d_i c_i(a) + \sum_{1 \le j < l \le |\wp(a)|} \frac{\lambda_j(a) + \lambda_l(a)}{2} d_{jl}. \end{aligned}$$

This together with $a \circ d = x$ establishes (2.4). □

Likewise, for the above a and $\wp(a)$, we define subspaces

$$\begin{aligned} J_a^0 &:= J(e_{\wp(a)}, 0) := \{x \in \mathcal{J} : x \circ e_{\wp(a)} = 0\}, \\ J_a^{\frac{1}{2}} &:= J(e_{\wp(a)}, \tfrac{1}{2}) := \{x \in \mathcal{J} : x \circ e_{\wp(a)} = \tfrac{1}{2} x\}. \end{aligned}$$

It is easy to see that $J_a^0 = J(e - e_{\wp(a)}, 1)$. Similarly, applying Lemma 20 in [10], any $x \in J_a^0$ can be expressed as

$$x = \sum_{i=|\wp(a)|+1}^{r} x_i c_i(a) + \sum_{|\wp(a)|+1 \le i < j \le r} x_{ij},$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in J_{ij}$. It is known that $\mathcal{J}$ is the orthogonal direct sum of spaces $J_a, J_a^{\frac{1}{2}}$ and J_a^0 (see Page 62 of [4]). From Theorem 2.2, we obtain

$$J_a = \bigoplus_{1\le j\le l\le |\wp(a)|} J_{jl}, \quad J_a^{\frac{1}{2}} = \bigoplus_{1\le j\le |\wp(a)|, |\wp(a)|+1\le l\le r} J_{jl}, \quad J_a^0 = \bigoplus_{|\wp(a)|+1\le j\le l\le r} J_{jl}. \tag{2.5}$$

Thus any $x \in \mathcal{J}$ can be expressed as $x = x^{(1)} + x^{(\frac{1}{2})} + x^{(0)}$ where $x^{(1)} \in J_a, x^{(\frac{1}{2})} \in J_a^{\frac{1}{2}}$ and $x^{(0)} \in J_a^0$. Observe that $a \circ x^{(\frac{1}{2})} \in J_a^{\frac{1}{2}}$. Moreover, $\mathcal{L}_a$ is a one-to-one mapping from $J_a^{\frac{1}{2}}$ to $J_a^{\frac{1}{2}}$, which is shown by the following.

Proposition 2.5. *Let $0 \neq a = \sum_{i=1}^r \lambda_i(a)c_i(a)$ with $\lambda_1(a) \geq \cdots \geq \lambda_{|\wp(a)|} > 0 = \lambda_{|\wp(a)|+1} = \cdots = \lambda_r(a)$. Then every $y \in J_a^{\frac{1}{2}}$ can be written as*

$$y = \sum_{1\le j\le |\wp(a)|, |\wp(a)|+1\le l\le r} y_{jl} \tag{2.6}$$

and

$$\mathcal{L}_a^{-1}(y) = \sum_{1\le j\le |\wp(a)|, |\wp(a)|+1\le l\le r} \frac{2}{\lambda_j(a)} y_{jl}, \tag{2.7}$$

where $y_{jl} \in J_{jl}$.

Proof. By Theorem 2.2 and (2.5), every $y \in J_a^{\frac{1}{2}}$ can be written in the form (2.6). Let $d := \mathcal{L}_a^{-1}(y)$. Then

$$d = \sum_{1\le j\le |\wp(a)|, |\wp(a)|+1\le l\le r} d_{jl},$$

for $d_{jl} \in J_{jl}$. As in the proof of Proposition 2.4, we have

$$\mathcal{L}_a^{-1}(y) = \sum_{1\le j\le |\wp(a)|, |\wp(a)|+1\le l\le r} \frac{2}{\lambda_j(a) + \lambda_l(a)} y_{jl}.$$

The desired conclusion follows from $\lambda_l(a) = 0$ for $|\wp(a)| + 1 \leq l \leq r$. $\square$

Next, we consider some continuity property of $\mathcal{L}_{a_\varepsilon}^{-1}$ where $a_\varepsilon := (a^2 + \varepsilon^2 e)^{\frac{1}{2}}$.

Proposition 2.6. *Let $0 \neq a = \sum_{i=1}^r \lambda_i(a)c_i(a)$ with $\lambda_1(a) \geq \cdots \geq \lambda_{|\wp(a)|} > 0 = \lambda_{|\wp(a)|+1} = \cdots = \lambda_r(a)$. Then for any $x \in J_a$ and $y \in J_a^{\frac{1}{2}}$, $\mathcal{L}_a^{-1}(x+y)$ is well-defined and*

$$\mathcal{L}_a^{-1}(x+y) = \mathcal{L}_a^{-1}(x) + \mathcal{L}_a^{-1}(y).$$

Let $a_\varepsilon := (a^2 + \varepsilon^2 e)^{\frac{1}{2}}$. Then

$$\lim_{\varepsilon\to 0} \mathcal{L}_{a_\varepsilon}^{-1}(x+y) = \mathcal{L}_a^{-1}(x+y). \tag{2.8}$$

Furthermore, we have

$$\lim_{\varepsilon\to 0} \mathcal{L}_{a_\varepsilon}^{-1}(x+y) \circ x = \mathcal{L}_a^{-1}(x+y) \circ x.$$

