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CONICALLY EQUIVALENT CONVEX SETS AND APPLICATIONS

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Abstract: Given a normed space X and a cone $K \subseteq X$, two closed, convex sets A and B in X^* are said to be K-equivalent if the support functions of A and B coincide on K. We characterize the greatest set in an equivalence class, analyze the equivalence between two sets, find conditions for the existence and the uniqueness of a minimal set, extending previous results. We give some applications to the study of gauges of convex radiant sets and of cogauges of convex coradiant sets. Moreover we study the minimality of a second order hypodifferential.

Key words: *convex sets, support function, convex radiant sets, Minkowski gauge, convex coradiant sets, cogauge, hypodifferentiable functions*

Mathematics Subject Classification: *52A07, 52A30, 46B20, 49J52*

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1 Introduction

It is well known that a lower semicontinuous sublinear function h defined on a normed space X is completely characterized by its support set

$$\partial h := \{\ell \in X^* : \langle x, \ell \rangle \le h(x), \forall x \in X\},$$

where ∂h stands for $\partial h(0)$, the subdifferential of h at the origin, which is a w^*-closed convex set in the topological dual space X^*; moreover one can recover h from ∂h by the formula

$$h(x) = \sup\{\langle x, \ell \rangle : \ell \in \partial h\},$$

that is, h is the support function of the convex set ∂h. Exploiting the above relations we obtain the well-known duality between the family $\mathcal{C}^*(X^*)$ of nonempty, w^*-closed, convex sets of X^* and the family $\mathcal{S}(X)$ of proper, lower semicontinuous, sublinear functions $p : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ (see e.g. [14]).

In a number of problems one needs to consider the restriction of h to some cone K. If K is not the whole space, there are several support functions which agree with h on K and two sets A and B in X^* are said to be equivalent with respect to K if their support functions coincide, when restricted to K; that is if it holds

$$\sigma_A(x) := \sup\{\langle x, \ell \rangle : \ell \in A\} = \sigma_B(x) := \sup\{\langle x, \ell \rangle : \ell \in B\}, \qquad \forall x \in K.$$

Given a sublinear function $h \in \mathcal{S}(X)$ and a cone $K \subseteq X$ we obtain an equivalence class, formed by those sets $A \subseteq X^*$ such that

$$\sigma_A(x) := \sup\{\langle x, \ell\rangle : \ell \in A\} = h(x), \qquad \forall x \in K.$$

Conically equivalent sets were studied in [5], where an application to the calculus of codifferentials was also given. We extend here some results from [5], by considering unbounded sets in infinite dimensional spaces, and give further results about conically equivalent sets.

In Section 2 we study the greatest set in some equivalence class and give some characterization for the equivalence between two sets. Special emphasis is given to the search for the sets which are minimal with respect to inclusion, and to the conditions which guarantee uniqueness.

The remaining sections are devoted to applications. In Section 3 we deal with the description of a closed, convex, radiant set $C \subseteq X$ by means of a sublinear gauge. Although the term gauge often includes the property of being nonnegative, we say here that a positively homogeneous function $p : X \to \mathbb{R}_\infty$ is a *gauge* for C if $C = [p \leq 1]$. This definition is meaningful for all radiant sets, but we only deal with convex sets and will only consider sublinear gauges. The Minkowski gauge (see the definition in Section 4) is shown to be the greatest sublinear gauge of a convex radiant set. We obtain two different characterizations, expressed in dual terms, for $p \in \mathcal{S}(X)$ to be a gauge of C and describe the support set of the least gauge of C. Moreover we study the conditions under which the Minkowski gauge is minimal. In this case it is the unique sublinear gauge of C. Such conditions apply for instance to bounded convex sets and, more generally, to continuous convex sets, as introduced by Gale and Klee [12]. In Section 4 we consider some results, presented in [23], about the existence of continuous cogauges of a convex coradiant set and the existence of a minimal cogauge. Such results are interpreted, and somewhat extended, in the light of the analysis of Section 2. Section 5 deals with the existence and the minimality of a second order hypodifferential, as introduced by Demyanov and Rubinov [7].

We consider a normed space X, in which the closed ball of radius δ centered in x is denoted by $B_\delta(x) = B(x, \delta)$; the closure, interior, boundary of some set $S \subseteq X$ are denoted by $\operatorname{cl} S$, $\operatorname{int} S$ and $\operatorname{bd} S$ respectively; the convex hull and the conic hull of S are denoted, respectively, as $\operatorname{conv} S$ and $\operatorname{cone} S = \{y = \lambda x : x \in S, \lambda > 0\}$. Let X^* be the topological dual space of X, endowed with the weak* topology and denote by $\langle x, \ell\rangle$, or equivalently $\ell(x)$, the usual bilinear pairing between $x \in X$ and $\ell \in X^*$. For a function $f : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$ and $r \in \mathbb{R}$ we denote by $[f \leq r]$ the sublevel set $\{x \in X : f(x) \leq r\}$ and by $[f \geq r]$ the superlevel set $\{x \in X : f(x) \geq r\}$. The symbols $[f < r]$ and $[f > r]$ have similar meanings.

