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FURTHER RESULTS ON THE LOWER SEMICONTINUITY OF EFFICIENT POINT MULTIFUNCTIONS*

T.D. Chuong, J.-C. Yao† and N.D. Yen

Abstract: Using the approach of Bednarczuk [1, 3] and introducing the new concepts of *local containment property*, *K-local domination property* and *uniformly local closedness* of a multifunction around a given point, we obtain further results on the lower semicontinuity of efficient point multifunctions taking values in Hausdorff topological vector spaces. The new theorems sharpen the corresponding ones in [1, 3].

Key words: *efficient point multifunction, lower semicontinuity, (local) containment property, (K-local) domination property, uniformly local closedness*

Mathematics Subject Classification: *49K40, 46N10, 90C29, 90C31*

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1 Introduction

This paper is motivated by the stability theory of parametric vector optimization problems. We first give some notations and definitions.

Let $f : X \times P_1 \to Y$ be a vector function, $C : P_2 \rightrightarrows X$ a multifunction, where X, Y are Hausdorff topological vector spaces, P_1 and P_2 are topological spaces. Given a pointed (i.e., $K \cap (-K) = \{0_Y\}$) closed convex cone $K \subset Y$, we consider the following *standard parametric vector optimization problem*

$$\min_K \left\{ f(x, p_1) \,|\, x \in C(p_2) \right\} \tag{1.1}$$

depending on the parameter pair $(p_1, p_2) \in P_1 \times P_2$.

Definition 1.1. We write $x \in \mathcal{S}(p_1, p_2)$ to indicate that x an *efficient solution* of (1.1) if

$$(f(x, p_1) - K) \cap f(C(p_2), p_1) = \{f(x, p_1)\},$$

where $f(C(p_2), p_1) = \{f(x, p_1) \,|\, x \in C(p_2)\}$ is the image of the *constraint set* $C(p_2)$ via the *objective function* $f(\cdot, p_1)$. We call $\mathcal{S} : P_1 \times P_2 \rightrightarrows X$ the *efficient solution map* (or the *Pareto solution map*) of (1.1).

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†Corresponding author.

Definition 1.2. We say that $y \in A$ is an *efficient point* of a subset $A \subset Y$ with respect to K, and write $y \in E(A|K)$, whenever $(y - K) \cap A = \{y\}$. Given a multifunction $F : P \rightrightarrows Y$, where P is a topological space, we put $\mathcal{F}(p) = E(F(p)|K)$ for all $p \in P$ and call $\mathcal{F} : P \rightrightarrows Y$ the *efficient point multifunction* corresponding to the quadruplet $\{F, P, Y, K\}$.

In view of Defs. 1.1 and 1.2, setting $P = P_1 \times P_2$ and $F(p) = f(C(p_2), p_1)$ for every $p = (p_1, p_2) \in P$, we have

$$\mathcal{S}(p) = \{x \in C(p_2) \,|\, f(x, p_1) \in \mathcal{F}(p)\} \quad \forall p = (p_1, p_2) \in P. \tag{1.2}$$

Formula (1.2) describes a way of computing the efficient solution set $\mathcal{S}(p)$ via the efficient point set $\mathcal{F}(p)$. Note that the efficient point multifunction $\mathcal{F}(\cdot)$ in Def. 1.2 is a special case of the efficient solution map $\mathcal{S}(\cdot)$ in Def. 1.1. To show this, for a given quadruplet $\{F, P, Y, K\}$, it suffices to put $P_1 = \{0_Y\}$, $P_2 = P$, $X = Y$, $f(x, 0_Y) = x$ for every $x \in X$, $C(p) = F(p)$ for every $p \in P$, and observe that

$$\mathcal{F}(p) = \mathcal{S}(0_Y, p) \quad \forall p \in P. \tag{1.3}$$

The identity in (1.3) allows us to interpret $\mathcal{F}(\cdot)$ as $\mathcal{S}(0_Y, \cdot)$ - the efficient solution map of a parametric vector optimization problem with the identity objective function.

Stability analysis in vector optimization has a long history. The papers by Naccache [14], Tanino and Sawaragi [21] are among the first results in this field. One of the main problems here is to find sufficient conditions for $\mathcal{S}(\cdot)$ and/or $\mathcal{F}(\cdot)$ to have a certain continuity properties. For instance, the lower (upper) semicontinuity of the efficient point multifunction have been examined by Penot [15, 16]. Various stability results on the efficient solution map and the efficient point multifunction can be found in the books by Tanino, Sawaragi and Nakayama [20], Luc [13]. More recently, Xiang and Zhou [18], Xiang and Yin [19] have characterized the lower and upper semicontinuity of $\mathcal{S}(\cdot)$ in the case where P_2 is a singleton and $f(\cdot, p_1)$, for every $p_1 \in P_1$, is an element of the space of continuous functions defined on a nonempty compact set with the values in $\mathbb{R}^m$. Using the so-called *domination property* and *containment property*, Bednarczuk [1, 2, 3, 5] studied the Hausdorff upper semicontinuity, the K-Hausdorff upper semicontinuity and the lower (upper) semicontinuity of $\mathcal{S}(\cdot)$ and/or $\mathcal{F}(\cdot)$.

The purpose of this paper is to show that most of the principal results of Bednarczuk in [1, 3] are valid under weaker assumptions. Namely, after presenting some preliminaries in Sect. 2, in Sects. 3–5 we propose new concepts called *local containment property*, *K-local domination property* and *uniformly local closedness* of a multifunction around a given point. Then, in those three sections, we derive various sufficient conditions for the lower semicontinuity of efficient point multifunctions taking values in Hausdorff topological vector spaces. The new theorems sharpen the corresponding ones in [1, 3]. A series of examples are provided for analyzing the obtained results and for comparing them with the preceding results of Bednarczuk.

2 Preliminaries

Let F, P, Y, K be as in Def. 1.2. The effective domain of the multifunction F is given by the formula $\mathrm{dom}F = \{p \in P \mid F(p) \neq \emptyset\}$. We denote by $\mathcal{N}(p)$ (resp., $\mathcal{N}(y)$) the set of all neighborhoods of $p \in P$ (resp., $y \in Y$). By $\mathcal{N}_B(0_Y)$ we denote the set of all balanced neighborhoods of 0_Y. Thus, $V \in \mathcal{N}_B(0_Y)$ iff V is a neighborhood of 0_Y and $\lambda V \subset V$ for all

$\lambda \in [-1, 1]$. It is well known (see W. Rudin, Functional Analysis, 1973, p. 12) that for any $W \in \mathcal{N}(0_Y)$ there exists $V \in \mathcal{N}_B(0_Y)$ satisfying $V + V \subset W$.

Given a subset $\Omega \subset Y$, we denote the interior and the closure of Ω respectively by $\mathrm{int}\Omega$ and $\mathrm{cl}\Omega$.

We shall encounter very frequently with the notions described in the next definition.

Definition 2.1. (i) F is *upper semicontinuous* (usc for brevity) at $p_0 \in P$ if for every open set V containing $F(p_0)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $F(p) \subset V$ for all $p \in U_0$.
(ii) F is said to be *lower semicontinuous* (lsc) at $p_0 \in \mathrm{dom}\, F$ if for any open set $V \subset Y$ satisfying $V \cap F(p_0) \neq \emptyset$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $V \cap F(p) \neq \emptyset$ for all $p \in U_0$.
(iii) F is said to be *continuous* at p_0 if it is both usc and lsc at p_0.
(iv) F is *Hausdorff upper semicontinuous* (H-usc) at p_0 if for every $W \in \mathcal{N}(0_Y)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $F(p) \subset F(p_0) + W$ for all $p \in U_0$.
(v) (This notion was called *inf-lower continuity* in [16].) F is said to be *K-lower semicontinuous* (K-lsc) at $p_0 \in \mathrm{dom}\, F$ if for any open set $V \subset Y$ satisfying $V \cap F(p_0) \neq \emptyset$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $(V + K) \cap F(p) \neq \emptyset$ for all $p \in U_0$.