Proof. The first part of the theorem is obvious by Propositions 2.4 and 2.5. For the second part, since $x \in J_a$ and $y \in J_a^{\frac{1}{2}}$, we can take x and y as in the forms (2.3) and (2.6), respectively. Noting that $a_\varepsilon = \sum_{i=1}^{r} \lambda_i(a_\varepsilon)c_i(a)$ with $\lambda_i(a_\varepsilon) = \sqrt{\lambda_i^2(a) + \varepsilon^2}$, and employing an argument similar to the one in the proof of Proposition 2.4, we have

$$\begin{aligned}\mathcal{L}_{a_\varepsilon}^{-1}(x+y) &= \sum_{i=1}^{|\wp(a)|} \frac{x_i}{\lambda_i(a_\varepsilon)} c_i(a) + \sum_{1\le j<l\le|\wp(a)|} \frac{2}{\lambda_j(a_\varepsilon)+\lambda_l(a_\varepsilon)} x_{jl} \\ &\quad + \sum_{1\le j\le|\wp(a)|,|\wp(a)|+1\le l\le r} \frac{2}{\lambda_j(a_\varepsilon)+\lambda_l(a_\varepsilon)} y_{jl}.\end{aligned}$$

This together with the facts $\lambda_l(a_\varepsilon) = |\varepsilon|$ for $|\wp(a)|+1 \le l \le r$, (2.4) and (2.7) yields (2.8).

Furthermore, note that

$$\begin{aligned}\mathcal{L}_{a_\varepsilon}^{-1}(x+y)\circ x &= \left(\sum_{i=1}^{|\wp(a)|} \frac{x_i}{\lambda_i(a_\varepsilon)} c_i(a)\right)\circ x + \left(\sum_{1\le j<l\le|\wp(a)|} \frac{2}{\lambda_j(a_\varepsilon)+\lambda_l(a_\varepsilon)} x_{jl}\right)\circ x \\ &\quad + \left(\sum_{1\le j\le|\wp(a)|,|\wp(a)|+1\le l\le r} \frac{2}{\lambda_j(a_\varepsilon)+\lambda_l(a_\varepsilon)} y_{jl}\right)\circ x,\end{aligned}$$

and

$$\lim_{\varepsilon\to 0}\left(\sum_{i=1}^{|\wp(a)|} \frac{x_i}{\lambda_i(a_\varepsilon)} c_i(a)\right)\circ x = \left(\sum_{i=1}^{|\wp(a)|} \frac{x_i}{\lambda_i(a)} c_i(a)\right)\circ x,$$

$$\lim_{\varepsilon\to 0}\left(\sum_{1\le j<l\le|\wp(a)|} \frac{2}{\lambda_j(a_\varepsilon)+\lambda_l(a_\varepsilon)} x_{jl}\right)\circ x = \left(\sum_{1\le j<l\le|\wp(a)|} \frac{2}{\lambda_j(a)+\lambda_l(a)} x_{jl}\right)\circ x,$$

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\sum_{1\le j\le|\wp(a)|,|\wp(a)|+1\le l\le r} \frac{2}{\lambda_j(a_\varepsilon)+\lambda_l(a_\varepsilon)} y_{jl}\right)\circ x = \\ &\qquad\qquad \left(\sum_{1\le j\le|\wp(a)|,|\wp(a)|+1\le l\le r} \frac{2}{\lambda_j(a)} y_{jl}\right)\circ x.\end{aligned}$$

It follows that $\lim_{\varepsilon\to 0} \mathcal{L}_{a_\varepsilon}^{-1}(x+y)\circ x = \mathcal{L}_a^{-1}(x+y)\circ x$, as desired. □

We end this section by presenting various useful inequalities on $\mathcal{L}^{-1}$.

Lemma 2.7. *For $x, y \in \mathcal{J}$, let $a_\varepsilon(x,y) := (x^2+y^2+\varepsilon^2 e)^{\frac{1}{2}}$ with $\varepsilon \neq 0$. Then for every $u, v \in \mathcal{J}$, we have*

$$\left\|\mathcal{L}_{a_\varepsilon(x,y)}^{-1}(x+y)\circ u\right\| \le 2\beta\|u\|,\quad \left\|\mathcal{L}_{a_\varepsilon(x,y)}^{-1}(v)\circ x\right\| \le \beta\|v\| \quad\text{and}\quad \left\|\mathcal{L}_{a_\varepsilon(x,y)}^{-1}(x\circ u)\right\| \le \gamma\|u\|,$$

where β and γ are positive constants only dependent on the rank of $\mathcal{J}$, which can be taken as $\beta = r^4$ and $\gamma = 5r^2$.

