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A NECESSARY OPTIMALITY CONDITION FOR FREE KNOTS LINEAR SPLINES: SPECIAL CASES

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Dedicated to the memory of Alexander M. Rubinov

Abstract: In this paper, we study the problem of best Chebyshev approximation by linear splines. We construct linear splines as a $\max - \min$ of linear functions. Then we apply nonsmooth optimisation techniques to analyse and solve the corresponding optimisation problems. This approach allows us to identify and introduce a new important property of linear spline knots (regular and irregular). Using this property, we derive a necessary optimality condition for the case of regular knots. This condition is stronger than those existing in the literature. We also present a numerical example which demonstrates the difference between the old and the new optimality conditions.

Key words: *nonsmooth optimization, polynomial spline,* $\max - \min$ *functions*

Mathematics Subject Classification: *49J35, 65D10, 65D07*

1 Introduction

Polynomial splines are an efficient tool for data and function approximation. In this paper, polynomial splines are thought to be continuous piece-wise polynomial functions in an approximation interval $[a, b]$. Their derivatives may be discontinuous at the points where the polynomials are joined together. In this paper we are focusing on linear splines.

Polynomial splines are more flexible approximation techniques than polynomials. The combination of the simplicity of polynomials and the flexibility allows the significant decrease of the degree of the corresponding polynomials and the reduction of severe oscillations of associated deviation functions.

Polynomial splines with fixed knots have been extensively studied. Optimality conditions for fixed knots polynomial spline approximations can be found in [5, 7, 9]. These conditions are generalisations of the Chebyshev theorem which represents necessary and sufficient optimality conditions for the case of polynomial approximation [2]. In the case of polynomial approximation and fixed knots polynomial spline approximation the corresponding optimisation problems are convex, and therefore necessary and sufficient optimality conditions coincide. In the case of free knots polynomial spline approximation the corresponding optimisation problems are nonconvex and therefore necessary conditions may not coincide with sufficient ones.

The problem of polynomial spline approximation with free knots has been studied by several researchers. An extensive review of the existing results can be found in [5]. This book also contains a characterization theorem for free knots polynomial spline approximation. In this paper, we derive a necessary optimality condition which can be treated as a complimentary condition to the one presented in [5]. Also, in this paper we show that in the case of free knots polynomial spline approximation internal knots may belong to two different categories (regular and irregular). By taking this into account, it may be possible to improve the existing necessary optimality condition [5]. This property for internal knots has not been identified before.

The necessary optimality condition, derived in this paper, is based on the notion of alternance. Similar to the case of polynomial and fixed knots polynomial spline approximation, this condition may be used for developing Remez-like algorithms for constructing polynomial splines which satisfy this necessary optimality condition.

This paper is constructed as follows. In section 2 we present necessary definitions and earlier obtained results for the case of polynomial and fixed knots polynomial spline approximation. In section 3 we formulate a necessary optimality condition for the case of free knots linear spline approximation. This condition is based on the notion of stationarity. Section 4 introduces an approach for the verification of this necessary optimality condition. This approach is based on the number of alternance points. Section 5 contains conclusions and further research directions.

2 Preliminaries

2.1 Definitions

There are several definitions for polynomial splines. In this paper we use the following approach to define polynomial splines: suppose that a polynomial spline $Sp(t)$ is determined in an interval $[a,b]$ (approximation interval), suppose also that this interval has been divided into n segments (subintervals), such that

$$a = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = b. \tag{2.1}$$

Definition 2.1. A polynomial spline (or spline) is a continuous piece-wise polynomial function, which is represented in each segment $T_i = [\theta_{i-1}, \theta_i]$, $i = 1, \dots, n$ by a polynomial of degree less than or equal to m, where m is the degree of the polynomial spline.

Definition 2.2. The points θ_i, $i = 0, \dots, n$ are called the polynomial spline knots (or knots). The points θ_i, $i = 1, \dots, n-1$ are called the internal knots. The points θ_0 and θ_n are called the external knots.

Definition 2.3. The difference between the degree of the spline and the order of the highest continuous derivative is called the defect of the spline.

Consider an example of polynomial spline construction (see [9]).

$$S_m(A,t) = a_0 + \sum_{i=1}^{n} \sum_{j=m-d+1}^{m} a_{ij}(t-\theta_{i-1})_+^j, \tag{2.2}$$

where m is the spline degree, d is the spline defect, $\theta_i, i = 0, \dots, n$ are the spline knots,

$$(\xi(x))_+ = \begin{cases} \xi(x), & \xi(x) > 0, \\ 0, & \xi(x) \le 0. \end{cases}$$

Definition 2.4. Vector $A = (a_0, a_{11}, \dots, a_{nm}) \in \mathbb{R}^{mn+1}$ is called the vector of spline parameters.