Consider the following pairs of closely-related assumptions on the upper/lower semicontinuity of F :
(A_1) F is H-usc and lsc at p_0;
(A_2) F is H-usc and $(-K)$-lsc at p_0.

It is easy to show that (A_1) implies (A_2).

Definition 2.2. (i) The *domination property*, denoted by (DP), is said to hold for $A \subset Y$ if

$$A \subset E(A|K) + K.$$

(ii) The *containment property*, denoted by (CP), is said to hold for $A \subset Y$ if for each $W \in \mathcal{N}(0_Y)$ there exists $V \in \mathcal{N}(0_Y)$ such that

$$[A \backslash (E(A|K) + W)] + V \subset E(A|K) + K.$$

The domination property has been used by many authors (see e.g. [1]–[5], [11],[13]).

The containment property was proposed by Bednarczuk in [1]. Later on, the property was used by Bednarczuk [2, 3, 4, 5], Dolecki and El Ghali in [9].

Relationships between (CP) and (DP) were established in [5]. An equivalent form of (CP) for cones with nonempty interiors was given in [4]. For the convenience of the reference of the reader, we recall this result with a proof.

Proposition 2.3 (see [4, Prop. 2.2]). *Let A be a subset of Y. If $\mathrm{int}K \neq \emptyset$, then the following two properties are equivalent:*
(i) (CP) *holds for A;*
(ii) *For each $W \in \mathcal{N}(0_Y)$ there is $W_0 \in \mathcal{N}(0_Y)$ such that for all*

$$y \in A \backslash (E(A|K) + W)$$

there exist $\eta_y \in E(A|K)$ and $k_y \in K$ satisfying

$$y = \eta_y + k_y, \quad k_y + W_0 \subset K.$$

Proof. (i) *implies* (ii): For each $W \in \mathcal{N}(0_Y)$, we put $K_W = \{k \in K | k + W \subset K\}$. Note that $\text{int}K = \bigcup_{W\in\mathcal{N}(0_Y)} K_W$. We shall show that for any $V \in \mathcal{N}(0_Y)$ there exists $W_V \in \mathcal{N}(0_Y)$ such that

$$\{y \in Y \mid y + V \subset E(A|K) + K\} \subset E(A|K) + K_{W_V}. \tag{2.1}$$

Indeed, since $0_Y \in (-K) = \text{cl}(-K) = \text{cl}(-\text{int}K)$, it follows that there exists $W_V \in \mathcal{N}(0_Y)$ satisfying $V \cap (-K_{W_V}) \neq \emptyset$. Choose $y_V \in V \cap (-K_{W_V})$. Take any $z \in \{y \in Y \mid y + V \subset E(A|K) + K\}$, i.e., $z + V \subset E(A|K) + K$. We have $z + y_V \in E(A|K) + K$, therefore

$$z \in E(A|K) + K + K_{W_V} \subset E(A|K) + K_{W_V}.$$

Next, take any $W \in \mathcal{N}(0_Y)$. Since (CP) holds for A there exists $V \in \mathcal{N}(0_Y)$ such that

$$[A\backslash(E(A|K) + W)] + V \subset E(A|K) + K. \tag{2.2}$$

By virtue of (2.1) we can find $W_V \in \mathcal{N}(0_Y)$ satisfying

$$\{y \in Y \mid y + V \subset E(A|K) + K\} \subset E(A|K) + K_{W_V}. \tag{2.3}$$

For each $y \in A\backslash(E(A|K) + W)$, it follows from (2.2) and (2.3) that

$$y + V \subset E(A|K) + K \subset E(A|K) + K_{W_V}.$$

Hence, there exist $\eta_y \in E(A|K)$ and $k_y \in K$ such that

$$y = \eta_y + k_y, \quad k_y + W_0 \subset K$$

where $W_0 := W_V$.
(ii) *implies* (i): This implication is obvious. □

Both properties (DP) and (CP) can serve well as *qualification conditions* for having the usc and lsc properties of $\mathcal{F}(\cdot)$ (see [1]–[3],[5],[9]). Further, an related notion to (CP), called the *containment rate function*, was used in [4] for studying the upper Hölder semicontinuity of the efficient point multifunction.

In relation to the multifunction F around a point $p_0 \in P$, the (DP) and (CP) properties can be formulated as follows.

Definition 2.4. We say the *domination property*, again denoted by (DP), *holds for F uniformly around* p_0 if there exists $U_0 \in \mathcal{N}(p_0)$ such that

$$F(p) \subset \mathcal{F}(p) + K \quad \forall p \in U_0.$$

Definition 2.5 (see [1]). The *containment property*, again denoted by (CP), *holds for F uniformly around* p_0 if $\forall W \in \mathcal{N}(0_Y)$, $\exists V \in \mathcal{N}(0_Y)$, $\exists U_0 \in \mathcal{N}(p_0)$ such that

$$[F(p)\backslash(\mathcal{F}(p) + W)] + V \subset \mathcal{F}(p) + K \quad \forall p \in U_0.$$

In the sequel, we will relax the above notions by introducing the concepts called the *local containment property*, the *K-local domination property* of a multifunction around a given point. With the help of these concepts, we will extend the main results of [1, 3].

$\boxed{3}$ Local Containment Property and the Lower Semicontinuity

The following definition gives a weaker form of the notion uniform containment property of a multifunction around a point in Def. 2.5.

Definition 3.1. We say the *local containment property*, denoted by (locCP), *holds for* F *uniformly around* p_0 if for each $y_0 \in \mathcal{F}(p_0)$ there exists $V_0 \in \mathcal{N}(0_Y)$ such that $\forall W \in \mathcal{N}(0_Y)$, $\exists V \in \mathcal{N}(0_Y)$, $\exists U_0 \in \mathcal{N}(p_0)$ satisfying

$$[((y_0 + V_0) \cap F(p)) \backslash (\mathcal{F}(p) + W)] + V \subset \mathcal{F}(p) + K \quad \forall p \in U_0.$$

By means of the same arguments as in the proof of Proposition 2.3 one can show that under the condition $\mathrm{int}K \neq \emptyset$, (locCP) holds for F uniformly around p_0 if and only if for each $y_0 \in \mathcal{F}(p_0)$ there exists $V_0 \in \mathcal{N}(0_Y)$ such that $\forall W \in \mathcal{N}(0_Y)$, $\exists W_0 \in \mathcal{N}(0_Y)$, $\exists U_0 \in \mathcal{N}(p_0)$ satisfying $\forall p \in U_0, \forall y \in [(y_0 + V_0) \cap F(p)] \backslash (\mathcal{F}(p) + W)$, $\exists \eta_y \in \mathcal{F}(p)$ $\exists k_y \in K$,

$$y = \eta_y + k_y, \quad k_y + W_0 \subset K.$$

It is clear that if (CP) holds for F uniformly around p_0 then (locCP) holds for F uniformly around p_0. The converse is not true in general (see Example 3.5 below). This means that (locCP) is really weaker than (CP).

Theorem 3.2. *Suppose that* $\mathrm{int}K \neq \emptyset$ *and* (locCP) *holds for* F *uniformly around* p_0. *If* (A_1) *is satisfied, then* $\mathcal{F}$ *is lsc at* p_0.