For $x, y \in \mathcal{J}$, define $a := (x^2+y^2)^{\frac{1}{2}}$ and let $a = \sum_{i=1}^r \lambda_i(a)c_i$. For $u, v \in \mathcal{J}$, by Theorem 2.2, we have

$$\begin{aligned} x &= \sum_{i=1}^r x_i c_i + \sum_{1\le j<l\le r} x_{jl}, \quad y = \sum_{i=1}^r y_i c_i + \sum_{1\le j<l\le r} y_{jl}, \\ u &= \sum_{i=1}^r u_i c_i + \sum_{1\le j<l\le r} u_{jl}, \quad v = \sum_{i=1}^r v_i c_i + \sum_{1\le j<l\le r} v_{jl}, \end{aligned}$$

where $x_i, y_i, u_i, v_i \in \mathbb{R}$ and $x_{jl}, y_{jl}, u_{jl}, v_{jl} \in J_{jl}$. Note that $a_\varepsilon(x,y) = (x^2+y^2+\varepsilon^2 e)^{\frac{1}{2}} = \sum_{i=1}^r \sqrt{\lambda_i^2(a)+\varepsilon^2}c_i$ and $a_\varepsilon^2(x,y) = x^2+y^2+\varepsilon^2 e$. Thus,

$$\begin{aligned} \lambda_i^2(a)+\varepsilon^2 &= \langle c_i, a_\varepsilon^2(x,y)\rangle \\ &= \langle c_i, x^2+y^2)\rangle + \varepsilon^2 \\ &= \langle x\circ c_i, x\rangle + \langle y\circ c_i, y\rangle + \varepsilon^2 \\ &= x_i^2 + y_i^2 + \frac{1}{2}\sum_{1\le j<l\le r, i\in\{j,l\}} (\|x_{jl}\|^2 + \|y_{jl}\|^2) + \varepsilon^2, \end{aligned}$$

where the second equality holds by $\langle c_i, e\rangle = 1$, the fourth follows from the facts that $x\circ c_i = x_i c_i + \frac{1}{2}\sum_{1\le j<l\le r, i\in\{j,l\}} x_{jl}$ by Theorem 2.2 and

$$\left\langle x_i c_i + \frac{1}{2}\sum_{1\le j<l\le r, i\in\{j,l\}} x_{jl}, x\right\rangle = x_i^2 + \frac{1}{2}\sum_{1\le j<l\le r, i\in\{j,l\}} \|x_{jl}\|^2.$$

This implies that

$$\sqrt{\lambda_i^2(a)+\varepsilon^2} \ge \max\left\{|x_i|, |y_i|, \frac{1}{\sqrt{2}}\|x_{jl}\|, \frac{1}{\sqrt{2}}\|y_{jl}\|,\ i\in\{j,l\}\right\}, \tag{2.9}$$

and for $j \ne l, j, l \in \{1, 2, \cdots, r\}$ we obtain

$$\sqrt{\lambda_j^2(a)+\varepsilon^2} + \sqrt{\lambda_l^2(a)+\varepsilon^2} \ge \max\left\{\sqrt{2}\|x_{jl}\|, \sqrt{2}\|y_{jl}\|\right\}. \tag{2.10}$$

From Proposition 2.4, we have

$$\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x) = \sum_{i=1}^r \frac{x_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i + \sum_{1\le j<l\le r} \frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2} + \sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}.$$

Thus, by Theorem 2.2, direct calculation yields

$$\begin{aligned}
\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x)\circ u &= \sum_{i=1}^r \frac{x_i u_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i + \left(\sum_{i=1}^r \frac{x_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i\right)\circ\left(\sum_{1\le j<l\le r} u_{jl}\right)\\
&\quad + \left(\sum_{i=1}^r u_i c_i\right)\circ\left(\sum_{1\le j<l\le r}\frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\right)\\
&\quad + \left(\sum_{1\le j<l\le r}\frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\right)\circ\left(\sum_{1\le j<l\le r} u_{jl}\right)\\
&= \sum_{i=1}^r \frac{x_i u_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i + \sum_{i=1}^r \sum_{1\le j<l\le r, i\in\{j,l\}} \frac{x_i}{2\sqrt{\lambda_i^2(a)+\varepsilon^2}} u_{jl}\\
&\quad + \sum_{i=1}^r \sum_{1\le j<l\le r, i\in\{j,l\}} \frac{u_i}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\\
&\quad + \sum_{1\le j<l\le r}\sum_{1\le i<k\le r} \frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\circ u_{ik},
\end{aligned}$$

where the second equality follows from $c_i \circ u_{jl} = \frac{1}{2} u_{jl}$ with $i \in \{j,l\}$. Therefore, we have

$$\begin{aligned}
\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x)\circ u\right\| &\le \sum_{i=1}^r \left\|\frac{x_i u_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i\right\| + \sum_{i=1}^r \sum_{1\le j<l\le r, i\in\{j,l\}} \left\|\frac{x_i}{2\sqrt{\lambda_i^2(a)+\varepsilon^2}} u_{jl}\right\|\\
&\quad + \sum_{i=1}^r \sum_{1\le j<l\le r, i\in\{j,l\}} \left\|\frac{u_i}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\right\|\\
&\quad + \sum_{1\le j<l\le r}\sum_{1\le i<k\le r} \left\|\frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\circ u_{ik}\right\|\\
&\le \sum_{i=1}^r |u_i| + \sum_{i=1}^r \sum_{1\le j<l\le r, i\in\{j,l\}} \frac{1}{2}\|u_{jl}\| + \sum_{i=1}^r \sum_{1\le j<l\le r, i\in\{j,l\}} \frac{|u_i|}{\sqrt{2}}\\
&\quad + \sum_{1\le j<l\le r}\sum_{1\le i<k\le r} \left\|\frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} x_{jl}\right\| \|u_{ik}\|\\
&\le r\|u\| + \frac{1}{2} r(r-1)\|u\| + \frac{\sqrt{2}}{2} r(r-1)\|u\| + \frac{\sqrt{2}}{4}\left[r(r-1)\right]^2\|u\|,
\end{aligned}$$

where the second inequality holds by Lemma 2.3, the third by the fact $\|u\| \ge \max\{|u_i|, \|u_{jl}\|\}$, (2.9) and (2.10). Let $\beta \ge r + \frac{1+\sqrt{2}}{2} r(r-1) + \frac{\sqrt{2}}{4}[r(r-1)]^2$. Then

$$\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x)\circ u\right\| \le \beta\|u\|.$$