Remark 2.5. Notice also that according to the Haar theorem (see [4] and references within) in the case of polynomial approximation the best polynomial approximation is unique, since $1, x, x^2, \dots, x^n$ form a Chebyshev system. In the case of polynomial spline approximation the best polynomial spline approximation is not necessarily unique, since $(t - \theta_{i-1})_+^j, i = 1, \dots, n, j = 1, \dots, m$ do not form a Chebyshev system.

In some cases, it is necessary to specify the degree of the polynomials which represent the spline in each segment more precisely. This approach may help to reduce the dimension of the corresponding optimisation problem.

Definition 2.6. If in each segment $[\theta_{i-1}, \theta_i]$ the spline is thought to be a polynomial of degree less than or equal to m_i then the vector $M = (m_1, m_2, \dots, m_n)$ is called the generalised (vector) degree of the polynomial spline.

A polynomial spline of generalised degree $M = (m_1, \dots, m_n)$ can be constructed as follows: in each segment $T_i = [\theta_{i-1}, \theta_i]$ it is presented by a polynomial $P_i(t)$, such that

$$P_1(t) = \sum_{j=1}^{m_1} a_{1j}(t-\theta_0)^j + a_0, \quad P_i(t) = \sum_{j=1}^{m_i} a_{ij}(t-\theta_{i-1})^j + P_{i-1}(\theta_{i-1}), \quad i = 2, \dots, n. \tag{2.3}$$

Definition 2.7. Vector $A = (a_0, a_{11}, a_{12}, \dots, a_{1m_1}, a_{21}, \dots a_{2m_2}, \dots, a_{nm_n}) \in \mathbb{R}^{\gamma+1}$, $\gamma = \sum_{j=1}^{n} m_j$ is called the vector of the parameters of the polynomial spline of generalised degree $M = (m_1, \dots, m_n)$.

A polynomial spline of generalised degree $M = (m_1, \dots, m_n)$ can be also presented as follows:

$$S_M(A, t) = a_0 + \sum_{i=1}^{n} \sum_{j=m_i-d_i+1}^{m_i} a_{ij}((\min\{t, \theta_i\} - \theta_{i-1})_+)^j, \tag{2.4}$$

where $A = (a_0, a_{11}, \dots, a_{nm_n}) \in \mathbb{R}^{\gamma+1}$ is the vector of spline parameters, $D = (d_1, \dots, d_n)$ is the defect vector, θ_i are the spline knots ($i = 0, \dots, n$, $a = \theta_0, b = \theta_n$),

$$(\xi(x))_+ = \begin{cases} \xi(x), & \xi(x) > 0, \\ 0, & \xi(x) \leq 0. \end{cases}$$

This presentation of polynomial splines is similar to (2.2). However, the meanings of spline parameters in (2.2) and (2.4) are not the same. There are two main advantages of (2.4) compared to (2.2).

Firstly, the presentation (2.4) allows for the construction of polynomial splines with various degree polynomials in different intervals, which is not straight forward in the case of (2.2) if, for example, $m_i > m_{i+1}$.

Secondly, the components of the vector of the spline parameters in (2.4) are the coefficients of the corresponding polynomials, namely

- $a_0, a_{11}, \dots, a_{1m_1}$ are the parameters of $P_1(t)$ from (2.3);
- $P_i(\theta_{i-1}), a_{i1}, \dots, a_{im_i}$ are the parameters of $P_i(t)$ from (2.3), $i = 2, \dots, n$.

Definition 2.8. Polynomial spline of generalised degree $M = (m_1, \ldots, m_n)$ and defect $D = (m_1, \ldots, m_n)$ are called the highest defect polynomial splines.

In this paper we are focusing on polynomial splines of degree one and defect one.

2.2 Necessary and Sufficient Optimality Conditions for Polynomial Splines with Fixed Knots

In this subsection we present a short review of existing results, covering optimality conditions for the case of polynomial and fixed knots polynomial spline approximation. These optimality conditions are based on the notion of alternance points.

Definition 2.9. A function $g(t)$ alternates p times in an interval $[a, b]$ if there exist $p+1$ points $t_i < t_{i+1} \in [a, b]$, such that

$$g(t_i) = -g(t_{i+1}) = \pm \max_{t \in [a,b]} |g(t)|.$$

Definition 2.10. Alternance points are the points where the absolute value of the deviation is maximal and the sign of the deviation at any two consequent points is opposite.

Necessary and sufficient optimality conditions in the case of polynomials have been obtained by Chebyshev (see [2]). Later they were generalised to some particular types of polynomial splines ([5, 7, 9]).

Theorem 2.11 (Chebyshev). *Necessary and sufficient conditions for a polynomial of degree m to be the best Chebyshev approximation are that the approximation interval contains at least $m+2$ alternance points.*

In 1967 this theorem was generalised to the case of polynomial splines of defect 1 (Rice, see [7], the characterization theorem).