2 Conically Equivalent Sets

Let us consider the family $\mathcal{S}(X)$ of all proper lower semicontinuous (l.s.c. for short) sublinear functions $h : X \to \mathbb{R}_\infty$. There exists a convex and w^*-closed subset A of X^* such that for all $u \in X$

$$h(u) = \sup_{\ell \in A} \langle u, \ell\rangle. \tag{2.1}$$

Let us denote by $\mathcal{C}^*(X^*)$ the family of all nonempty, convex and w^*-closed subsets of X^* (and analogously $\mathcal{C}(X)$ for the family of all nonempty, closed, convex subsets of X). In the sequel we will often drop the prefix w^*- as no other topology for X^* will be used.

Definition 2.1. Given a cone $K \subseteq X$, the sets A,$B \in \mathcal{C}^*(X^*)$ are said to be *equivalent* w.r.t. the cone K, or K-equivalent, denoted $A \sim_K B$, if

$$\sup_{\ell \in A} \langle u, \ell \rangle = \sup_{\ell \in B} \langle u, \ell \rangle \qquad \forall u \in K. \tag{2.2}$$

Since any proper l.s.c. sublinear function vanishes at the origin, it is obvious that (2.2) always holds for $u = 0$. For the same reason, it is not relevant if the origin belongs to K or not. Moreover the requirement that K is a cone is not restrictive, since, due to positive homogeneity, for any set $C \subseteq X$ it holds

$$\sigma_A(u) = \sigma_B(u) \quad \forall u \in C$$

if and only if the same equality holds for all u in the conic hull of C.

Given any proper, l.s.c. sublinear function h and its support set A, we indicate by $\mathcal{E}_K(A) \subset \mathcal{C}^*(X^*)$ the family of all sets equivalent to A, with respect to K:

$$\mathcal{E}_K(A) := \left\{ C \in \mathcal{C}^*(X^*) : \, h(u) = \sup_{\ell \in C} \langle u, \ell \rangle \quad \forall u \in K \right\}.$$

Observe that, if $K = X$, then $\mathcal{E}_K(A) = \{A\}$, but if $K \neq X$ the family $\mathcal{E}_K(A)$ can be quite rich. We consider here several problems related to an equivalence class $\mathcal{E}_K(\,\cdot\,)$: describe the conditions ensuring the equivalence $A \sim_K B$, characterize the minimal sets inside this class, give conditions for uniqueness and find a formula to compute such a minimal set.

We start with the description of the greatest set in an equivalence class $\mathcal{E}_K(\,\cdot\,)$. Given the cone $K \subseteq X$ and the function $h \in \mathcal{S}(X)$, with $\partial h = A$, let us consider the following set:

$$G_K(A) := \{\ell \in X^* : \, \langle u, \ell \rangle \leq h(u), \quad \forall u \in K\}.$$

Such a set is convex and closed, being the intersection of closed halfspaces in X^*, but in general it is not bounded.

Denoting by $\iota_S : X \to \mathbb{R}_\infty$ the *indicator function* of the set S, that is $\iota_S(x) = 0$ if $x \in S$ and $\iota_S(x) = +\infty$ for $x \notin S$, and though ι_K is a convex function only when the cone K is convex, we can rewrite $G_K(A)$ as the support set of the function $s(x) = \sigma_A(x) + \iota_K(x)$, that is

$$G_K(A) = \partial s = \partial(\sigma_A + \iota_K). \tag{2.3}$$

Note that the sum $\sigma_A + \iota_K$ is convex if and only if $b(A) \cap K$ is convex, where $b(A) = \{x \in X : \sigma_A(x) < +\infty\}$ is the *barrier cone* of A.

For every $x \in X$ and $\alpha \in \mathbb{R}$, denote by

$$H^-[x, \alpha] := \{a \in X^* : \, \langle x, a \rangle \leq \alpha\}$$

the lower halfspace determined by x and α. By convention let, for every $x \in X$, $H^-[x, +\infty] = X^*$.

For $A \in \mathcal{C}(X^*)$, we have by definition

$$G_K(A) = \bigcap_{k \in K} H^-[k, \sigma_A(k)] = \bigcap_{k \in K \cap b(A)} H^-[k, \sigma_A(k)] = G_{K \cap b(A)}(A), \tag{2.4}$$

since for $k \notin b(A)$ we have $H^-[k, \sigma_A(k)] = X^*$.

Some straightforward properties of $G_K(\cdot)$, whose verification is trivial, are the following:

1. for every set $A \in X^*$, it holds $G_K(A) = G_K(\operatorname{cl}\operatorname{conv}(A))$;

2. for every set $A \in \mathcal{C}^*(X^*)$ it holds $G_K(A) \in \mathcal{E}_K(A)$, that is the restriction to K of the support function of $G_K(A)$ concides with $\sigma_A|_K$;
3. for all $B \in \mathcal{E}_K(A)$, it holds $B \subseteq G_K(A)$;
4. if $B \in \mathcal{C}^*(X^*)$ and $A \subseteq B$, then $B \in \mathcal{E}_K(A)$ if and only if $B \subseteq G_K(A)$;
5. if $K_1 \subseteq K_2$, then $G_{K_2}(A) \subseteq G_{K_1}(A)$;
6. if A is bounded (i.e. if σ_A is continuous on X), then $G_{cl\,K}(A) = G_K(A)$, so that $G_K(A) = A$ (and consequently $\mathcal{E}_K(A) = \{A\}$), if K is dense in X.

We also note that in general $G_K(A) \neq G_{conv\,K}(A)$, as shown for instance by Example 2.4 below.