Likewise, we have $\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(y)\circ u\right\| \le \beta\|u\|$. Hence,

$$\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u\right\| \le \left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x)\circ u\right\| + \left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(y)\circ u\right\| \le 2\beta\|u\|.$$

Similarly, noting that

$$\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(v) = \sum_{i=1}^{r} \frac{v_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i + \sum_{1\le j<l\le r} \frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} v_{jl},$$

we obtain

$$\left\| \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(v) \circ x \right\| \le \beta \|v\|.$$

We next show $\left\| \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x \circ u) \right\| \le \gamma \|u\|$ with γ only dependent on r. Note that

$$\begin{aligned}
x \circ u &= \sum_{i=1}^{r} x_i u_i c_i + \left(\sum_{i=1}^{r} x_i c_i\right) \circ \left(\sum_{1\le j<l\le r} u_{jl}\right) + \left(\sum_{i=1}^{r} u_i c_i\right) \circ \left(\sum_{1\le j<l\le r} x_{jl}\right) \\
&\quad + \left(\sum_{1\le j<l\le r} x_{jl}\right) \circ \left(\sum_{1\le j<l\le r} u_{jl}\right) \\
&= \sum_{i=1}^{r} x_i u_i c_i + \sum_{1\le j<l\le r} \frac{x_j + x_l}{2} u_{jl} + \sum_{1\le j<l\le r} \frac{u_j + u_l}{2} x_{jl} \\
&\quad + \left(\sum_{1\le i<j\le r} \sum_{1\le k<l\le r, \{i,j\}\ne\{k,l\}} x_{ij} \circ u_{kl} + \sum_{1\le j<l\le r} x_{jl} \circ u_{jl} \right) \\
&= \sum_{i=1}^{r} x_i u_i c_i + \sum_{1\le j<l\le r} \left(\frac{x_j + x_l}{2} u_{jl} + \frac{u_j + u_l}{2} x_{jl} \right) \\
&\quad + \sum_{1\le i<j<k\le r} (x_{ij} \circ u_{jk} + x_{jk} \circ u_{ij}) + \sum_{1\le j<l\le r} x_{jl} \circ u_{jl}.
\end{aligned}$$

By Theorem 2.2, we can write $x_{jl} \circ u_{jl} = f_1^{jl} c_j + f_2^{jl} c_l$ with $f_1^{jl}, f_2^{jl} \in \mathbb{R}$. Thus, by Proposition 2.4,

$$\begin{aligned}
\mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left(\sum_{1\le j<l\le r} x_{jl} \circ u_{jl} \right) &= \mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left(\sum_{1\le j<l\le r} (f_1^{jl} c_j + f_2^{jl} c_l) \right) \\
&= \sum_{1\le j<l\le r} \left(\frac{f_1^{jl}}{\sqrt{\lambda_j^2(a)+\varepsilon^2}} c_j + \frac{f_2^{jl}}{\sqrt{\lambda_l^2(a)+\varepsilon^2}} c_l \right).
\end{aligned}$$

Observe that since $\langle c_j, c_l\rangle = 0$, with $\theta_{jl} := \min\left\{\sqrt{\lambda_j^2(a)+\varepsilon^2}, \sqrt{\lambda_l^2(a)+\varepsilon^2}\right\}$, we deduce

$$\begin{aligned}
\left\|\frac{f_1^{jl}}{\sqrt{\lambda_j^2(a)+\varepsilon^2}}c_j + \frac{f_2^{jl}}{\sqrt{\lambda_l^2(a)+\varepsilon^2}}c_l\right\|^2 &= \left\|\frac{f_1^{jl}}{\sqrt{\lambda_j^2(a)+\varepsilon^2}}c_j\right\|^2 + \left\|\frac{f_2^{jl}}{\sqrt{\lambda_l^2(a)+\varepsilon^2}}c_l\right\|^2 \\
&\le \left\|\frac{f_1^{jl}}{\theta_{jl}}c_j\right\|^2 + \left\|\frac{f_2^{jl}}{\theta_{jl}}c_l\right\|^2 \\
&= \frac{1}{\theta_{jl}^2}\left(\|f_1^{jl}c_j\|^2 + \|f_2^{jl}c_l\|^2\right) \\
&= \frac{1}{\theta_{jl}^2}\|f_1^{jl}c_j + f_2^{jl}c_l\|^2 \quad (\text{by } \langle c_j, c_l\rangle = 0) \\
&= \frac{1}{\theta_{jl}^2}\|x_{jl}\circ u_{jl}\|^2 \\
&= \left\|\frac{1}{\theta_{jl}}x_{jl}\circ u_{jl}\right\|^2 \\
&\le \left\|\frac{1}{\theta_{jl}}x_{jl}\right\|^2 \|u_{jl}\|^2 \quad (\text{by Lemma 2.3}) \\
&\le 2\|u_{jl}\|^2 \quad \left(\theta_{jl} \ge \frac{1}{\sqrt{2}}\|x_{jl}\| \text{ by (2.9)}\right) \\
&\le 2\|u\|^2.
\end{aligned}$$