Theorem 2.12 (Rice). *Let $f(t)$ be continuous in $[a, b]$ and let $S(A, t)$ be a polynomial spline of defect 1 and degree m. Then necessary and sufficient conditions for $S(A^*, t)$ to be a best Chebyshev approximation to $f(t)$ is that there exists a subinterval $[\theta_i, \theta_{i+p}]$, which contains at least $m + p + 1$ alternance points.*

Definition 2.13. A minimal length subsequence of intervals, where the condition of Rice theorem are satisfied is called a minimal subsequence (see [9]).

In the case of higher degree fixed knots polynomial splines necessary and sufficient optimality conditions can be found in [5, 9].

Apart from their theoretical importance, these optimality conditions can be used to develop an algorithm of optimal spline construction (Remez-type algorithm).

The classical Remez algorithm [6] plays an important role in the area of polynomial approximation. At each iteration of this algorithm necessary and sufficient optimality conditions for the case of polynomial approximation are verified. This method has been generalised to the case of fixed knots polynomial splines (see [5, 8]). The generalisations in [8] also require the verification of the necessary and sufficient optimality conditions for polynomial spline approximation.

3 Free Knots Linear Spline Approximation: Necessary Conditions

3.1 Formulation

In [1] the author uses piece-wise linear functions to separate two sets of points. These piece-wise linear functions have been constructed as a max min of some linear functions. In our research we use a similar approach to construct polynomial splines of degree one, which are simply piece-wise linear functions. The corresponding approximation problem can be formulated as follows:

$$\text{minimise} \quad \Psi(A,B) \quad \text{subject} \quad \text{to} \quad A \in \mathbb{R}^n, B \in \mathbb{R}^n, \tag{3.1}$$

where n is the number of subintervals,

$$\Psi(A,B) = \max_{t\in[a,b]} \max\{\varphi(A,B,t), -\varphi(A,B,t)\},$$

$$\varphi(A,B,t) = \max_{i\in I} \min_{j\in J_i}(a_{ij}t - b_{ij}) - f(t),$$

$J_i, i \in I$ are partitions.

Each partition is a subsequence of successive intervals, such that the corresponding piece-wise linear functions are concave (after taking min) and the number of intervals in the subsequence is maximal (maximal length subsequence). Then, by taking max over concave functions one gets the whole spline. Each interval can be identified as a pair $(i,j) : i \in I, j \in J_i$, basing on the partition. This pair is called an identification pair. Henceforth the following notation is used:
$T(i,j)$ is the interval with the identification pair (i,j),

$$\varphi_{i,j}(A,B,t) = (a_{ij}t - b_{ij}) - f(t), i \in I, j \in J_i; \tag{3.2}$$

$$I = \{1,\ldots,|I|\}, J_i = \{1,\ldots,j_i\}, j_i = |J_i|.$$

Remark 3.1. It is essential that the lengths of the corresponding subsequences are maximal.

Remark 3.2. In most practical problems optimal partitions are unknown. The goal of our study is to obtain a necessary optimality condition for polynomial spline approximation. Therefore, our goal is not to find an optimal partition, but to assess whether the obtained partition may be optimal or not.

Remark 3.3. Also notice that in the case of free knots polynomial splines necessary optimality conditions may not coincide with sufficient optimality conditions as the optimisation problem is not convex.

Now let us highlight some important characteristics of the points located in an approximation interval. These characteristics are direct consequences of partitioning.

- Points which are not internal knots (external knots and internal interval points) enjoy the following properties:
  - at each point there is exactly one $i \in I$, such that the max is reached;
  - at each point there is exactly one $j \in J_i$, such that the min is reached;

- Polynomial spline internal knots enjoy the following properties:
  - at each internal knot there are no more than two $i \in I$, such that the max is reached;
  - at each internal knot there are no more than two $j \in J_i$, such that the min is reached.

Definition 3.4. If for an internal knot θ there exists exactly one pair of indexes $i_1, i_2 \in I$ $(j_1, j_2 \in J_i)$, such that the max (min) is reached then this knot is a responsible for max (min) knot.

Definition 3.5. If for an internal knot θ there exist exactly two indexes $i_1, i_2 \in I$ $(j_1, j_2 \in J_i)$, such that max (min) is reached then this knot is a responsible for max (min) knot.

Definition 3.6. If an internal knot joins two linear functions with the same linear coefficients then the spline is smooth at this knot and the knot is called a smooth knot.

Notice that each internal knot is either responsible for a max or for a min or this knot is smooth.

Example 3.7.

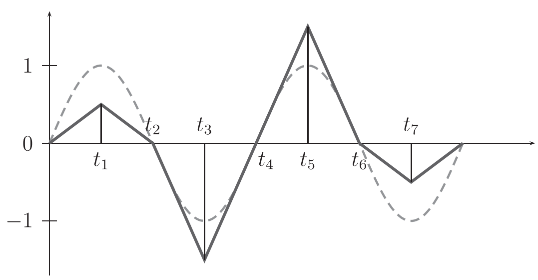

Figure 1: Example of regular and irregular alternance points.