Proof. Let $y_0 \in \mathcal{F}(p_0)$ and $W \in \mathcal{N}(0_Y)$. The proof will be compeleted if we can show that there exists $U_W \in \mathcal{N}(p_0)$ such that

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_W. \tag{3.1}$$

Take any $W_1 \in \mathcal{N}_B(0_Y)$ satisfying $W_1 + W_1 \subset W$. By our assumptions there exist $V_0 \in \mathcal{N}(0_Y)$, $W_0 \in \mathcal{N}(0_Y)$ and $U_0 \in \mathcal{N}(p_0)$ such that for all $p \in U_0$ and $y \in [(y_0 + V_0) \cap F(p)] \backslash (\mathcal{F}(p) + W_1)$ we can find $\eta_y \in \mathcal{F}(p)$ and $k_y \in K$ satisfying

$$y = \eta_y + k_y, \quad k_y + W_0 \subset K. \tag{3.2}$$

Choose $W_2 \in \mathcal{N}_B(0_Y), W_2 + W_2 \subset W_0$. Since F is lsc at p_0, there exists $U_1 \in \mathcal{N}(p_0)$, $U_1 \subset U_0$, such that

$$[y_0 + (V_0 \cap W_1 \cap W_2)] \cap F(p) \neq \emptyset \quad \forall p \in U_1.$$

For each $p \in U_1$, we take a point

$$y_p \in [y_0 + (V_0 \cap W_1 \cap W_2)] \cap F(p). \tag{3.3}$$

As F is H-usc at p_0, there exists $U_2 \in \mathcal{N}(p_0)$, $U_2 \subset U_1$, such that

$$F(p) \subset F(p_0) + (V_0 \cap W_1 \cap W_2) \quad \forall p \in U_2. \tag{3.4}$$

Suppose first that there exists $\bar{U} \in \mathcal{N}(p_0), \bar{U} \subset U_0$, such that

$$y_p \in \mathcal{F}(p) + W_1 \quad \forall p \in \bar{U}. \tag{3.5}$$

For each $p \in U_1 \cap \bar{U}$, from (3.3) and (3.5) it follows that there exist $w_p \in V_0 \cap W_1 \cap W_2, \eta_p \in \mathcal{F}(p)$ and $\bar{w}_p \in W_1$ satisfying $y_p = y_0 + w_p = \eta_p + \bar{w}_p$. Thus,

$$\eta_p = y_0 + w_p - \bar{w}_p \in y_0 + W_1 + W_1 \subset y_0 + W.$$

Then we have

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_1 \cap \bar{U}$$

which establishes (3.1) with $U_W := U_1 \cap \bar{U}$.

Next, suppose that for all $U \in \mathcal{N}(p_0), U \subset U_2$, there exists $p \in U$ such that

$$y_p \notin \mathcal{F}(p) + W_1. \tag{3.6}$$

Combining (3.6) with (3.3) we get $y_p \in [(y_0 + V_0) \cap F(p)] \backslash (\mathcal{F}(p) + W_1)$. By (3.2) we can find $\eta_p \in \mathcal{F}(p)$ and $k_p \in K$ with the properties that

$$y_p = \eta_p + k_p, \quad k_p + W_0 \subset K. \tag{3.7}$$

From (3.4) and the relation $\eta_p \in \mathcal{F}(p) \subset F(p)$ it follows that there exist $z_0 \in F(p_0)$ and $w_0 \in V_0 \cap W_1 \cap W_2$ satisfying

$$\eta_p = z_0 + w_0. \tag{3.8}$$

By (3.3) there exists $w_p \in V_0 \cap W_1 \cap W_2$ such that

$$y_p = y_0 + w_p. \tag{3.9}$$

Using (3.9), (3.7) and (3.8) we obtain $y_0 + w_p = \eta_p + k_p = z_0 + w_0 + k_p$. This implies $y_0 = z_0 + k_p + w_0 - w_p$. Besides,

$$\begin{aligned} k_p + w_0 - w_p &\in k_p + (V_0 \cap W_1 \cap W_2) - (V_0 \cap W_1 \cap W_2) \\ &\subset k_p + W_2 + W_2 \subset k_p + W_0 \subset K. \end{aligned}$$

Hence $y_0 = z_0 + k_0$, where $k_0 := k_p + w_0 - w_p \in k_p + W_0 \subset K$. Since there is no loss of generality in assuming that W_0 is an open neighborhood, we can assert that $y_0 \in z_0 + \mathrm{int}K$. This contradicts the minimality of y_0 and completes the proof. □

In Theorem 3.2, the assumption that (locCP) holds for F uniformly around p_0 of is essential.

Example 3.3. Let $P = [0,1]$, $Y = \mathbb{R}^2, K = \mathbb{R}^2_+$. Let $F : P \rightrightarrows \mathbb{R}^2$ be defined by setting $F(0) = \{(y_1, y_2) \mid -y_1 \leq y_2 \leq -y_1 + 2\}$ and

$$F(p) = \{(y_1, y_2) \mid f_1(y_1) \leq y_2 \leq -y_1 + 2\} \backslash \left\{(y_1, y_2) \mid y_2 = -y_1 + p,\ y_1 \leq \frac{1}{p}\right\}$$

for every $p \in P \backslash \{0\}$, where

$$f_1(t) = \begin{cases} -t + p & \text{if } t \leq \frac{1}{p} \\ p - \frac{1}{p} & \text{if } \frac{1}{p} < t \leq \frac{1}{p} + 2 - p \\ -t + 2 & \text{if } t > \frac{1}{p} + 2 - p \end{cases}$$

for all $t \in \mathbb{R}$. We have

$$\mathcal{F}(0) = \{(y_1, y_2) \mid y_2 = -y_1\},$$

$$\mathcal{F}(p) = \{(y_1, y_2) \mid y_2 = -y_1 + 2, y_1 > \frac{1}{p} + 2 - p\}.$$

Note that F is H-usc and lsc at $p_0 = 0$. It is not hard to see that (locCP) does not hold for F uniformly around $p_0 = 0$. Observe that $\mathcal{F}$ is not lsc at $p_0 = 0$.

The following result is due to Bednarczuk.

Theorem 3.4 (see [1, Theorem 4]). *Suppose that* $\mathrm{int}K \neq \emptyset$ *and* (CP) *holds for* F *uniformly around* p_0. *If* (A_1) *is satisfied, then* $\mathcal{F}$ *is lsc at* p_0.

The next example shows that Theorem 3.2 is a proper extension of Theorem 3.4.

Example 3.5. Let $P = [0,1]$, $Y = \mathbb{R}^2, K = \mathbb{R}^2_+$. Let $F : P \rightrightarrows \mathbb{R}^2$ be defined by setting $F(0) = \{(y_1, y_2) \mid -y_1 \leq y_2 \leq -y_1 + 2\}$ and

$$F(p) = \{(y_1, y_2) \mid f_1(y_1) \leq y_2 \leq -y_1 + 2\}$$

for every $p \in P\backslash\{0\}$, where

$$f_1(t) = \begin{cases} -t + p & \text{if } t \leq \frac{1}{p} \\ p - \frac{1}{p} & \text{if } \frac{1}{p} < t \leq \frac{1}{p} + 2 - p \\ -t + 2 & \text{if } t > \frac{1}{p} + 2 - p. \end{cases}$$

for all $t \in \mathbb{R}$. We have

$$\mathcal{F}(0) = \{(y_1, y_2) \mid y_2 = -y_1\},$$

$$\begin{aligned} \mathcal{F}(p) = &\left\{(y_1, y_2) \mid y_2 = -y_1 + p, y_1 \leq \frac{1}{p}\right\} \\ &\cup \left\{(y_1, y_2) \mid y_2 = -y_1 + 2, y_1 > \frac{1}{p} + 2 - p\right\}. \end{aligned}$$

It is easy to show that F is continuous at $p_0 = 0$, and (CP) holds for $F(0)$. But (CP) does not hold for $F(p)$ with any $p \in P\backslash\{0\}$. It is not hard to see that (locCP) holds for F uniformly around $p_0 = 0$. Thus, Theorem 3.4 is not applicable to F at $p_0 = 0$. Meanwhile, Theorem 3.2 works well for the multifunction at p_0.