That is,

$$\left\|\frac{f_1^{jl}}{\sqrt{\lambda_j^2(a)+\varepsilon^2}}c_j + \frac{f_2^{jl}}{\sqrt{\lambda_l^2(a)+\varepsilon^2}}c_l\right\| \le \sqrt{2}\|u\|.$$

Therefore, we have

$$\begin{aligned}
\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left(\sum_{1\le j<l\le r} x_{jl}\circ u_{jl}\right)\right\| &= \left\|\sum_{1\le j<l\le r}\left(\frac{f_1^{jl}}{\sqrt{\lambda_j^2(a)+\varepsilon^2}}c_j + \frac{f_2^{jl}}{\sqrt{\lambda_l^2(a)+\varepsilon^2}}c_l\right)\right\| \\
&\le \sum_{1\le j<l\le r}\left\|\frac{f_1^{jl}}{\sqrt{\lambda_j^2(a)+\varepsilon^2}}c_j + \frac{f_2^{jl}}{\sqrt{\lambda_l^2(a)+\varepsilon^2}}c_l\right\| \\
&\le \sum_{1\le j<l\le r}\sqrt{2}\|u\| \\
&= \frac{\sqrt{2}}{2}r(r-1)\|u\|. \qquad (2.11)
\end{aligned}$$

Set $\xi := \sum_{i=1}^r x_iu_ic_i + \sum_{1\le j<l\le r}\left(\frac{x_j+x_l}{2}u_{jl} + \frac{u_j+u_l}{2}x_{jl}\right) + \sum_{1\le i<j<k\le r}(x_{ij}\circ u_{jk} + x_{jk}\circ u_{ij})$. Note that by Theorem 2.2, $(x_{ij}\circ u_{jk} + x_{jk}\circ u_{ij}) \in J_{ik}$. Similarly, by Proposition 2.4, we

have

$$\begin{aligned}
\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(\xi) &= \sum_{i=1}^{r} \frac{x_i u_i}{\sqrt{\lambda_i^2(a)+\varepsilon^2}} c_i \\
&+ \sum_{1\le j<l\le r} \frac{2}{\sqrt{\lambda_j^2(a)+\varepsilon^2}+\sqrt{\lambda_l^2(a)+\varepsilon^2}} \left(\frac{x_j+x_l}{2} u_{jl} + \frac{u_j+u_l}{2} x_{jl} \right) \\
&+ \sum_{1\le i<j<k\le r} \frac{2}{\sqrt{\lambda_i^2(a)+\varepsilon^2}+\sqrt{\lambda_k^2(a)+\varepsilon^2}} \left(x_{ij}\circ u_{jk} + x_{jk}\circ u_{ij} \right).
\end{aligned}$$

Thus, we obtain from Lemma 2.3 and inequalities (2.9) and (2.10) that

$$\begin{aligned}
\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(\xi)\| &\le \sum_{i=1}^{r} |u_i| + \sum_{1\le j<l\le r} (\|u_{jl}\| + |u_j| + |u_l|) + \sum_{1\le i<j<k\le r} (\|u_{jk}\| + \|u_{ij}\|) \\
&\le r\|u\| + \frac{1+2\sqrt{2}}{2} r(r-1)\|u\| + \sqrt{2} r(r-1)\|u\|. \qquad (2.12)
\end{aligned}$$

So, combining the above inequalities (2.11) and (2.12), we have

$$\begin{aligned}
\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right\| &= \left\| \mathcal{L}^{-1}_{a_\varepsilon(x,y)} \left(\xi + \sum_{1\le j<l\le r} x_{jl}\circ u_{jl} \right) \right\| \\
&\le \left[r\|u\| + \frac{1+2\sqrt{2}}{2} r(r-1)\|u\| + \sqrt{2} r(r-1)\|u\| \right] + \frac{\sqrt{2}}{2} r(r-1)\|u\| \\
&= \left(\frac{1+5\sqrt{2}}{2} r^2 - \frac{5\sqrt{2}-1}{2} r \right) \|u\|.
\end{aligned}$$

Letting $\gamma \ge \frac{1+5\sqrt{2}}{2} r^2 - \frac{5\sqrt{2}-1}{2} r$, we obtain the desired inequality. □

3 Lipschitz Continuity of FB Function

In this section, we establish the Lipschitz continuity of FB C-function. For this purpose, we need the following result about the derivative of $x^{\frac{1}{2}}$ in $\mathrm{int}(K)$, the *interior* of K.

Lemma 3.1. *The function $x^{\frac{1}{2}}$ is smooth at every $x \in \mathrm{int}(K)$. Moreover, it holds*

$$\nabla(x^{\frac{1}{2}}) = \frac{(\mathcal{L}(x^{\frac{1}{2}}))^{-1}}{2} \quad \text{for every} \quad x \in \mathrm{int}(K). \qquad (3.1)$$

Proof. Clearly, $x^{\frac{1}{2}}$ is smooth at $x \in \mathrm{int}(K)$. For the second part of the lemma, suppose that $(x+h)^{\frac{1}{2}} - x^{\frac{1}{2}} = Sh + o(\|h\|)$ for some linear operator S. Multiplying both sides of this equation by $(x+h)^{\frac{1}{2}} + x^{\frac{1}{2}}$, we have

$$((x+h)^{\frac{1}{2}} + x^{\frac{1}{2}}) \circ ((x+h)^{\frac{1}{2}} - x^{\frac{1}{2}}) = ((x+h)^{\frac{1}{2}} + x^{\frac{1}{2}}) \circ (Sh + o(\|h\|)).$$