To discuss the boundedness of $G_K(A)$ it is important to separate two important cases, namely that the cone K be contained in some closed halfspace or not. If the *polar cone* $K^+ = \{\ell \in X^* : \langle k, \ell\rangle \geq 0, \forall k \in K\}$ contains some nonzero element (or, equivalently, K is contained in a closed halfspace), then $G_K(A)$ is unbounded, as it holds

$$A - K^+ \subseteq G_K(A).$$

In particular we have the following result.

Proposition 2.2. *If $A \in \mathcal{C}(X^*)$ is bounded and the cone K is convex, then*

$$G_K(A) = A - K^+.$$

Proof. The inclusion $\supseteq$ is obvious. So suppose that $x^* \notin A - K^+$, which is w^*-closed, since A is w^*-compact and K^+ is w^*-closed. By the separation theorem, there exist $x \in X$ and $\alpha \in \mathbb{R}$ such that

$$\langle x, y^* - z^*\rangle \leq \alpha < \langle x, x^*\rangle, \qquad \forall y^* \in A,\ z^* \in K^+. \tag{2.5}$$

If it were $\langle x, z^*\rangle < 0$ for some nonzero $z^* \in K^+$, as the latter is a cone, we would have a contradiction to (2.5). Hence $\langle x, z^*\rangle \geq 0$ for all $z^* \in K^+$ and $x \in K^{++} = \operatorname{cl} K$. By taking $z^* = 0$ in (2.5), we obtain

$$\langle x, y^*\rangle \leq \alpha < \langle x, x^*\rangle, \qquad \forall y^* \in A,$$

that is $\sigma_A(x) \leq \alpha < \langle x, x^*\rangle$ and hence $x^* \notin G_{cl\,K}(A) = G_K(A)$. □

As the referee pointed out to us, the reader more acquainted with convex analysis would easily prove Proposition 2.2 recalling (2.3) and showing that the equalities

$$G_K(A) = G_{cl\,K}(A) = \partial\sigma_A + \partial\iota_{cl\,K} = A - K^+$$

hold under the given assumptions, as σ_A is convex continuous and ι_K is convex.

If $K^+ = \{0\}$, that is if K is not contained in any halfspace, then we may wonder if $G_K(A)$ is bounded, at least if so is A. This is indeed true in finite dimensional spaces.

Proposition 2.3. *Let $X = \mathbb{R}^n$ and A be bounded. If $K^+ = \{0\}$, then $G_K(A)$ is bounded.*

Proof. As $K^+ = \{0\}$, the cone K is not contained in any halfspace and hence $\operatorname{conv} K = \mathbb{R}^n$. Thus there exists a finite set of vectors $Y = \{x^1, x^2, ..., x^s\} \subset K$ such that $0 \in \operatorname{int} S$, where $S = \operatorname{conv}\{x^1, x^2, ..., x^s\}$.

To prove that $G_K(A)$ is contained in some bounded set, consider the polar set of S:

$$S^\circ := \{v \in \mathbb{R}^n : \langle x, v\rangle \le 1, \forall x \in S\}.$$

We have that S° is a polyhedral convex set (see [19, Sect. 19]) and $0 \in \operatorname{int} S$ implies that S° is a bounded set (see [19, Cor.14.5.1]). Actually it holds

$$S^\circ = \{v \in \mathbb{R}^n : \langle x^i, v\rangle \le 1, \forall i = 1, ..., s\} = Y^\circ.$$

Since A is bounded, each linear functional $\langle x^i, \cdot\rangle$ is bounded on A. Let

$$\alpha_i = \sup\{\langle x^i, v\rangle, v \in A\} = \sigma_A(x^i), \qquad i = 1, ..., s$$

and $\alpha = \max\{\alpha_1, \alpha_2, ..., \alpha_s\} > 0$. Setting $K' = \operatorname{cone} Y$, we have

$$G_K(A) \subseteq G_{K'}(A) = \bigcap_{i=1}^{s} H^-[x^i, \alpha_i] \subseteq \alpha S^\circ$$

and $G_K(A)$ is bounded. □

The above result cannot be extended in general to infinite dimensional spaces, as shown by the following example, which was suggested to us by J. E. Martínez-Legaz.

Example 2.4. Consider the Hilbert space $X = l_2$, with the usual norm $\|x\| = \left(\sum_{i=1}^{+\infty} x_i^2\right)^{1/2}$. Let $A = B_X$, the unit ball, and let K be the cone of all sequences having at most one nonzero term. Obviously A is bounded and K is not contained in any halfspace. On the other hand $G_K(A)$ consists of all sequences which belong to l_2 and whose terms belong to the interval $[-1, 1]$. This set is unbounded, as it contains the sequence $\{x^n\}$, whose general element has the first n components equal to 1 and all the remaining components equal to 0 (we have $\|x^n\| = \sqrt{n}$).

To have an impression of the type of questions we will deal with, consider the following simple example.

Example 2.5. Let us consider $X = \mathbb{R}^2$, the cone $K = \{(x, y) \in \mathbb{R}^2 : y = 0, x \ge 0\}$ and the function $h(x, y) = 0$ for all $(x, y) \in X$, with $\partial h = \{0\} = A$. In this case we have

$$G_K(A) = \{(u, v) \in \mathbb{R}^2 : ux \le 0, \forall x > 0\} = \{(u, v) \in \mathbb{R}^2 : u \le 0\}$$

and $B \in \mathcal{E}_K(A)$ if and only if $\sup\{u : (u, v) \in B\} = 0$. Each singleton set $\{(0, v)\}$, $v \in \mathbb{R}$ is minimal by inclusion inside the class $\mathcal{E}_K(A)$.