$\boxed{4}$ K-local Domination Property and the Lower Semicontinuity

In this section, we shall need the notion of normality of an ordering cone in a Hausdorff topological vector space.

Definition 4.1 (see [17]). The convex cone K is called *normal* if there is a base of neighborhoods of 0_Y consisting of sets S with the property that $S = (S + K) \cap (S - K)$.

It was shown by Borwein [6] that if Y is a locally convex space and K has a bounded base, then K is normal. Recall [12, p. 9] that a nonempty convex subset $\Theta \subset K$ is said to be a *base* of K if each $v \in K\backslash\{0\}$ can be represented uniquely in the form $v = \lambda\theta$ $(\lambda > 0,\ \theta \in \Theta)$.

We now introduce a weaker form of the property (DP) described by Def. 2.4.

Definition 4.2. We say the *K-local domination property*, denoted (K-locDP), holds for F uniformly around p_0 if for each $y_0 \in \mathcal{F}(p_0)$, there exist $V_0 \in \mathcal{N}(0_Y)$ and $U_0 \in \mathcal{N}(p_0)$ such that

$$(y_0 + V_0 - K) \cap F(p) \subset \mathcal{F}(p) + K \quad \forall p \in U_0.$$

Clearly, (DP) implies (K-locDP). Example 4.5 given below will show that the reverse implication is not valid in general.

Our first result in this section is stated as follows.

Theorem 4.3. *Let K be a normal cone in Y. Suppose that $F(p_0)$ is closed,* $\mathrm{cl}\mathcal{F}(p_0)$ *is compact,* (DP) *holds for the set $F(p_0)$, and* (K-locDP) *holds for F uniformly around p_0. If* (A_2) *is satisfied, then $\mathcal{F}$ is lsc at p_0.*

Proof. Let $y_0 \in \mathcal{F}(p_0)$. Fist, we observe that

$$\begin{cases} \forall W \in \mathcal{N}(0_Y)\ \exists V \in \mathcal{N}(0_Y) \text{ such that} \\ [((\mathcal{F}(p_0)+K)\backslash(y_0+W))+V] \cap (y_0-K)=\emptyset. \end{cases} \tag{4.1}$$

Indeed, suppose on the contrary that there exists some $W \in \mathcal{N}(0_Y)$ such that for any $V \in \mathcal{N}(0_Y)$, $\exists k_V \in K$, $\exists k_V^1 \in K$, $\exists \eta_V \in \mathcal{F}(p_0)$, $\exists s_V \in V$ satisfying

$$y_0 - k_V = \eta_V + k_V^1 + s_V \ \text{ and } \ \eta_V + k_V^1 \notin y_0 + W. \tag{4.2}$$

There is no loss of generality in assuming that the net $\{s_V\}_{V\in\mathcal{N}(0_Y)}$ tends to 0_Y. Here the direction of net is defined by the natural set-inclusion in the family $\mathcal{N}(0_Y)$:

$$s_{V_1} \preccurlyeq s_{V_2} \Leftrightarrow V_2 \subset V_1$$

for any $V_1, V_2 \in \mathcal{N}(0_Y)$. Since $\mathrm{cl}\mathcal{F}(p_0)$ is compact, there exists a subnet of $\{\eta_V\}$ converging to an element $\eta \in \mathrm{cl}\mathcal{F}(p_0) \subset F(p_0)$ (recall that $F(p_0)$ is closed by our assumption). Without loss of generality, we may assume that $\eta_V \to \eta$. By (4.2), $k_V + k_V^1 = y_0 - \eta_V - s_V$. Hence

$$k_V + k_V^1 \to y_0 - \eta, \tag{4.3}$$

and it follows from the closedness of K that $y_0 - \eta \in K$. The minimality of y_0 and the last inclusion implies that $y_0 = \eta$. Thus from (4.3) it follows that $k_V + k_V^1 \to 0_Y$. Since K is normal, using [17, Proposition 1.3] we obtain $k_V \to 0_Y$ and $k_V^1 \to 0_Y$. Given any $W_1 \in \mathcal{N}(0_Y)$ with $W_1 + W_1 \subset W$, we find $\widetilde{U} \in \mathcal{N}(0_Y)$ such that

$$\eta_V + k_V^1 \subset (\eta + W_1) + W_1 \subset y_0 + W$$

for every $U \subset \widetilde{U}$. This contradicts the property $\eta_V + k_V^1 \notin y_0 + W$. We have thus obtained the relation (4.1).

Now, take any $W \in \mathcal{N}(0_Y)$. To obtain the lower semicontinuity of $\mathcal{F}$ at p_0, we need to show that there exists $U_W \in \mathcal{N}(p_0)$ such that

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_W. \tag{4.4}$$

Let $W_1 + W_1 \subset W$. Due to (4.1), there exists $V \in \mathcal{N}(0_Y)$ such that

$$[((\mathcal{F}(p_0)+K)\backslash(y_0+W_1))+V] \cap [(y_0 - K] = \emptyset$$

Hence, for any $V_1 \in \mathcal{N}_B(0_Y)$ satisfying $V_1 + V_1 \subset V$, it holds

$$[((\mathcal{F}(p_0)+K)\backslash(y_0+W_1))+V_1] \cap [(y_0+V_1) - K] = \emptyset \tag{4.5}$$

As (DP) holds for the set $F(p_0)$, it follows that

$$F(p_0) \subset [(\mathcal{F}(p_0)+K)\backslash(y_0+W_1)] \cup (y_0+W_1).$$

Hence,

$$F(p_0) + (V_1 \cap W_1) \subset [((\mathcal{F}(p_0) + K)\backslash(y_0 + W_1)) + (V_1 \cap W_1)] \cup [(y_0 + W_1) + (V_1 \cap W_1)].$$

Since $(y_0 + W_1) + (V_1 \cap W_1) \subset y_0 + W_1 + W_1 \subset y_0 + W$, it follows that

$$F(p_0) + (V_1 \cap W_1) \subset [((\mathcal{F}(p_0) + K)\backslash(y_0 + W_1)) + (V_1 \cap W_1)] \cup (y_0 + W). \tag{4.6}$$

Since F is H-usc at p_0, there is $U_1 \in \mathcal{N}(p_0)$ such that

$$F(p) \subset F(p_0) + (V_1 \cap W_1) \quad \forall p \in U_1. \tag{4.7}$$

Combining (4.6) with (4.7) we get

$$F(p) \subset [((\mathcal{F}(p_0) + K)\backslash(y_0 + W_1)) + (V_1 \cap W_1)] \cup (y_0 + W), \tag{4.8}$$

for all $p \in U_1$. Since (K-locDP) holds for F uniformly around p_0, there exist $V_0 \in \mathcal{N}(0_Y)$ and $U_0 \in \mathcal{N}(p_0)$ such that

$$(y_0 + V_0 - K) \cap F(p) \subset \mathcal{F}(p) + K \quad \forall p \in U_0. \tag{4.9}$$

Since F is $(-K)$-lsc at p_0, there must exist $U_2 \in \mathcal{N}(p_0)$ with the property that

$$[y_0 + (V_0 \cap V_1 \cap W_1) - K] \cap F(p) \neq \emptyset \quad \forall p \in U_2.$$

For each $p \in U_2$ we can choose

$$y_p \in [y_0 + (V_0 \cap V_1 \cap W_1) - K] \cap F(p). \tag{4.10}$$

Observe that

$$y_p - K \subset [y_0 + (V_0 \cap V_1 \cap W_1) - K] - K \subset y_0 + (V_0 \cap V_1 \cap W_1) - K. \tag{4.11}$$