Direct computation yields $h = ((x+h)^{\frac{1}{2}} + x^{\frac{1}{2}}) \circ Sh + o(\|h\|)$ or $h = 2x^{\frac{1}{2}} \circ (Sh) + o(\|h\|)$, using $(x+h)^{\frac{1}{2}} = x^{\frac{1}{2}} + Sh + o(\|h\|)$ and $Sh \circ Sh = o(\|h\|)$. That is, $h = \mathcal{L}(2x^{\frac{1}{2}})(Sh) + o(\|h\|)$. Hence, $S = (\mathcal{L}(2x^{\frac{1}{2}}))^{-1} = \frac{(\mathcal{L}(x^{\frac{1}{2}}))^{-1}}{2}$ by the linearity of Lyapunov transformation. □

We now prove the Lipschitz property of $(x^2+y^2)^{\frac{1}{2}}$.

Lemma 3.2. *The function $(x^2+y^2)^{\frac{1}{2}}$ is globally Lipschitz continuous everywhere in $\mathcal{J}\times\mathcal{J}$.*

Proof. Fix $x, y \in \mathcal{J}$, let $a(x,y) := (x^2+y^2)^{\frac{1}{2}}$ and $a_\varepsilon(x,y) := (x^2+y^2+\varepsilon^2 e)^{\frac{1}{2}}$ for $\varepsilon \neq 0$. Note that for any $u, v \in \mathcal{J}$,

$$\begin{aligned}
& \|a_\varepsilon(x+u,y+v)-a_\varepsilon(x,y)\| \\
= \; & \|a_\varepsilon(x+u,y+v)-a_\varepsilon(x,y+v)+a_\varepsilon(x,y+v)-a_\varepsilon(x,y)\| \\
= \; & \left\|\int_0^1 \mathcal{L}^{-1}_{a_\varepsilon(x+tu,y+v)}\mathcal{L}(x+tu)u dt + \int_0^1 \mathcal{L}^{-1}_{a_\varepsilon(x,y+tv)}\mathcal{L}(y+tv)v dt\right\| \\
= \; & \left\|\int_0^1 \mathcal{L}^{-1}_{a_\varepsilon(x+tu,y+v)}((x+tu)\circ u) dt + \int_0^1 \mathcal{L}^{-1}_{a_\varepsilon(x,y+tv)}((y+tv)\circ v) dt\right\| \\
\leq \; & \int_0^1 \left\|\mathcal{L}^{-1}_{a_\varepsilon(x+tu,y+v)}((x+tu)\circ u)\right\| dt + \int_0^1 \left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y+tv)}((y+tv)\circ v)\right\| dt \\
\leq \; & \int_0^1 \gamma\|u\| dt + \int_0^1 \gamma\,\|v\|\, dt \\
= \; & \gamma(\|u\|+\|v\|) \\
\leq \; & \sqrt{2}\gamma\|(u,v)\|,
\end{aligned}$$

where the second equality holds by the Mean Value Theorem and Lemma 3.1, the first inequality holds by Lemma 2.7 and γ is only dependent on r, and the last inequality follows from the fact $\|u\|+\|v\| \leq \sqrt{2}\sqrt{\|u\|^2+\|v\|^2} = \sqrt{2}\|(u,v)\|$. Thus, we deduce

$$\|a_\varepsilon(x+u,y+v)-a_\varepsilon(x,y)\| \quad \leq \quad \sqrt{2}\gamma\|(u,v)\|.$$

The desired conclusion follows by taking $\varepsilon \to 0$ in the inequality above. □

As a consequence of the lemma above, we immediately obtain the Lipschitz continuity of FB function.

Theorem 3.3. *The FB function Φ_{FB} (given by (1.3)) is globally Lipschitz continuous everywhere in $\mathcal{J}\times\mathcal{J}$.*

4 Lipschitz Continuity of the Derivatives of Ψ_{FB}

This section deals with Lipschitz continuity of the derivatives of the squared norm of FB C-function. The main result is stated below.

Theorem 4.1. *The derivatives of the squared norm of the Fischer-Burmeister function Ψ_{FB} (given by (1.6)) are Lipschitz continuous everywhere in $\mathcal{J}\times\mathcal{J}$.*

Our proof relies on four lemmas. First, we focus on $\mathcal{L}_a^{-1}(x+y)\circ x$ where $a=(x^2+y^2)^{\frac{1}{2}}$ in the subsequent analysis. Observe that a may have eigenvalues that are zero. For the sake of simplicity, we look at a "smoothed" counterpart $a_\varepsilon(x,y)$. Let $S_\varepsilon(x,y) := \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ x$, we have the following.