The most remarkable fact in Example 2.5 is that, while the greatest element $G_K(A)$ is well defined, the family $\mathcal{E}_K(A)$ contains many different sets which are minimal w.r.t. inclusion and a least element does not exist.

In order to give further results on conically equivalent sets and discuss, in particular, the question of minimality, we need to introduce one more concept.

Definition 2.6. Given $A \in \mathcal{C}^*(X^*)$ and the cone $K \subset X$, we say that the point $\ell \in A$ is *illuminated* by K (or K-illuminated), and denote it by $\ell \in \operatorname{ill}_K(A)$, if there exists some $u \in K\backslash\{0\}$ such that $\langle u, \ell\rangle = \sigma_A(u)$. Then u is said to illuminate A.

The set $\mathrm{ill}_K(A)$ is formed by all w^*-*support points* of A, for which the w^*-support functionals are elements of K. We will use the notation $SF(A)$ to denote the set of w^*-support functionals of A. And will say that a pair $(a, x) \in X^* \times X$ is a *supporting pair* for A, if $a \in A$ is a support point in A with w^*-support functional x.

It is easy to verify that $K_1 \subseteq K_2$ implies $\mathrm{ill}_{K_1}(A) \subseteq \mathrm{ill}_{K_2}(A)$, for every $A \in \mathcal{C}^*(X^*)$.

If we want to obtain uniqueness for the minimal set inside a given equivalence class and find a formula to compute it starting from a set A we need additional hypotheses on the cone K.

We will assume in the sequel that K satisfies the following condition:

$$\mathrm{int}\ K = K \setminus \{0\}\,. \tag{2.6}$$

Since every support function σ satisfies $\sigma(0) = 0$, there is no loss in generality if we suppose that K does not contain the origin and substitute (2.6) with the requirement that K is open.

Theorem 2.7. *Let the cone $K \subseteq X$ be open. If $A, B \in \mathcal{C}^*(X^*)$ and $A \sim_K B$, then*

$$\mathrm{ill}_K\ (A) = \mathrm{ill}_K\ (B)\,. \tag{2.7}$$

Proof. Let us assume that (2.7) does not hold, i.e there exists $\bar{a} \in \mathrm{ill}_K\ (A)$ s.t. $\bar{a} \notin \mathrm{ill}_K\ (B)$. Since $\bar{a} \in \mathrm{ill}_K\ (A)$, there exists $\bar{u} \in K$ s.t.

$$\langle \bar{u}, \bar{a}\rangle = \sigma_A(\bar{u})\,.$$

On the other hand $\bar{a} \notin \mathrm{ill}_K\ (B)$ and $\sigma_A(\bar{u}) = \sigma_B(\bar{u})$ imply $\bar{a} \notin B$. By the separation theorem there exist $x \in X$ and $\delta > 0$ s.t.

$$\langle x, b - \bar{a}\rangle \leq -\delta < 0, \qquad \forall b \in B.$$

If we consider the point

$$x_\lambda = (1-\lambda)\,\bar{u} + \lambda x$$

with $\lambda \in (0, 1]$, we have that

$$\langle x_\lambda, b - \bar{a}\rangle = (1-\lambda)\,\langle \bar{u}, b - \bar{a}\rangle + \lambda\langle x, b - \bar{a}\rangle, \qquad \forall b \in B.$$

As

$$\langle \bar{u}, b\rangle \leq \sigma_B(\bar{u}) = \sigma_A(\bar{u}) = \langle \bar{u}, \bar{a}\rangle, \qquad \forall b \in B,$$

it follows that

$$\langle x_\lambda, b - \bar{a}\rangle \leq -\delta\lambda < 0,$$

i.e. for all $b \in B$

$$\langle x_\lambda, b\rangle \leq -\delta\lambda + \langle x_\lambda, \bar{a}\rangle. \tag{2.8}$$

Since $\bar{u} \in \mathrm{int}\ K$, then for $\lambda > 0$ small enough we have that $x_\lambda \in K$. Moreover, as $\bar{a} \in A$, we have

$$\langle x_\lambda, \bar{a}\rangle \leq \sigma_A(x_\lambda)\,.$$

As (2.8) holds, then

$$\sigma_B(x_\lambda) = \sup_{b\in B}\langle x_\lambda, b\rangle \leq -\delta\lambda + \langle x_\lambda, \bar{a}\rangle \leq \sigma_A(x_\lambda) - \delta\lambda < \sigma_A(x_\lambda)\,,$$

which is a contradiction. □

Theorem 2.7 extends a similar result from [5]. The proof is given for the sake of completeness, although the arguments are not new.

With the aim of giving a converse to Theorem 2.7, we start by studying the relation between the support function of a set $A \in \mathcal{C}^*(X^*)$ and the one of its illuminated points. This analysis needs at least two warnings. First of all, as initially discovered by V. Klee [15] and then analyzed e.g. by Borwein and Tingley [4] and Fonf [11], in every incomplete normed space X there exist instances of closed, bounded, convex sets with no support points. Thus the set of illuminated points may be empty. Moreover the set of illuminated points may fail to conveniently describe a convex set A when a w^*-functional $u \in K$ is unbounded on A. The following example illustrate this situation.