Using (4.11), (4.5) and the obvious inclusion $V_0 \cap V_1 \cap W_1 \subset V_1$, we obtain

$$(y_p - K) \cap [((\mathcal{F}(p_0) + K)\backslash(y_0 + W_1)) + (V_1 \cap W_1)] = \emptyset. \tag{4.12}$$

Hence

$$[(y_p - K) \cap F(p)] \cap [((\mathcal{F}(p_0) + K)\backslash(y_0 + W_1)) + (V_1 \cap W_1)] = \emptyset. \tag{4.13}$$

From (4.8) it follows that

$$[(y_p - K) \cap F(p)] \subset [((\mathcal{F}(p_0) + K)\backslash(y_0 + W_1)) + (V_1 \cap W_1)] \cup (y_0 + W) \tag{4.14}$$

for each $p \in U_1 \cap U_2$. By (4.13) and (4.14), for each $p \in U_1 \cap U_2$ we have

$$(y_p - K) \cap F(p) \subset y_0 + W. \tag{4.15}$$

According to (4.10) and (4.9), for each $p \in U_0 \cap U_1 \cap U_2$ there exist $\eta_p \in \mathcal{F}(p)$ and $k_p \in K$ such that $y_p = \eta_p + k_p$. Therefore, for each $p \in U_0 \cap U_1 \cap U_2$, by (4.15) and the fact that $\mathcal{F}(p) \subset F(p)$ we get

$$\eta_p \in (y_p - K) \cap F(p) \subset y_0 + W.$$

Hence

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_0 \cap U_1 \cap U_2.$$

This shows that (4.4) holds for $U_W := U_0 \cap U_1 \cap U_2$ and completes the proof. $\square$

Let us compare Theorem 4.3 with the following result of Bednarczuk.

Theorem 4.4 (see [3, Theorem 4.3]). *Let K be a normal cone in Y. Assume that $F(p_0)$ is closed,* $\mathrm{cl}\mathcal{F}(p_0)$ *is compact, and* (DP) *holds for F uniformly around p_0. If* (A_2) *is satisfied, then $\mathcal{F}$ is lsc at p_0.*

As it is clear from the next example, Theorem 4.3 has a wider range of applicability than that of Theorem 4.4.

Example 4.5. Let $P = [0,1]$, $Y = \mathbb{R}^2, K = \mathbb{R}^2_+$. Let $F : P \rightrightarrows \mathbb{R}^2$ be given by setting $F(0) = \{(y_1, y_2) \mid 1 - y_1 \le y_2 \le 1, 0 \le y_1 \le 1\} \cup \{(y_1, y_2) \mid 0 \le y_2 \le 1, y_1 > 1\}$ and

$$F(p) = \{(y_1, y_2) \mid f_1(y_1) \le y_2 \le 1\}$$

for every $p \in P\backslash\{0\}$, where

$$f_1(t) = \begin{cases} -t + p + 1 & \text{if } p \le t \le p + 1 \\ 0 & \text{if } p + 1 < t \le p + 1 + \frac{1}{p} \\ -p & \text{if } t > p + 1 + \frac{1}{p}. \end{cases}$$

for all $t \in \mathbb{R}$. We have

$$\mathcal{F}(0) = \{(y_1, y_2) \mid y_2 = 1 - y_1, 0 \le y_1 \le 1\},$$

$$\mathcal{F}(p) = \{(y_1, y_2) \mid y_2 = -y_1 + p + 1, p \le y_1 \le p + 1\}.$$

Note that $F(0)$ is closed, $\mathrm{cl}\mathcal{F}(0)$ is compact, (DP) holds for $F(0)$, and (K-locDP) holds for F uniformly around $p_0 = 0$. Besides, it is easy to see that $K = \mathbb{R}^2_+$ is normal cone, F is $(-K)$-lsc and H-usc at $p_0 = 0$. By Theorem 4.3, $\mathcal{F}$ is lsc at $p_0 = 0$. Meanwhile, since (DP) does not hold for any $F(p)$ with $p \in P\backslash\{0\}$, Theorem 4.4 does not work for the problem under consideration.

The assumption that (K-locDP) holds for F uniformly around p_0 cannot be dropped from the formulation of Theorem 4.3.

Example 4.6. Let $P = [0,1]$, $Y = \mathbb{R}^2, K = \mathbb{R}^2_+$. Let $F : P \rightrightarrows \mathbb{R}^2$ be defined by setting $F(0) = \{(y_1, y_2) \mid 1 - y_1 \le y_2 \le 1, 0 \le y_1 \le 1\} \cup \{(y_1, y_2) \mid 0 \le y_2 \le 1, y_1 > 1\}$ and

$$F(p) = \{(y_1, y_2) \mid f_1(y_1) \le y_2 \le 1\}\backslash\{(y_1, y_2) \mid y_2 = -y_1 + p + 1, p < y_1 \le p + 1\}$$

for every $p \in P\backslash\{0\}$, where

$$f_1(t) = \begin{cases} -t + p + 1 & \text{if } p \le t \le p + 1 \\ 0 & \text{if } p + 1 < t \le p + 1 + \frac{1}{p} \\ -p & \text{if } t > p + 1 + \frac{1}{p}. \end{cases}$$

for all $t \in \mathbb{R}$. We have

$$\mathcal{F}(0) = \{(y_1, y_2) \mid y_2 = 1 - y_1, 0 \le y_1 \le 1\}$$

and

$$\mathcal{F}(p) = \{(p, 1)\}.$$

One can verify that all assumptions of Theorem 4.3, except for (K-locDP), are satisfied. Note that $\mathcal{F}$ is not lsc at $p_0 = 0$.

To proceed further, we need to consider some notions of proper efficiency. As shown by Bednarczuk [1], properly efficient points are very useful for studying the lower semicontinuity of the efficient point multifunction.

Definition 4.7. Let A be a subset of Y.
(i) (see [10]) An element $y_0 \in E(A|K)$ is said to be a *properly efficient point* of A, in the sense of Henig, if there exists a closed convex cone K_0 with $K\backslash\{0\} \subset \mathrm{int}K_0$ such that $y_0 \in E(A|K_0)$.
(ii) (see [1]) An element $y_0 \in E(A|K)$ is a *strongly properly efficient point* of A if there exists a closed convex cone K_0 with $K\backslash\{0\} \subset \mathrm{int}K_0$ such that $y_0 \in E(A|K_0)$ and, for each $W \in \mathcal{N}(0_Y)$, one can find $V \in \mathcal{N}(0_Y)$ such that

$$(K\backslash W) + V \subset K_0.$$

(iii) (see [3]) An element $y_0 \in A$ is a *strictly efficient point* of A if for each $W \in \mathcal{N}(0_Y)$ there exists $V \in \mathcal{N}(0_Y)$ such that

$$[(A\backslash(y_0 + W)) + V] \cap (y_0 - K) = \emptyset.$$

The set of the properly efficient points (resp., strongly properly efficient points, strictly efficient points) of A is denoted by $E^{He}(A|K)$ (resp., $E^{\mathrm{sHe}}(A|K), E_1(A|K)$).

It can be proved that $E_1(A|K) \subset E(A|K)$. The reverse inclusion may fail to hold (see [3, Example 3.1]). According to [3, Proposition 3.1],

$$E^{\mathrm{sHe}}(A|K) \subset E_1(A|K). \tag{4.16}$$

Obviously, $E^{\mathrm{sHe}}(A|K) \subset E^{He}(A|K)$. We now give a sufficient condition for the last inclusion to become an equality.