Figure 1 represents a linear spline approximating the dashed curve. The spline can be constructed as follows:

$$\max\{\min\{\ell_1, \ell_2, \ell_3\}, \min\{\ell_4, \ell_5, \ell_6\}, \min\{\ell_7\}, \min\{\ell_8\}\},$$

where $\ell_k, k = 1, \dots, 8$ are the linear functions corresponding to the spline in the kth interval. Then $I = \{1, 2, 3, 4\}$, $J_1 = \{1, 2, 3\}$, $J_2 = \{1, 2, 3\}$, $J_3 = \{1\}$, and $J_4 = \{1\}$: ℓ_1, ℓ_2, ℓ_2 correspond to J_1, ℓ_4, ℓ_5, ℓ_6 correspond to J_2, ℓ_7 corresponds to J_3,ℓ_8 corresponds to J_4.

Functions ℓ_4 and ℓ_5 coincide. Hence the knot t_4 is a smooth knot. Points t_3, t_6 and t_7 are max-knots; t_1, t_2 and t_5 are min-knots. Unlike the other knots, t_2, t_4 and t_6 are not points of maximal deviation, and so are not considered in the construction of the quasidifferentials.

3.2 Characterization Theorem for Free Knots Linear Spline Approximation: Existing Results

An extensive review of existing results in the area of free knots polynomial spline approximation can be found in [5]. In this book one can find a characterization theorem obtained for free knots polynomial spline approximation. Since in this paper we are focusing on polynomial splines of degree one, we reformulate the characterization theorem from [5] to the case of linear splines.

Theorem 3.8 (Characterization theorem). *Suppose that a continuous or discrete function (data) is approximated by free knots polynomial splines of degree one. A necessary condition for a linear spline to be optimal is that there exists a subsequence of k subintervals of the original interval $[a, b]$, such that this subsequence contains at least $2k + 1$ alternance points.*

3.3 Optimality Conditions for Free Knots Linear Spline Approximation

The objective function in optimisation problem (3.1) is nonsmooth and nonconvex. This function is quasidifferentiable (see [3]). A necessary condition for a point (A^*, B^*) to be a local minimum of $\Psi(A, B)$ is as follows [3]:

$$-\overline{\partial}\Psi(A^*, B^*) \subset \underline{\partial}\Psi(A^*, B^*), \tag{3.3}$$

where $\underline{\partial}\Psi(A^*, B^*)$ and $\overline{\partial}\Psi(A^*, B^*)$ are sub- and superdifferential of $\Psi(A^*, B^*)$ at (A^*, B^*). The condition (3.3) is also known as a stationarity condition.

Definition 3.9. For a given pair (A^*, B^*) points t_k, where $\varphi(A^*, B^*, t_k)$ reaches its maximal absolute value are called maximal deviation points.

A subdifferential $\underline{\partial}\Psi(A, B)$ at (A^*, B^*) can be constructed as follows:

$$\underline{\partial}\Psi(A^*, B^*) = \mathrm{co}\left\{ \bigcup_{k\in K^+} \underline{\partial}\varphi(A^*, B^*, t_k) - \left(\sum_{i\in K^+, i\neq k} \overline{\partial}\varphi(A^*, B^*, t_i) - \sum_{i\in K^-} \underline{\partial}\varphi(A^*, B^*, t_i) \right), \right. \tag{3.4}$$

$$\left. \bigcup_{k\in K^-} -\overline{\partial}\varphi(A^*, B^*, t_k) - \left(\sum_{i\in K^+} \overline{\partial}\varphi(A^*, B^*, t_i) - \sum_{i\in K^-, i\neq k} \underline{\partial}\varphi(A^*, B^*, t_i) \right) \right\}$$

and the corresponding superdifferential $\overline{\partial}\Psi(A, B)$ at (A^*, B^*) can be constructed as follows:

$$\overline{\partial}\Psi(A^*, B^*) = \sum_{k\in K^+} \overline{\partial}\varphi(A^*, B^*, t_k) - \sum_{k\in K^-} \underline{\partial}\varphi(A^*, B^*, t_k), \tag{3.5}$$

where $\underline{\partial}\varphi(A^*, B^*, t_k)$ and $\overline{\partial}\varphi(A^*, B^*, t_k)$ are sub- and superdifferentials of $\varphi(A, B, t)$ at (A^*, B^*, t_k), K^+ and K^- are sets of indexes, such that

- $t_k : k \in K^+$ are maximal deviation points and $\varphi(A^*, B^*, t_k) > 0$;
- $t_k : k \in K^-$ are maximal deviation points and $\varphi(A^*, B^*, t_k) < 0$.

Definition 3.10. A maximal deviation point t_k is regular if

$$k \notin (K^+ \bigcap L_{min}) \bigcup (K^- \bigcap L_{max}).$$

Otherwise, a maximal deviation point t_k is irregular.