Example 2.8. Let $X = \mathbb{R}^2$ with $K = \operatorname{int}\mathbb{R}^2_+ \cup \operatorname{int}\mathbb{R}^2_-$ and $A = \mathbb{R}^2_+$. In this case all w^*-functionals in $\operatorname{int}\mathbb{R}^2_-$ support A at the origin (and at no other point), while all w^*-functionals in $\mathbb{R}^2_+$ are unbounded above on A. Hence $\operatorname{ill}_K(A) = \{0\}$.

On the other hand we have that $\sigma_A(x_1, x_2) = 0$ for all pairs $(x_1, x_2) \in \mathbb{R}^2_-$ and $\sigma_A(x_1, x_2) = +\infty$ otherwise, so that the sets A and $B := \operatorname{ill}_K(A)$ have the same sets of illuminated points but not the same support functions.

In order to overcome these problems and prove that a set $A \in \mathcal{C}^*(X^*)$ and its set of illuminated points have the same support functions, we will have to make some assumptions. First of all, we will have to require that the set of w^*-supporting functionals have some density property. This can be proved in various situations. For instance by assuming that A is w^*-compact, so that all elements in K are illuminating, or using a results by Phelps [18] (see also [3]), who proved that the set of w^*-supporting functionals of any set $A \in \mathcal{C}^*(X^*)$ is norm dense among those which are bounded above on A, provided X is a Banach space. Another possibility, exploited in Section 4, is based on the correspondence between supporting pairs for a convex set $C \subseteq X$ and those of its (reverse) polar set.

To overcome the second type of problems, we will have to pay attention to w^*-functionals $u \in K$, which are unbounded above on A and bounded above on $\operatorname{ill}_K(A)$.

Theorem 2.9. *Let X be a normed space and consider $A \in \mathcal{C}^*(X^*)$ and the open cone $K \subseteq X$. Suppose that $SF(A)$ is dense in $b(A)$. Setting $L := \operatorname{ill}_K(A)$ and $K_A := K \cap b(A)$, we have the following:*

a) $G_K(A) = G_{K_A}(L)$;

b) $\sigma_A(x) = \sigma_L(x), \qquad \forall x \in K_A.$

Proof. (a) Since $L \subseteq A$, and recalling (2.4), it holds

$$G_{K_A}(L) \subseteq G_{K_A}(A) = G_K(A).$$

To prove the converse relation, let $\bar{a} \notin G_{K_A}(L)$. Then there exists $k \in K \cap b(A)$ such that

$$\langle k, \bar{a}\rangle > \sup_{a\in L}\langle k, a\rangle = \sigma_L(k). \tag{2.9}$$

Consider the set $K' \subseteq K$ of all $k \in K$ which are w^*-supporting functionals for A, that is all $k \in K$ for which there exist $a_k \in A$ with $\langle k, a_k\rangle = \sigma_A(k)$. The vector a_k is indeed an illuminated point of A, and hence $a_k \in L$. This implies that $\sigma_A(k) = \sigma_L(k)$, for all $k \in K'$.

Since K is open, K' is dense in $K \cap b(A)$, and (2.9) implies that there exist $\bar{k} \in K'$ and $a_{\bar{k}} \in A$ such that

$$\langle \bar{k}, \bar{a}\rangle > \sup_{a\in A}\langle \bar{k}, a\rangle = \langle \bar{k}, a_{\bar{k}}\rangle = \sup_{l\in L}\langle \bar{k}, l\rangle,$$

so that $\bar{a} \notin G_{K\cap b(A)}(A) = G_K(A)$.

To prove (b) it is enough to recall that, for any set $C \in \mathcal{C}(X^*)$ and any cone K, $G_K(C) \in \mathcal{E}_K(C)$, so that (a), with $L = \mathrm{ill}_K(A)$, $B = \mathrm{cl\,conv}\, L$ and $x \in K \cap b(A)$, yields

$$\sigma_A(x) = \sigma_{G_K(A)}(x) = \sigma_{G_{K_A}(L)}(x) = \sigma_{G_{K_A}(B)}(x) = \sigma_B(x) = \sigma_L(x).$$

□

To illustrate what we learn from Theorem 2.9, let us return to Example 2.8. It holds $b(A) = \mathbb{R}^2_-$ and $K \cap b(A) = \mathrm{int}\,\mathbb{R}^2_-$ so that

$$A = G_K(A) = G_{K_A}(A) = G_{K_A}(L) \neq G_K(L) = \{0\}.$$

Moreover, for all $k \in K \cap b(A) = \mathrm{int}\,\mathbb{R}^2_-$, it holds $\sigma_L(k) = \sigma_A(k)$, while for $k \in K \setminus b(A) = \mathrm{int}\,\mathbb{R}^2_+$, it holds $\sigma_A(k) = +\infty \neq \sigma_L(k) = 0$.

Given the set $A \in \mathcal{C}^*(X^*)$, let us consider the set

$$M_K(A) := \mathrm{cl\,conv}\,(\mathrm{ill}_K(A)).$$

The next result shows how the information conveyed by the set $M_K(A)$ can be used in order to obtain the values of σ_A and gives a partial converse to Theorem 2.7.