Lemma 4.2. *Let $u, v \in \mathcal{J}$ be given. Then for every $x, y \in \mathcal{J}$ and $\varepsilon \neq 0$ we have*

$$[\nabla_x S_\varepsilon(x,y)]\, u = \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u + \mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left[u - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right]\circ x,$$

$$[\nabla_y S_\varepsilon(x,y)]\, v = \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ v + \mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left[v - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(y\circ v)\right]\circ y.$$

Proof. Fix $u \in \mathcal{J}$. Set $z := a_\varepsilon(x+u,y) - a_\varepsilon(x,y)$ and $w := 2x \circ u + u^2$. Noting that

$$a_\varepsilon(x+u,y) = [(x+u)^2 + y^2 + \varepsilon^2 e]^{\frac{1}{2}} = [(x^2+y^2+\varepsilon^2 e) + 2x\circ u + u^2]^{\frac{1}{2}},$$

we have $z = [a_\varepsilon^2(x,y) + w]^{\frac{1}{2}} - a_\varepsilon(x,y)$. Note that $J_{a_\varepsilon(x,y)} = \mathcal{J}$ from (2.5) and $|\wp(a)| = r$. From Lemma 6.6(2) in [16], it follows that

$$z = \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(2x\circ u + u^2) + o(\|u\|) = 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u) + o(\|u\|).$$

Thus $z \to 0$ as $u \to 0$ and $z = O(\|u\|)$. Let

$$\eta := \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y) \ \text{ and } \ \eta + h := \mathcal{L}^{-1}_{a_\varepsilon(x,y)+z}(x+u+y).$$

It is easy to see that $a_\varepsilon(x,y)\circ\eta = x+y$ and $[a_\varepsilon(x,y)+z]\circ(\eta+h) = x+u+y$. So, $a_\varepsilon(x,y)\circ h = u - z\circ\eta - z\circ h$, or

$$h = \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(u - z\circ\eta) - \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(z\circ h).$$

Since $z\to 0$ as $u \to 0$ and $z = O(\|u\|)$, $h\to 0$ as $u\to 0$ and $h\circ z = o(\|z\|) = o(\|u\|)$. We deduce

$$\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(z\circ h) = o(\|u\|) \ \text{ and } h = \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(u - z\circ\eta) + o(\|u\|).$$

Thus, direct calculation yields

$$\begin{aligned}
& S_\varepsilon(x+u,y) - S_\varepsilon(x,y)\\
= \ & \mathcal{L}^{-1}_{a_\varepsilon(x+u,y)}(x+u+y)\circ(x+u) - \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ x\\
= \ & (\eta+h)\circ(x+u) - \eta\circ x\\
= \ & \eta\circ u + h\circ(x+u)\\
= \ & \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u + \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(u - z\circ\eta)\circ(x+u) + o(\|u\|)\circ(x+u)\\
= \ & \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u + \mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left[u - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right]\circ(x+u) + o(\|u\|)\\
= \ & \mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u + \mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left[u - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right]\circ x + o(\|u\|).
\end{aligned}$$

This establishes the formula for $[\nabla_x S_\varepsilon(x,y)]u$. The formula for $[\nabla_y S_\varepsilon(x,y)]v$ follows from the symmetry of x and y. □

Lemma 4.3. *For every x, y, u and $v \in \mathcal{J}$,*

$$\|[\nabla_x S_\varepsilon(x,y)]u\| \le \rho\|u\|, \quad \|[\nabla_y S_\varepsilon(x,y)]v\| \le \rho\|v\|$$

where ρ is only dependent on the rank of $\mathcal{J}$, which can be taken as $\rho = 20r^{10} + 3r^4$.

Proof. Using Lemmas 2.7 and 4.2, we deduce

$$\begin{aligned}
& \|[\nabla_x S_\varepsilon(x,y)]u\|\\
= \ & \left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u + \mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left[u - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right]\circ x\right\|\\
\le \ & \left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ u\right\| + \left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}\left[u - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right]\circ x\right\|\\
\le \ & 2\beta\|u\| + \beta\left\|u - 2\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right\|\\
\le \ & 2\beta\|u\| + \beta\left(\|u\| + 2\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x+y)\circ\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right\|\right)\\
\le \ & 2\beta\|u\| + \beta\left(\|u\| + 4\beta\left\|\mathcal{L}^{-1}_{a_\varepsilon(x,y)}(x\circ u)\right\|\right)\\
\le \ & 2\beta\|u\| + \beta(\|u\| + 4\beta\gamma\|u\|),
\end{aligned}$$

where β, γ are given as in Lemma 2.7. Therefore, by taking $\rho := 3\beta + 4\beta^2\gamma$, we obtain that $\|[\nabla_x S_\varepsilon(x,y)]u\| \le \rho\|u\|$ with ρ only dependent on r and $\rho = 20r^{10} + 3r^4$ if $\beta = r^4$ and $\gamma = 5r^2$. As in the proof of the last lemma, $\|[\nabla_y S_\varepsilon(x,y)]v\| \le \rho\|v\|$ follows from symmetry. □

Now we are in a position to prove the Lipschitz continuity of $\mathcal{L}_a^{-1}(x+y) \circ x$.