Henceforth, the following notation is used:

$$\nabla_C \varphi(C,t) = \frac{\partial \varphi_{ij}(C,t)}{\partial C},$$

where $C = (A, B)$.

Sub- and superdifferentials of $\varphi(A,B,t)$ at (A^*,B^*,t_k) can be constructed as follows:

1. Let $t_k : k \in L_{smooth}$ be an internal point of one of the intervals, one of the external knots or an internal knot, which joins linear functions with the same slope (smooth knot). Suppose that the identification pair of this interval is (i, j). Then

$$\underline{\partial}\varphi(A^*,B^*,t_k) = \operatorname{sign}(\varphi_{i,j}(A^*,B^*,t_k))\nabla_C\varphi_{i,j}(A^*,B^*,t_k),$$

$$\overline{\partial}\varphi(A^*,B^*,t_k) = 0_{2n},$$

where n is the number of subintervals.

2. Let $t_k : k \in L_{max}$ is be internal knot which is responsible for max. Suppose that the identification pairs for the corresponding intervals are (i, j_i) and $(i + 1, 1)$. Then

$$\underline{\partial}\varphi(A^*,B^*,t_k) = \operatorname{sign}(\varphi_{i,j_i}(A^*,B^*,t_k))\operatorname{co}\{\nabla_C\varphi_{i,j_i}(A^*,B^*,t_k), \nabla_C\varphi_{i+1,1}(A^*,B^*,t_k)\},$$

$$\overline{\partial}\varphi(A^*,B^*,t_k) = 0_{2n},$$

where n is the number of subintervals.

3. Let $t_k : k \in L_{min}$ is be internal knot which is responsible for min. Suppose that the identification pairs for the corresponding intervals are (i, j) and $(i, j + 1)$. Then

$$\underline{\partial}\varphi(A^*,B^*,t_k) = 0_{2n},$$

$$\overline{\partial}\varphi(A^*,B^*,t_k) = \operatorname{sign}(\varphi_{i,j}(A^*,B^*,t_k))\operatorname{co}\{\nabla_C\varphi_{i,j}(A^*,B^*,t_k), \nabla_C\varphi_{i,j+1}(A^*,B^*,t_k)\},$$

where n is the number of subintervals.

Therefore, the superdifferential can be calculated as follows

$$\overline{\partial}\Psi(A^*,B^*) = 0_{2n} + \sum_{k\in K^+\bigcap L_{min}} \operatorname{co}\{\nabla_C\varphi_{i_k,j_k}(A^*,B^*,t_k), \nabla_C\varphi_{i_k,j_k+1}(A^*,B^*,t_k)\} - \tag{3.6}$$

$$- \sum_{k\in K^-\bigcap L_{max}} \operatorname{co}\{\nabla_C\varphi_{i_k,j_{i_k}}(A^*,B^*,t_k), \nabla_C\varphi_{i_k+1,1}(A^*,B^*,t_k)\} =$$

$$= 0_{2n} + \Sigma^+ - \Sigma^-.$$

and the subdifferential can be calculated as follows

$$\underline{\partial}\Psi(A^*,B^*) = \operatorname{co}\left\{ \bigcup_{k\in K^+\bigcap L_{smooth}} (\nabla_C\varphi_{i_k,j_k}(A^*,B^*,t_k) - \overline{\partial}\Psi(A^*,B^*)), \right. \tag{3.7}$$

$$\bigcup_{k\in K^+\bigcap L_{max}} (\operatorname{co}\{\nabla_C\varphi_{i_k,j_{i_k}}(A^*,B^*,t_k), \nabla_C\varphi_{i_k+1,1}(A^*,B^*,t_k)\} - \overline{\partial}\Psi(A^*,B^*)),$$

$$\bigcup_{k\in K^+\bigcap L_{min}} (0_{2n} - \sum_{l\in K^+\bigcap L_{min}, l\neq k} \operatorname{co}\{\nabla_C\varphi_{i_l,j_{i_l}}(A^*,B^*,t_l), \nabla_C\varphi_{i_l+1,1}(A^*,B^*,t_l)\} + \Sigma^-),$$

$$\bigcup_{k\in K^- \bigcap L_{smooth}} (-\nabla_C \varphi_{i_k,j_k}(A^*,B^*,t_k) - \overline{\partial}\Psi(A^*,B^*)),$$

$$\bigcup_{k\in K^- \bigcap L_{max}} (0_{2n} - \Sigma^+ + \sum_{l\in K^- \bigcap L_{max}, l\neq k} \mathrm{co}\{\nabla_C \varphi_{i_l,j_{i_l}}(A^*,B^*,t_l), \nabla_C \varphi_{i_l+1,1}(A^*,B^*,t_l)\}),$$

$$\left.\bigcup_{k\in K^- \bigcap L_{min}} (-\mathrm{co}\{\nabla_C \varphi_{i_k,j_k}(A^*,B^*,t_k), \nabla_C \varphi_{i_k,j_k+1}(A^*,B^*,t_k)\} - \overline{\partial}\Psi(A^*,B^*))\right\}.$$

Notice that

$$\nabla_C \varphi_{i,j}(A^*,B^*,t_k) = \mathrm{sign}\,(\varphi_{i,j}(A^*,B^*,t_k))(0,\dots,1,t_k,0\dots,0)^T, \tag{3.8}$$

the non-zero block corresponds to the coefficients of the interval with the identification pair (i,j).