Theorem 2.10. *Let X be a normed space and consider $A \in \mathcal{C}^*(X^*)$ and the open cone $K \subseteq X$. Suppose that $SF(A)$ is dense in $b(A)$. Then, setting $M = M_K(A)$, it holds*

a)
$$\sigma_A(u) = \sigma_M(u) + \iota_{b(A)}(u) \qquad \forall u \in K. \tag{2.10}$$

b) *For all sets $B \in \mathcal{C}^*(X^*)$ it holds*
$$B \sim_K A \iff M_K(A) \subseteq B \subseteq G_K(A) \quad \text{and} \quad b(B) \cap K = b(A) \cap K.$$

c) *If A is bounded, then $M_K(A)$ is the least element w.r.t. inclusion in $\mathcal{E}_K(A)$.*

Proof. (a) follows immediately from Theorem 2.9 and the fact that $\sigma_A(u) = +\infty$ when $u \notin b(A)$, irrespective of the value $\sigma_M(u)$. The 'only if' implication in (b) follows from Theorem 2.7, while the converse follows from part a). Finally c) is an immediate consequence of b), as $b(A) = X$ when A is bounded and $M_K(A) \subseteq A$ yields $b(A) \subseteq b(M_K(A))$. □

If we return to Example 2.8 we observe that the family $\mathcal{E}_K(A)$ contains infinitely many minimal sets and no least element exists. Indeed every set

$$B_\alpha = \{(v_1, v_2) \in \mathbb{R}^2 : \, v_2 = \alpha v_1, \, v_1 \geq 0\}, \qquad \alpha \geq 0,$$

together with $B_\infty = \{(v_1, v_2) : \, v_1 = 0, \, v_2 \geq 0\}$ satisfies $\mathrm{ill}_K(B_\alpha) = \{(0,0)\}$ and $b(B_\alpha) \cap K = \mathrm{int}\,\mathbb{R}^2_-$, so that

$$A \sim_K B_\alpha, \qquad \forall \alpha \in [0, +\infty].$$

Moreover each set B_α is minimal and no least element exists.

Theorem 2.10 (c) proves the existence of a least element in some class $\mathcal{E}_K(A)$ under the assumption that A is bounded. We will see in Sections 3 and 4 that the existence of a least element can be proved under less restrictive assumptions when the sets A and K have some special structure.

In [5] the issues of conical equivalence and minimality are studied under less restrictive assumptions on K than openness. More precisely the cone K is required to satisfy the condition $\mathrm{cl}\,(\mathrm{int}\, K) = \mathrm{cl}\, K$. Further results are given for cones with nonempty interior. We address the interested reader to [5] for more details.

3 Gauges of Convex Radiant Sets

This section and the following one show how the concepts developed in Section 2 can be used to discuss the functional representation (in primal terms) of some particular classes of convex sets. If C is a closed, convex set of X containing the origin, the Minkowski gauge μ_C is used for the description of C, in that it satisfies $C = [\mu_C \leq 1]$, and this equality finds several applications in Functional Analysis and in the theory of normed spaces. More generally we will call *gauge* of C any positively homogeneous function $p : X \to \mathbb{R}_\infty$ such that $C = [p \leq 1]$. In the next section we will consider a set C which is closed, convex and coradiant (see the definition below); in this case C can be described in functional terms by means of a *cogauge*, that is a positively homogeneous function q such that $C = [q \leq -1]$.

We introduce here some concepts which will be used in the next two sections.

Definition 3.1. The set $A \subseteq X$ is called *radiant* if $x \in A$, $t \in [0, 1]$ imply that $tx \in A$. It is called *coradiant* if its complement $A^C = X \backslash A$ is radiant, that is if either $A = X$ or $0 \notin A$ and $x \in A$, $t \geq 1$ imply that $tx \in A$.

We deduce that the empty set $\emptyset$ and the set X are both radiant and coradiant. Alternative definitions of a radiant and coradiant set can be given in terms of their kernel or outer kernel.

Definition 3.2. [20] The *kernel* of a set $A \subseteq X$ is the set of points

$$\ker A = \{z \in X : z + t(x - z) \in A, \forall x \in A, \forall t \in (0, 1]\}.$$

The *outer kernel* of a set $A \subseteq X$, $\operatorname{oker} A$, is the kernel of its complement A^C, that is the set

$$\operatorname{oker} A = \{z \in X : z + t(x - z) \notin A, \forall x \notin A, \forall t \in (0, 1]\}.$$

It is obvious that a set $A \subseteq X$ including the origin is radiant if and only if $0 \in \ker A$ and that a proper set A excluding the origin is coradiant if and only if $0 \in \operatorname{oker} A$.

Given a set $A \subseteq X$, we call *shadow* of A the set

$$\operatorname{shw} A = \{x \in X : x = ta, a \in A, t \geq 1\}.$$

If $0 \notin A$ then the set $B = \operatorname{shw} A$ is coradiant; it is indeed the smallest coradiant set containing A, that is the coradiant hull of A. It follows from the definition that, if $0 \in A$, then its coradiant hull coincides with X.

We are particularly interested in those radiant or coradiant sets which are also convex and, given a normed space X, will denote by $\mathcal{C}_0(X)$ the sets in $\mathcal{C}(X)$ which are radiant and by $\mathcal{C}_\infty(X)$ the sets in $\mathcal{C}(X)$ which are coradiant. It is easy to see that a convex set is radiant if and only if it contains the origin.