Proposition 4.8. *Let Y be a normed space and $A \subset Y$. If $K \subset Y$ is a closed convex cone having a compact base Θ, then*

$$E^{\mathrm{sHe}}(A|K) = E^{He}(A|K).$$

Proof. We need only to prove $E^{He}(A|K) \subset E^{\mathrm{sHe}}(A|K)$. For any $y_0 \in E^{He}(A|K)$, there exists a closed convex cone K_0 with $K\backslash\{0\} \subset \mathrm{int}K_0$ such that $y_0 \in E(A|K_0)$. To verify the inclusion $y_0 \in E^{\mathrm{sHe}}(A|K)$, it suffices to show that for each $W \in \mathcal{N}(0_Y)$ there exists $V \in \mathcal{N}(0_Y)$ such that

$$K\backslash W + V \subset K_0. \tag{4.17}$$

For each $\theta \in \Theta$, we have $\theta \in K\backslash\{0\}$, hence there exists $W_\theta \in \mathcal{N}(0_Y)$ satisfying $\theta + W_\theta \subset K_0$. Choose $V_\theta \in \mathcal{N}_B(0_Y), V_\theta + V_\theta \subset W_\theta$. The family $\{\theta + V_\theta\}_{\theta\in\Theta}$ is a cover of Θ. By the compactness of Θ, there exists a finite subcover $\{\theta_i + V_{\theta_i}\}_{i=1}^m$. Clearly, there exists $\lambda_0 > 0$ such that $\lambda\Theta \subset W$ for all $0 \le \lambda \le \lambda_0$. Put $V_0 = \bigcap_{i=1}^m V_{\theta_i}$. We choose some $V \in \mathcal{N}(0_Y)$ satisfying $V \subset \lambda_0 V_0$. For any $y \in K\backslash W$, there exist $\theta_y \in \Theta$ and $\lambda_y > \lambda_0$ such that $y = \lambda_y\theta_y$. Therefore,

$$y + V \subset \lambda_y\theta_y + \lambda_0 V_0 = \lambda_y\left(\theta_y + \frac{\lambda_0}{\lambda_y}V_0\right) \subset \lambda_y(\theta_y + V_0) \subset K_0.$$

Hence $K\backslash W + V \subset K_0$, i.e., (4.17) is satisfied and the proof is complete. □

If the K-local domination property (see Def. 4.2) holds and the strictly efficient point set is "good" in the sense that its closure contains the efficient point set, then we may hope that $\mathcal{F}$ is lsc at p_0. Namely, we have following theorem.

Theorem 4.9. *Suppose that* (K-locDP) *holds for* F *uniformly around* p_0 *and*

$$\mathcal{F}(p_0) \subset \mathrm{cl}E_1(F(p_0)|K). \tag{4.18}$$

If (A_2) *is satisfied, then* $\mathcal{F}$ *is lsc at* p_0*.*

Proof. Let $y_0 \in \mathcal{F}(p_0)$ and $W \in \mathcal{N}(0_Y)$. The proof will be completed if we can show that there exists $U_W \in \mathcal{N}(p_0)$ such that

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_W. \tag{4.19}$$

Choose $W_1, W_2 \in \mathcal{N}(0_Y)$ such that $W_1 + W_1 \subset W$ and $W_2 + W_2 \subset W_1$. By (4.18), we have $y_0 \in \mathrm{cl}E_1(F(p_0)|K)$. Therefore, we can pick an $y_1 \in E_1(F(p_0)|K) \cap (y_0 + W_2)$. Since $y_1 \in E_1(F(p_0)|K) \subset \mathcal{F}(p_0)$ and (K-locDP) holds for F uniformly around p_0, there exist $V_0 \in \mathcal{N}(0_Y)$ and $U_0 \in \mathcal{N}(p_0)$ such that

$$(y_1 + V_0 - K) \cap F(p) \subset \mathcal{F}(p) + K \quad \forall p \in U_0. \tag{4.20}$$

By the strict minimality of y_1, there is $V \in \mathcal{N}(0_Y)$ such that

$$[(F(p_0)\backslash(y_1 + W_2)) + V] \cap (y_1 - K) = \emptyset.$$

Therefore, for any $V_1 \in \mathcal{N}_B(0_Y)$ satisfying $V_1 + V_1 \subset V$, it holds

$$[((F(p_0)\backslash(y_1 + W_2)) + V_1] \cap [(y_1 + V_1) - K] = \emptyset. \tag{4.21}$$

It is obvious that

$$F(p_0) \subset [(F(p_0)\backslash(y_1 + W_2)] \cup (y_1 + W_2).$$

Hence

$$F(p_0) + (V_1 \cap W_2) \subset [(F(p_0)\backslash(y_1 + W_2)) + (V_1 \cap W_2)] \cup [(y_1 + W_2) + (V_1 \cap W_2)].$$

Since $(y_1 + W_2) + (V_1 \cap W_2) \subset y_1 + W_2 + W_2 \subset y_1 + W_1$, it follows that

$$F(p_0) + (V_1 \cap W_2) \subset [(F(p_0)\backslash(y_1 + W_2)) + (V_1 \cap W_2)] \cup (y_1 + W_1). \tag{4.22}$$

As F is H-usc at p_0, there exists $U_1 \in \mathcal{N}(p_0)$ such that

$$F(p) \subset F(p_0) + (V_1 \cap W_2) \quad \forall p \in U_1. \tag{4.23}$$

Combining (4.22) with (4.23) gives

$$F(p) \subset [((F(p_0)\backslash(y_1 + W_2)) + (V_1 \cap W_2)] \cup (y_1 + W_1) \quad \forall p \in U_1. \tag{4.24}$$

Since F is $(-K)$-lsc at p_0, there exists $U_2 \in \mathcal{N}(p_0)$ satisfying

$$[y_1 + (V_0 \cap V_1 \cap W_2) - K] \cap F(p) \neq \emptyset \quad \forall p \in U_2.$$

For each $p \in U_2$ we choose some

$$y_p \in [y_1 + (V_0 \cap V_1 \cap W_2) - K] \cap F(p) \tag{4.25}$$

and observe that

$$y_p - K \subset [y_1 + (V_0 \cap V_1 \cap W_2) - K] - K \subset y_1 + (V_0 \cap V_1 \cap W_2) - K. \tag{4.26}$$

Using (4.26), (4.21) and the obvious inclusion $V_0 \cap V_1 \cap W_2 \subset V_1$, we get

$$(y_p - K) \cap [(F(p_0)\backslash(y_1 + W_2)) + (V_1 \cap W_2)] = \emptyset. \tag{4.27}$$

Hence

$$[(y_p - K) \cap F(p)] \cap [(F(p_0)\backslash(y_1 + W_2)) + (V_1 \cap W_2)] = \emptyset. \tag{4.28}$$

For each $p \in U_1 \cap U_2$, by (4.24) we have

$$[(y_p - K) \cap F(p)] \subset [(F(p_0)\backslash(y_1 + W_2)) + (V_1 \cap W_2)] \cup (y_1 + W_1). \tag{4.29}$$

According to (4.28) and (4.29),

$$(y_p - K) \cap F(p) \subset y_1 + W_1 \quad \forall y \in U_1 \cap U_2. \tag{4.30}$$

Thus

$$\begin{aligned}(y_p - K) \cap F(p) \subset y_1 + W_1 \subset (y_0 + W_2) + W_1 \subset y_0 + W_1 + W_1 \\ \subset y_0 + W. \end{aligned} \tag{4.31}$$

From (4.25) and (4.20), it follows that for each $p \in U_0 \cap U_1 \cap U_2$ there exist $\eta_p \in \mathcal{F}(p)$ and $k_p \in K$ such that $y_p = \eta_p + k_p$. Therefore, for each $p \in U_0 \cap U_1 \cap U_2$, by (4.31) and the inclusion $\mathcal{F}(p) \subset F(p)$ we may conclude that

$$\eta_p \in (y_p - K) \cap F(p) \subset y_0 + W.$$

Hence

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_0 \cap U_1 \cap U_2$$

and (4.19) holds for $U_W := U_0 \cap U_1 \cap U_2$. □

The following result can be found in [3, Theorem 4.1].