Notice that if $(K^+ \bigcap L_{min}) \bigcup (K^- \bigcap L_{max})$ is empty then the stationarity condition (3.3) at (A^*,B^*) can be simplified to the following:

$$0_{2n} \in \underline{\partial}\Psi(A^*,B^*). \tag{3.9}$$

Notice also, that the verification of (3.3) is not easy, especially when

$$(K^+ \bigcap L_{min}) \bigcup (K^- \bigcap L_{max}) \neq \emptyset.$$

In the next section we present an approach which allows one to simplify the verification procedure in the case of $\overline{\partial}\Psi(A^*,B^*) = 0_{2n}$ and some special cases of $\overline{\partial}\Psi(A^*,B^*) \neq 0_{2n}$. Similar to the case of fixed knots spline approximation, this verification approach is based on the notion of alternance.

4 Necessary Condition Verification

4.1 Regular Maximal Deviation Points

In this subsection we are concentrating on a necessary optimality condition when

$$(K^+ \bigcap L_{min}) \bigcup (K^- \bigcap L_{max}) = \emptyset.$$

In this case all maximal deviation points are regular.

Notice that

$$\mathrm{co}\{X_1,\dots,X_p,\mathrm{co}\{Y_1,\dots,Y_q\}\} = \mathrm{co}\{X_1,\dots,X_p,Y_1,\dots,Y_q\},$$

where $X_i, Y_j, i = 1,\dots,p, j = 1,\dots,q$ are arbitrary vectors in $\mathbb{R}^n$. Then the condition (3.9) can be presented in the following way: there exists a nonnegative solution of a linear homogeneous system

$$T\Lambda = 0_{2n}, \tag{4.1}$$

where T is a matrix whose columns correspond to the points of maximal deviation t_k, namely:

- $k \in L_{smooth}$ and t_k belongs to an interval with an identification pair (i,j) then the corresponding column is

$$\mathrm{sign}\,(\varphi_{i,j}(A^*,B^*,t_k))\nabla_C \varphi_{i,j}(A^*,B^*,t_k);$$

- $k \in K^+ \bigcap L_{max}$ and t_k belongs to an interval with identification pairs (i, j_i) and $(i+1, 1)$ then the corresponding columns are

$$\nabla_C \varphi_{i,j_i}(A^*, B^*, t_k), \quad \nabla_C \varphi_{i+1,1}(A^*, B^*, t_k);$$

- $k \in K^- \bigcap L_{min}$ and t_k belongs to an interval with identification pairs (i, j) and $(i, j+1)$ then the corresponding columns are

$$-\nabla_C \varphi_{i,j}(A^*, B^*, t_k), \quad -\nabla_C \varphi_{i,j+1}(A^*, B^*, t_k).$$

Define T as a matrix whose columns correspond to maximal deviation points from a subsequence of intervals. Then, similar to fixed knots polynomial spline approximation, minimal subsequence can be defined as follows.

Definition 4.1. A minimal length subsequence of intervals, such that 0_{2n} can be constructed with the columns of the corresponding matrix T is called a minimal subsequence of intervals.

Suppose that in our case the minimal subsequence starts at the $p-$th interval and ends at the $q-$th interval.

Each interval $l : p \leq l \leq q$ with m_l maximal deviation points corresponds to a block T_l, which can be presented as follows:

$$\left(\text{sign}\,(\varphi_{i,j}(A^*, B^*, t_1^l)) \begin{pmatrix} 1 \\ t_1^l \end{pmatrix}, \quad \dots, \quad \text{sign}\,(\varphi_{i,j}(A^*, B^*, t_{m_l}^l)) \begin{pmatrix} 1 \\ t_{m_l}^l \end{pmatrix} \right) \tag{4.2}$$