If $C \subseteq X$ is any radiant set, a number of features of C can be described by means of its Minkowski gauge $\mu_C : X \to \mathbb{R}_\infty$, where

$$\mu_C(x) = \inf\{\lambda > 0 : x \in \lambda C\},$$

which is a nonnegative positively homogeneous function which satisfies $[\mu_C < 1] \subseteq C \subseteq [\mu_C \leq 1]$. This relation specifies to $C = [\mu_C \leq 1]$ provided C is closed and this assumption will always be standing in what follows. Moreover the function μ_C is sublinear if and only if C is convex, and our attention in this paper restricts to this situation. In the latter case the support set $\partial\mu_C$ coincides with the polar set

$$C^\circ = \{\ell \in X^* : \langle x, \ell\rangle \leq 1, \forall x \in C\}.$$

For $C \in \mathcal{C}(X)$, μ_C is continuous (or equivalently finite valued) on X if and only if $0 \in \operatorname{int} C$ and equivalently if and only if C° is bounded.

It is easy to see that $\mu_C(x) = 0$ if the ray $R_x := \{y = \lambda x, \lambda > 0\}$ is contained in C and consequently it holds

$$[\mu_C = 0] = \operatorname{Rec} C, \tag{3.1}$$

where $\operatorname{Rec} C = \{d \in X : x + td \in C, \forall x \in C, \forall t \geq 0\}$ is the *recession cone* of C, a closed convex cone. Conversely it holds $[\mu_C > 0] = X \setminus \operatorname{Rec} C$.

Although the Minkowski gauge has so often and so succesfully been used to give a functional description of a convex radiant set C, it suffers some drawback if it is used in the framework of the separation theory for coradiant sets, as expressed in the following result, whose proof can be found in [22].

Theorem 3.3. *For a proper subset $F \subseteq X$ the following are equivalent:*

a) *F is closed and coradiant;*
b) *for every $x \notin F$ there exists an open convex radiant set G such that $x \in G$ and $F \cap G = \emptyset$;*
c) *for every $x \notin F$ there exists a continuous and sublinear function $p : X \to \mathbb{R}$ such that $p(x) > -1$ and $p(a) \leq -1$ for all $a \in F$.*

Theorem 3.3 shows that convex radiant sets can be used to separate points from coradiant sets and this can be expressed in functional terms using sublevel sets of a continuous sublinear function as the separating set. Part (c) of Theorem 3.3 can be easily proved by taking $p = \mu_G$, but this choice has a drawback: as the Minkowski gauge is always nonnegative, it can never become a linear function. We want to show that the separation property expressed in Theorem 3.3 is a true extension of the classical separation result for convex sets and, to reach this aim, we need to show that a different definition can be given for p, in a way that, in those cases in which the separating set G is a halfspace, its functional description gives a linear function.

As we are only interested here in convex sets, we will restrict our attention to sublinear representations and, given $C \in \mathcal{C}_0(X)$, we will see that the Minkowski gauge is not, in general, the only sublinear gauge of the set C.

For instance all functions

$$p_\alpha(x) = \begin{cases} \alpha x & x \leq 0 \\ x & x > 0 \end{cases} \qquad \alpha \in [0, 1]$$

are continuous sublinear gauges of the set $C = (-\infty, 1]$ and among them $\mu_C = p_0$ is the greatest, while the least one is given by $p_1(x) = x$, which is linear.

We will see in this section how the search for equivalent gauges of a convex radiant set, and in particular one which is minimal, is related to the results of Section 2. This topic is analysed further in [24]. Since the separating set G in Theorem 3.3 always has the origin as an interior point, we are particularly interested in this case, but will not restrict to it.

The following result, whose proof is straightforward, explains to what extent a sublinear gauge of some set $C \in \mathcal{C}(X)$ can differ from the Minkowski gauge.

Proposition 3.4. *Let $C \in \mathcal{C}_0(X)$. Then $p \in \mathcal{S}(X)$ is a gauge of C if and only if*

$$[p \leq 0] = \operatorname{Rec} C \qquad \text{and} \qquad (\, x \notin \operatorname{Rec} C \implies p(x) = \mu_C(x)\,)\,. \tag{3.2}$$

Since $\mu_C(x) = 0$ for all $x \in \operatorname{Rec} C$, we deduce from Proposition 3.4 that $p \in \mathcal{S}(X)$ is a gauge of C if and only if it holds

$$p \leq \mu_C \qquad \text{and} \qquad p(x) = \mu_C(x) \qquad \forall x \in X \backslash \operatorname{Rec} C,$$

and this can immediately be translated in terms of conical equivalence: indeed for $C \in \mathcal{C}_0(X)$ it holds $\mu_C = \sigma_{C^\circ}$ and $p \in \mathcal{S}(X)$ is a gauge of C if and only if

$$\partial p \subseteq \partial \mu_C = C^\circ \qquad \text{and} \qquad \partial p \sim_K C^\circ, \quad \text{with } K = X \backslash \operatorname{Rec} C. \tag{3.3}$$

In this case we are not interested in the determination of the set $G_K(C^\circ)$, since, if the condition $p \leq \mu_C$ is not satisfied, then p is not a gauge of C. Conversely the Minkowski gauge is always the greatest gauge of a set $C \in \mathcal{C}_0(X)$. The following example shows that the support set C° needs not be the greatest element in $\mathcal{E}_K(C^\circ)$.