Theorem 4.10. *Suppose that* (DP) *holds for* F *uniformly around* p_0 *and*

$$\mathcal{F}(p_0) \subset \mathrm{cl}E_1(F(p_0)|K).$$

If (A_2) *is satisfied, then* $\mathcal{F}$ *is lsc at* p_0.

It is easy to verify that Theorem 4.9 is applicable to Example 4.5 (note that $\mathcal{F}(0) = E_1(F(0)|K)$), but Theorem 4.10 does not work for the example.

The forthcoming corollary, which extends [3, Theorem 4.2] and [1, Theorem 7], follows directly from Theorem 4.9 and (4.16).

Corollary 4.11. *Suppose that* (K-locDP) *holds for* F *uniformly around* p_0 *and*

$$\mathcal{F}(p_0) \subset \mathrm{cl}E^{\mathrm{sHe}}(F(p_0)|K).$$

If (A_2) *is satisfied, then* $\mathcal{F}$ *is lsc at* p_0.

We remark that Corollary 4.11 can be applied to Example 4.5 (here it holds $\mathcal{F}(0) = E^{\mathrm{sHe}}(F(0)|K)$), but [3, Theorem 4.2] and [1, Theorem 7] are not applicable to this example.

We need to recall a notion of efficiency due to Borwein and Zhuang [7].

Definition 4.12 (see [7]). Let Y is a normed space and $A \subset Y$. A element $y_0 \in A$ is a *super efficient point* of A with respect to the ordering cone K if there exists $\gamma > 0$ such that

$$\mathrm{cl}[\mathrm{cone}(A - y_0)] \cap (B - K) \subset \gamma B$$

where B denotes the closed unit ball of Y. We abbreviate the super efficient point set of A w.r.t. K by $E^{\mathrm{BoZh}}(A|K)$.

As shown in [3, Prop. 5.2] and [3, Theorem 5.1],

$$E^{\mathrm{BoZh}}(A|K) \subset E_1(A|K). \tag{4.32}$$

Moreover, if K has a bounded base,

$$E^{\mathrm{BoZh}}(A|K) = E^{\mathrm{sHe}}(A|K). \tag{4.33}$$

By Theorem 4.9 and (4.32) we can obtain the following generalization of [3, Theorem 5.2].

Corollary 4.13. *Let Y be a normed space. Assume that* (K-locDP) *holds for F uniformly around p_0 and*

$$\mathcal{F}(p_0) \subset \mathrm{cl}E^{\mathrm{BoZh}}(F(p_0)|K).$$

If (A_2) *is satisfied, then $\mathcal{F}$ is lsc at p_0.*

Since $\mathcal{F}(0) = E^{\mathrm{BoZh}}(F(0)|K)$, we see that Corollary 4.13 is applicable to Example 4.5. Meanwhile, since (DP) does not hold for any $F(p)$ with $p \in P\backslash\{0\}$, [3, Theorem 5.2] does not work for the example.

We want to stress that the assumption that (K-locDP) holds for F uniformly around p_0 is essential for Theorem 4.9, Corollary 4.11 and Corollary 4.13. To see this, it suffices to consider Example 4.6 again.

$\boxed{5}$ Uniformly Local Closedness and the Lower semicontinuity

We now introduce a notion on uniformly local closeness of multifunctions.

Definition 5.1. We say the F is *uniformly local closed* around p_0 if for each $y_0 \in \mathcal{F}(p_0)$ there exists $V_0 \in \mathcal{N}(0)$ and $U_0 \in \mathcal{N}(p_0)$ such that $\mathrm{cl}(y_0 + V_0) \cap F(p)$ is closed for all $p \in U_0\backslash\{p_0\}$.

Clearly, if $F(p)$ is closed for all $p \neq p_0$ from a neighborhood of p_0 then F is uniformly locally closed around p_0. The converse is not true in general. In Example 4.5, we have encountered with a multifunction F where $F(p)$ is not closed for any $p \in P\backslash\{p_0\}$.

Theorem 5.2. *Let Y be a locally compact topological vector space. Suppose that*

$$\mathcal{F}(p_0) \subset \mathrm{cl}E_1(F(p_0)|K)$$

and F is uniformly locally closed around p_0. If (A_2) *is satisfied, then $\mathcal{F}$ is lsc at p_0.*

Proof. Let $y_0 \in \mathcal{F}(p_0)$ and $W \in \mathcal{N}(0_Y)$. We need to show that there exists $U_W \in \mathcal{N}(p_0)$ such that

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in U_W. \tag{5.1}$$

Let $W_1 \in \mathcal{N}(0_Y)$ be such that $W_1 + W_1 \subset W$. By $y_0 \in \mathrm{cl}E_1(F(p_0)|K)$, we can choose $y_1 \in E_1(F(p_0)|K) \cap (y_0 + W_1)$. Since $y_1 \in E_1(F(p_0)|K) \subset \mathcal{F}(p_0)$ and F is uniformly locally closed around p_0, there exist $V_0 \in \mathcal{N}(0_Y)$ and $U_0 \in \mathcal{N}(p_0)$ such that $\mathrm{cl}(y_1 + V_0) \cap F(p)$ is closed for all $p \in U_0 \backslash \{p_0\}$. As Y is a locally compact topological vector space, there is $W_2 \in \mathcal{N}(0_Y)$ such that $\mathrm{cl}W_2$ is compact and

$$\mathrm{cl}W_2 \subset W_1 \cap V_0. \tag{5.2}$$

Let $W_3 \in \mathcal{N}(0_Y)$ be such that $W_3 + W_3 \subset W_2$. By the strict minimality of y_1, we can find $V \in \mathcal{N}(0_Y)$ with the property that

$$[(F(p_0)\backslash(y_1 + W_3)) + V] \cap (y_1 - K) = \emptyset.$$

Therefore, for any $V_1 \in \mathcal{N}_B(0_Y)$ satisfying $V_1 + V_1 \subset V$, it holds

$$[((F(p_0)\backslash(y_1 + W_3)) + V_1] \cap [(y_1 + V_1) - K] = \emptyset. \tag{5.3}$$

The inclusion

$$F(p_0) \subset [(F(p_0)\backslash(y_1 + W_3)] \cup (y_1 + W_3)$$

yields

$$F(p_0) + (V_1 \cap W_3) \subset [(F(p_0)\backslash(y_1 + W_3)) + (V_1 \cap W_3)] \cup [(y_1 + W_3) + (V_1 \cap W_3)].$$

Thus

$$F(p_0) + (V_1 \cap W_3) \subset [(F(p_0)\backslash(y_1 + W_3)) + (V_1 \cap W_3)] \cup (y_1 + W_2). \tag{5.4}$$

By (A_2), F is H-usc at p_0, so there exists $U_1 \in \mathcal{N}(p_0)$ such that

$$F(p) \subset F(p_0) + (V_1 \cap W_3) \quad \forall p \in U_1.$$

Combining this with (5.4) gives

$$F(p) \subset [((F(p_0)\backslash(y_1 + W_3)) + (V_1 \cap W_3)] \cup (y_1 + W_2) \quad \forall p \in U_1. \tag{5.5}$$

Since F is $(-K)$-lsc at p_0, there is $U_2 \in \mathcal{N}(p_0)$ such that for each $p \in U_2$ we can find an $y_p \in [y_1 + (V_1 \cap W_3) - K] \cap F(p)$. Observe that

$$y_p - K \subset [y_1 + (V_1 \cap W_3) - K] - K \subset y_1 + (V_1 \cap W_3) - K. \tag{5.6}$$