The corresponding components $\lambda_{l,1}, \dots, \lambda_{l,m_l}$ of the vector Λ are nonnegative. Consider a new auxiliary subsystem

$$T_l \Lambda_l = 0_2, \tag{4.3}$$

where $\Lambda_l = (\lambda_{l,1}, \dots, \lambda_{l,m_l})^T$. Therefore, in this interval there are at least three maximal deviation points (assume for simplicity that they are the points t_1^l, t_2^l, t_3^l and $\lambda_{l,1}, \lambda_{l,2}, \lambda_{l,3}$ are the corresponding coefficients), such that the following system has a positive solution:

$$\begin{pmatrix} \text{sign}\,(\varphi(A^*, B^*, t_1^l)) & \text{sign}\,(\varphi(A^*, B^*, t_2^l)) \\ \text{sign}\,(\varphi(A^*, B^*, t_1^l)) t_1^l & \text{sign}\,(\varphi(A^*, B^*, t_2^l)) t_2^l \end{pmatrix} \begin{pmatrix} \lambda_{l,1} \\ \lambda_{l,2} \end{pmatrix} = \tag{4.4}$$

$$= -\lambda_{l,3} \text{sign}\,(\varphi(A^*, B^*, t_3^l)) \begin{pmatrix} 1 \\ t_3^l \end{pmatrix}$$

Equivalently, system (4.4) has a positive solution if and only if the following system has a nonzero solution

$$\begin{pmatrix} 1 & 1 \\ t_1^l & t_2^l \end{pmatrix} \begin{pmatrix} \bar{\lambda}_{l,1} \\ \bar{\lambda}_{l,2} \end{pmatrix} = -\bar{\lambda}_{l,3} \begin{pmatrix} 1 \\ t_3^l \end{pmatrix}, \tag{4.5}$$

where $\bar{\lambda}_{l,i} = \text{sign}\,(\varphi(A^*, B^*, t_i^l)) \lambda_{l,i}, i = 1, 2, 3$. The transpositions of the corresponding system matrices are Vandermonde matrices. Using Cramer's rules deduce that each interval of a minimal subsequence should contain at least three alternance points. Therefore, the following theorem holds.

Theorem 4.2. *Suppose that a continuous or discrete function (data) is approximated by a free knots linear spline piece-wise linear function). Suppose also that all the maximal deviation points are regular and there exists an interval with at least three alternance points t_k. Then the stationarity condition (3.3) is satisfied.*

4.2 Irregular Maximal Deviation Points

In this subsection we assume that not all the maximal deviation points are regular. First we have to prove the following lemma.

Lemma 4.3. *Suppose that $\bar{B}, \bar{A}_i, i = 1, \ldots, m$ are convex compact sets in $\mathbb{R}^n$ then*

$$\bar{B} \subset \text{co}\{\bigcup_{i=1,\ldots,m} (\bar{A}_i + \bar{B})\} \Leftrightarrow 0_n \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}.$$

Proof. 1. $0_n \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\} \Rightarrow \bar{B} \subset \text{co}\{\bigcup_{i=1,\ldots,m}(\bar{A}_i + \bar{B})\}$.

Since $0_n \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}$ there exist vectors $\bar{a}_i \in \bar{A}_i, i = 1, \ldots, m$ and coefficients $\alpha_i \geq 0$, $i = 1, \ldots, m$, $\sum_{i=1}^m \alpha_i = 1$, such that $0_n = \sum_{i=1}^m \alpha_i \bar{a}_i$.

Then an arbitrary vector $\bar{b} \in \bar{B}$ can be constructed as follows:

$$\bar{b} = \bar{b} + 0_n = \bar{b} + \sum_{i=1}^m \alpha_i \bar{a}_i = \sum_{i=1}^m \alpha_i (\bar{a}_i + \bar{b}) \in \text{co}\{\bigcup_{i=1,\ldots,m} (\bar{A}_i + \bar{B})\}.$$

2. $\bar{B} \subset \text{co}\{\bigcup_{i=1,\ldots,m}(\bar{A}_i + \bar{B})\} \Rightarrow 0_n \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}$.

Suppose that $\bar{B} \subset \text{co}\{\bigcup_{i=1,\ldots,m}(\bar{A}_i + \bar{B})\}$, but $0_n \notin \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}$.

Define $\bar{a}_0 \in \text{co}\{\bigcup_{i=1,\ldots,m}(\bar{A}_i)\}$ as follows

$$\|\bar{a}_0\| = \min_{\bar{a} \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}} \|\bar{a}\|.$$

Then $\forall \bar{a} \in \text{co}\{\bigcup_{i=1,\ldots,n} \bar{A}_i\}, \quad \bar{a}_0(\bar{a} - \bar{a}_0) \geq 0 \Rightarrow \bar{a}_0 \bar{a} \geq \|\bar{a}_0\|^2 > 0.$

Define

$$b_0 = \text{argmin}_{\,b \in B} \bar{a}_0 b,$$

therefore

$$\bar{a}_0(\bar{b} - \bar{b}_0) \geq 0 \quad \forall \bar{b} \in \bar{B}.$$

Then, for $\bar{c} \in (\bar{B} + \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\})$ obtain

$$\bar{a}_0(\bar{c} - \bar{b}_0) = \bar{a}_0(\bar{a} + \bar{b} - \bar{b}_0) = \bar{a}_0 \bar{a} + \bar{a}_0(\bar{b} - \bar{b}_0) \geq \bar{a}_0 \bar{a} > 0$$

$$\forall \bar{a} \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}, \quad \bar{b} \in \bar{B}.$$