Lemma 4.4. *$\mathcal{L}_a^{-1}(x+y) \circ x$ is globally Lipschitz continuous for all $x, y \in \mathcal{J}$.*

Proof. We first show that $S_\varepsilon(x,y) = \mathcal{L}_{a_\varepsilon(x,y)}^{-1}(x+y) \circ x$ is globally Lipschitz continuous for all $x, y \in \mathcal{J}$. For every $u, v \in \mathcal{J}$,

$$S_\varepsilon(x+u, y+v) - S_\varepsilon(x,y) = S_\varepsilon(x+u, y+v) - S_\varepsilon(x, y+v) + S_\varepsilon(x, y+v) - S_\varepsilon(x,y).$$

By Lemma 4.2, $\nabla_x S_\varepsilon(x,y)$ is continuous in x. Thus, it follows by the Mean Value Theorem that

$$S_\varepsilon(x+u, y+v) - S_\varepsilon(x, y+v) = \int_0^1 [\nabla_x S_\varepsilon(x+tu, y+v)]udt.$$

By Lemma 4.3, it follows that $\|[\nabla_x S_\varepsilon(x+tu, y+v)]u\| \le \rho\|u\|$. Hence,

$$\begin{aligned}
\|S_\varepsilon(x+u, y+v) - S_\varepsilon(x, y+v)\| &= \left\| \int_0^1 [\nabla_x S_\varepsilon(x+tu, y+v)]udt \right\| \\
&\le \int_0^1 \|[\nabla_x S_\varepsilon(x+tu, y+v)]u\|\, dt \\
&\le \int_0^1 \rho\|u\|dt = \rho\|u\|.
\end{aligned}$$

That is, $\|S_\varepsilon(x+u, y+v) - S_\varepsilon(x, y+v)\| \le \rho\|u\|$ with ρ only dependent on r.

Likewise, we have $\|S_\varepsilon(x, y+v) - S_\varepsilon(x,y)\| \le \rho\|v\|$. We therefore obtain that $\|S_\varepsilon(x+u, y+v) - S_\varepsilon(x,y)\| \le \rho(\|u\| + \|v\|)$, or

$$\left\| \mathcal{L}_{a_\varepsilon(x+u,y+v)}^{-1}(x+u+y+v) \circ (x+u) - \mathcal{L}_{a_\varepsilon(x,y)}^{-1}(x+y) \circ x \right\| \le \rho(\|u\| + \|v\|).$$

Note that $\mathcal{L}_{a_\varepsilon(x,y)}^{-1}(x+y) \circ x \to \mathcal{L}_a^{-1}(x+y) \circ x$ as $\varepsilon \to 0$ by Proposition 2.6. Letting $\varepsilon \to 0$ in the inequality above, we obtain the desired result. □

Before proving Theorem 4.1, we need to recall a lemma by Liu, Zhang and Wang [16].

Lemma 4.5. *(Lemma 6.7, [16]) $\Psi_{FB}(x,y)$ is differentiable at every $(x,y) \in \mathcal{J} \times \mathcal{J}$, and if $(x,y) = (0,0)$, then $\nabla_x \Psi_{FB}(0,0) = \nabla_y \Psi_{FB}(0,0) = 0$; if $(x,y) \ne (0,0)$, then*

$$\begin{aligned}
\nabla_x \Psi_{FB}(x,y) &= \mathcal{L}_a^{-1}(a-x-y) \circ (x-a), \\
\nabla_y \Psi_{FB}(x,y) &= \mathcal{L}_a^{-1}(a-x-y) \circ (y-a),
\end{aligned}$$

where $a = (x^2+y^2)^{\frac{1}{2}}$.

Proof of Theorem 4.1. It is easy to see that

$$\Psi_{FB}(x,y) = \frac{1}{2}\langle x^2+y^2, e\rangle + \frac{1}{2}\langle (x+y)^2, e\rangle - \langle (x^2+y^2)^{\frac{1}{2}}, x+y\rangle.$$

Let $R(x,y) := \langle (x^2+y^2)^{\frac{1}{2}}, x+y\rangle$. Then

$$R(x,y) = \frac{1}{2}\langle x^2+y^2, e\rangle + \frac{1}{2}\langle (x+y)^2, e\rangle - \Psi_{FB}(x,y).$$

Set $a := (x^2+y^2)^{\frac{1}{2}}$. By Lemma 4.5, it is straightforward to derive that

$$\begin{array}{rcl} \nabla_x R(x,y) & = & x + (x+y) - \nabla_x \Psi_{FB}(x,y) \\ & = & 2x + y - \mathcal{L}_a^{-1}(a-x-y)\circ x + \mathcal{L}_a^{-1}(a-x-y)\circ a \\ & = & 2x + y - \left[\mathcal{L}_a^{-1}(a) - \mathcal{L}_a^{-1}(x+y)\right]\circ x + (a-x-y) \\ & = & \mathcal{L}_a^{-1}(x+y)\circ x + (x^2+y^2)^{\frac{1}{2}}, \end{array}$$

where the third equality holds by $\mathcal{L}_a^{-1}(a-x-y)\circ a = a-x-y$ because Lemma 6.6(1) of [16] implies that $a-x-y$ lies in the space J_a where $\mathcal{L}_a^{-1}$ makes sense, and the fourth by $\mathcal{L}_a^{-1}(a)\circ x = x$. Combining Lemmas 3.2 and 4.4, we conclude that $\nabla_x R(x,y)$ is globally Lipschitz.

By symmetry in x and y, $\nabla_y R(x,y)$ is also globally Lipschitz continuous. □

$\boxed{5}$ Final Remarks

In this article, we studied some properties of the Lyapunov operator, and using these properties we established Lipschitz continuity of FB function and the derivatives of squared norm of FB function.

Sun and Sun [18] showed that the FB function is strongly semismooth everywhere in the cases of SDCP and SOCCP. However, it is not clear whether FB function Φ_{FB} (given by (1.3)) is strongly semismooth? We leave this question as a future research topic.

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