Using (5.6) and (5.3) we have

$$(y_p - K) \cap [(F(p_0)\backslash(y_1 + W_3)) + (V_1 \cap W_3)] = \emptyset. \tag{5.7}$$

Hence

$$[(y_p - K) \cap F(p)] \cap [(F(p_0)\backslash(y_1 + W_3)) + (V_1 \cap W_3)] = \emptyset. \tag{5.8}$$

For each $p \in U_1 \cap U_2$, from (5.5) it follows that

$$[(y_p - K) \cap F(p)] \subset [(F(p_0)\backslash(y_1 + W_3)) + (V_1 \cap W_3)] \cup (y_1 + W_2). \tag{5.9}$$

For each $p \in U_1 \cap U_2$, by (5.8) and (5.9) we have

$$(y_p - K) \cap F(p) \subset y_1 + W_2. \tag{5.10}$$

Hence

$$(y_p - K) \cap F(p) \subset y_1 + W_2 \subset y_1 + W_1 \subset (y_0 + W_1) + W_1 \subset y_0 + W.$$

For each $p \in U_1 \cap U_2$ using (5.10) and (5.2) we get

$$(y_p - K) \cap F(p) \subset y_1 + \mathrm{cl}W_2 \subset y_1 + V_0 \subset \mathrm{cl}(y_1 + V_0). \tag{5.11}$$

Therefore,

$$(y_p - K) \cap F(p) = [(y_p - K) \cap F(p)] \cap \mathrm{cl}(y_1 + V_0). \tag{5.12}$$

For each $p \in (U_0 \cap U_1 \cap U_2)\backslash\{p_0\}$, by the closedness of K, the closedness of $\mathrm{cl}(y_1 + V_0) \cap F(p)$ and (5.12), the set $(y_p - K) \cap F(p)$ is closed. In view of the (5.11) and the compactness of $y_1 + \mathrm{cl}W_2$ we see that $(y_p - K) \cap F(p)$ is a compact set. It follows from [12, Theorem 6.3 (c)] that the section $(y_p - K) \cap F(p)$ contains an element $\eta_p \in \mathcal{F}(p)$. Hence

$$\eta_p \in (y_p - K) \cap F(p) \subset y_0 + W.$$

This yields

$$(y_0 + W) \cap \mathcal{F}(p) \neq \emptyset \quad \forall p \in (U_0 \cap U_1 \cap U_2)\backslash\{p_0\}. \tag{5.13}$$

Obviously, $(y_0 + W) \cap \mathcal{F}(p_0) \neq \emptyset$. So, this and (5.13) imply that (5.1) holds for $U_W := U_0 \cap U_1 \cap U_2$. □

We can restate [1, Theorem 8] as follows.

Theorem 5.3. *Let Y be a locally compact topological vector space. Suppose that*

$$\mathcal{F}(p_0) = E^{\mathrm{sHe}}(F(p_0)|K)$$

and there a neighborhood U_0 of p_0 such that $F(p)$ is closed for every $p \in U_0\backslash\{p_0\}$. If (A_1) is satisfied, then $\mathcal{F}$ is lsc at p_0.

From (4.16) we see that Theorem 5.2 is a refinement of Theorem 5.3. One can apply the first theorem to Example 4.5, but one cannot do this with the second theorem.

In the finite-dimensional case, Theorem 5.2 gives us the following sufficient condition for the lsc property of the efficient point multifunction, which complements [1, Theorem 9].

Corollary 5.4. *Let $Y = \mathbb{R}^m$ and $F : P \rightrightarrows \mathbb{R}^m$. Suppose that*

$$\mathcal{F}(p_0) \subset \mathrm{cl}E_1(F(p_0)|K)$$

and F is uniformly locally closed around p_0. If (A_2) is satisfied, then $\mathcal{F}$ is lsc at p_0.

The assumption that F is uniformly locally closed around p_0 is essential for the validity of conclusions of Theorem 5.2 and Corollary 5.4; see Example 4.6.

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References

[1] E.M. Bednarczuk, Berge-type theorems for vector optimization problems, *Optimization* 32 (1995) 373–384.

[2] E.M. Bednarczuk, Some stability results for vector optimization problems in partially ordered topological vector, in *Proceedings of the first world congress on World congress of nonlinear analysts*, volume III table of contents, Tampa, Florida, United States, 1996, pp. 2371–2382.

[3] E.M. Bednarczuk, A note on lower semicontinuity of minimal points, *Nonlinear Anal.* 50 (2002) 285–297.

[4] E.M. Bednarczuk, Upper Hölder continuity of minimal points, *J. Convex Anal.* 9 (2002) 327–338.

[5] E.M. Bednarczuk, Continuity of minimal points with applications to parametric multiple objective optimization, *European J. Oper. Res.* 157 (2004) 59–67.

[6] J.M. Borwein, Continuity and differentiability properties of convex operators, *Proc. London Math. Soc.* 44 (1982) 420–444.

[7] J.M. Borwein and D. Zhuang, Super efficiency in vector optimization, *Trans. AMS* 338 (1993) 105–122.

[8] S. Dolecki and C. Malivert, Stability of minimal sets: continuity of mobile polarities, *Nonlinear Anal.* 12 (1988) 1461–1486.

[9] S. Dolecki and B. El Ghali, Some old and new results on lower semicontinuity of efficient points, *Nonlinear Anal.* 39 (2000) 599–609.

[10] M.I. Henig, Proper efficiency with respect to cones, *J. Optim. Theory Appl.* 36 (1982) 387–407.

[11] M.I. Henig, The domination property in multicriteria optimization, *J. Math. Anal. Appl.* 114 (1986) 7–16.

[12] J. Jahn, Vector Optimization, *Theory, Applications, and Extensions*, Springer-Verlag, Berlin, 2004.

[13] D.T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, 319, Springer-Verlag, Berlin, 1989.

[14] P.H. Naccache, Stability in multicriteria optimization, *J. Math. Anal. Appl.* 68 (1979) 441–453.

[15] J.-P. Penot and A. Sterna-Karwat, Parametrized multicriteria optimization; continuity and the closedness of optimal multifunctions, *J. Math. Anal. Appl.* 120 (1986) 150–168.

[16] J.-P. Penot and A. Sterna-Karwat, Parametrized multicriteria optimization; order continuity of the marginal multifunctions, *J. Math. Anal. Appl.* 144 (1989) 1–15.

[17] A.L. Peressini, *Ordered Topological Vector Spaces*, Harper and Row, New York, 1967.

[18] S.W. Xiang and Y.H. Zhou, Continuity properties of solutions of vector optimization, *Nonlinear Anal.* 64 (2006) 2496–2506.

[19] S.W. Xiang and W.S. Yin, Stability results for efficient solutions of vector optimization problems, *J. Optim. Theory Appl.* 134 (2007) 385–398.

[20] Y. Sawaragi, H. Nakayama and T. Tanino, *Theory of Multiobjective Optimization*, Mathematics in Science and Engineering, 176. Academic Press, Inc., Orlando, FL, 1985.

[21] T. Tanino and Y. Sawaragi, Stability of nondominated solutions in multicriteria decision-making. *J. Optim. Theory Appl.* 30 (1980) 229–253.

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T.D. Chuong
Department of Mathematics, Dong Thap University
Cao Lanh City, Dong Thap Province, Vietnam
E-mail address: chuongthaidoan@yahoo.com

J.-C. Yao
Department of Applied Mathematics, National Sun Yat-Sen University
Kaohsiung 804, Taiwan
E-mail address: yaojc@math.nsysu.edu.tw

N.D. Yen
Institute of Mathematics, Vietnamese Academy of Science and Technology
18 Hoang Quoc Viet, Hanoi 10307, Vietnam
E-mail address: ndyen@math.ac.vn