Therefore $\bar{a}_0(\bar{c} - \bar{b}_0) > 0 \quad \forall \bar{c} \in (\bar{B} + \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\})$.
Since $\bar{a}_0(\bar{b}_0 - \bar{b}_0) = 0$, $\bar{b}_0 \in \bar{B}$, but $\bar{b}_0 \notin (\bar{B} + \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\})$. This contradicts the original assumption and therefore

$$\bar{B} \subset \text{co}\{\bigcup_{i=1,\ldots,m} (\bar{A}_i + \bar{B})\} \Rightarrow 0_n \in \text{co}\{\bigcup_{i=1,\ldots,m} \bar{A}_i\}.$$

□

The following theorem is a direct consequence of theorem 4.2 and lemma 4.3.

Theorem 4.4. *Suppose that a continuous or discrete function (data) is approximated by a free knots linear spline. Suppose also that there exists an interval with at least three regular alternance points. Then the stationary condition (3.3) is satisfied.*

Proof. Assume that the set $-\bar{B}$ from lemma 4.3 coincides with $\overline{\partial}\Psi(A^*, B^*)$ and vectors $\bar{A}_i, i = 1, \dots, m$ from lemma 4.3 are the vertices of the set $\bigcup_{t^* \in \mathrm{Reg}} \underline{\partial}\varphi(A, B, t^*)$, where Reg is the set of all regular alternance points. The combination of lemma 4.3 and theorem 4.2 completes the proof. □

Comparing the results of theorem 4.4 with the existing results (theorem 3.8, see also [5]), one can notice that in the case of one-interval subsequences the existence of three alternance points is enough to satisfy the condition of theorem 3.8. However, theorem 4.4 still questions about the optimality of this spline. The following example shows that the "doubts" are reasonable.

Example 4.5. The approximation interval $[a, b]$ contains two subintervals. The second interval contains three alternance points $t_1 < t_2 < t_3$. The deviation at t_1 is positive. Suppose that t_1 is an irregular alternance point, t_2 and t_3 are regular points. Then for stationarity the following condition should satisfy:

$$\mathrm{co}\left\{\begin{array}{cc} -1 & 0 \\ -t_1 & 0 \\ 0 & -1 \\ 0 & -t_1 \end{array}\right\} \subset \mathrm{co}\left\{\begin{array}{ccccc} -1 & 0 & -1 & 0 & 0 \\ -t_1 & 0 & -t_1 & 0 & 0 \\ -1 & -2 & 1 & 0 & 0 \\ -t_2 & -t_1 - t_2 & t_3 & -t_1 + t_3 & 0 \end{array}\right\}.$$

However, the first vector from the left hand side of the inclusion does not belong to the right hand side of the inclusion. Therefore, this spline, indeed, cannot be optimal.

5 Conclusions and Further Research Directions

In this paper a new necessary condition for a particular type of free knots linear splines has been obtained. This condition improves the earlier known condition. Also, in this paper we show that internal knots may belong to two different categories (regular and irregular). This characteristic for internal knots has not been identified before. We insist that this characteristic is very important and may be used for further improvements of the existing optimality condition in the case of higher degree polynomial splines.

Further research directions are to obtain

- optimality conditions for piece-wise linear approximations with irregular alternance points (general case);
- optimality conditions for higher dimension and higher degree polynomial splines;
- a modification of the Remez algorithm, based on the obtained optimality conditions.

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References

[1] A. Bagirov, Max-min separability, *Optimization methods and Software* 20 (2005) 271–290.

[2] P.L. Chebyshev, The theory of mechanisms known as parallelograms, Selected works, Publishing House of the USSR Academy of Sciences, Moscow 1955, pp. 611-648 (Russian).

[3] V.F. Demyanov and A.M. Rubinov, *Quasidifferentiability and Related Topics*, Kluwer Academic Publisher, 2000, 400 p.

[4] R. DeVore and G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[5] G. Nürnberger, *Approximation by Spline Functions*, Springer-Velgar, Berlin, 1989.

[6] E.Y. Remez, General computational methods of Chebyshev approximation. Problems with linear real parameters, Izv. Akad. Nauk Ukrain. SSR, Kiev, 1957. Translation available from United States Atomic Commission, Washington, D.C., 1962.

[7] J. Rice. *Characterization of Chebyshev approximation by splines*, *SIAM J. Numer. Anal.* 4 (1967) 557–567.

[8] N.V. Soukhoroukova, A generalisation of Vallèe-Poussin theorem and exchange basis rules to the case of Polynomial splines, in *Proceedings of Stability and Control Processes SCP2005*, Dedicated to the 75-years anniversary of V.I. Zubov, Vol.2 , pp. 948–958.

[9] M.G. Tarashnin, *Application of the theory of quasidifferentials to solving approximation problems*, – PhD thesis, St-Petersburg State University, 1996, 119 p.

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