Example 3.5. Take any linear continuous functional $0 \neq \ell \in X^*$ and let $C = [\ell \leq 1]$. Then we have $\operatorname{Rec} C = [\ell \leq 0]$ and

$$\mu_C(x) = \begin{cases} 0 & \text{if } \langle x, \ell \rangle < 0 \\ \alpha & \text{if } \langle x, \ell \rangle = \alpha \quad \geq 0 \end{cases}$$

Moreover $C^\circ = \{\alpha \ell,\, \alpha \in [0, 1]\}$, $K = [\ell > 0]$ and, according to Proposition 2.2, we have

$$G_K(C^\circ) = C^\circ - K^+ = \{\beta \ell,\, \beta \in (-\infty, 1]\}.$$

It is easy to see that the support function $\sigma_G(x)$ of the set $G := G_K(C^\circ)$, given by

$$\sigma_G(x) = \begin{cases} +\infty & \text{if } \langle x, \ell \rangle < 0 \\ \alpha & \text{if } \langle x, \ell \rangle = \alpha \quad \geq 0 \end{cases}$$

is not a gauge of C.

We pass now to characterize those sublinear functions which are gauges of a given set $C \in \mathcal{C}_0(X)$, and this will be done in terms of conical equivalence of support sets. As we always consider closed sets C, the recession cone $\operatorname{Rec} C$ is closed and $K = X \backslash \operatorname{Rec} C$ is open, thus allowing us to use the results in Section 2. Observe that the assumption of density in Theorem 3.6 and in the ones below is certainly satisfied if either X is Banach or if $0 \in \operatorname{int} C$.

Theorem 3.6. *Let X be a normed space and $C \in \mathcal{C}_0(X)$. Suppose that $SF(C^\circ)$ is dense in $b(C^\circ)$. Then for the l.s.c. sublinear function $p : X \to \mathbb{R}_\infty$ the following are equivalent:*

a) *p is a gauge of C;*

b) $\operatorname{ill}_K(C^\circ) + \operatorname{Rec}(C^\circ) \subseteq \partial p \subseteq C^\circ$, *with $K = X \setminus \operatorname{Rec} C$;*

c) $\operatorname{cl} \operatorname{conv}(\partial p \cup \{0\}) = C^\circ$.

Proof. Recalling Theorem 2.10 (b) and condition (3.3), we obtain that $p \in \mathcal{S}(X)$ is a gauge of C if and only if

$$b(\partial p) \setminus \operatorname{Rec} C = b(C^\circ) \setminus \operatorname{Rec} C \tag{3.4}$$

and

$$\operatorname{ill}_K(C^\circ) \subseteq \partial p \subseteq C^\circ, \tag{3.5}$$

with $K = X \setminus \operatorname{Rec} C$.

Since the equality $b(B) = \operatorname{cone}(B^\circ)$ holds for every $B \in \mathcal{C}^*(X^*)$, we have that $b(C^\circ) = \operatorname{cone}((C^\circ)^\circ) = \operatorname{cone} C = \operatorname{dom} \mu_C$ and $b(\partial p) = \operatorname{dom} p$, just by comparing the definitions. Then (3.4) is equivalent to

$$\operatorname{dom} p = \operatorname{cone} C, \tag{3.6}$$

since $\operatorname{Rec} C$ is contained in both sets.

In order to show that relations (3.6) and (3.5) are equivalent to the one in (b), we start by assuming that (b) holds. Then (3.5) is obvious. To obtain (3.6), observe that $\partial p \subseteq C^\circ$ implies $p \leq \mu_C$ and $\operatorname{cone} C = \operatorname{dom} \mu_C \subseteq \operatorname{dom} p$. Moreover (b) yields

$$\operatorname{Rec}(C^\circ) \subseteq \operatorname{Rec}(\partial p)$$

and consequently

$$(\operatorname{Rec}(\partial p))^- \subseteq (\operatorname{Rec}(C^\circ))^- = \operatorname{cone} C,$$

where $K^- = -K^+$ is the negative polar cone of a set $K \subseteq X$. To prove that $\operatorname{dom} p \subseteq \operatorname{cone} C$, it is enough to show that $\operatorname{dom} p \subseteq (\operatorname{Rec}(\partial p))^-$. Suppose that $\langle x, \ell \rangle > 0$ holds, for some $\ell \in \operatorname{Rec}(\partial p)$ and some $x \in X$. Since $p(x) \geq x^*(x) + t\ell(x)$ holds for all $t > 0$ and all $x^* \in \partial p$, then $p(x)$ cannot be finite valued, and $x \notin \operatorname{dom} p$.

Now we need to show that (3.5) and (3.6) imply (b). We only need to prove that $\operatorname{Rec}(C^\circ) \subseteq \operatorname{Rec}(\partial p)$. To this aim, take $\ell \in \operatorname{Rec}(C^\circ) = (\operatorname{cone} C)^-$, $t > 0$ and $x^* \in \partial p$. The inequality

$$p(x) \geq x^*(x) + t\ell(x)$$

is certainly true if $x \notin \operatorname{dom} p$. If $x \in \operatorname{dom} p$, since $\operatorname{dom} p = \operatorname{cone} C$, we have $\ell(x) \leq 0$ and again the inequality holds, and $\ell \in \operatorname{Rec}(\partial p)$.

To prove that (a) is equivalent to (c), observe that, taking into account (3.1), p is a gauge of C if and only if

$$\mu_C(x) = \max(p(x), 0), \qquad \forall x \in X.$$

Indeed if p is a gauge then the inequality $p(x) < 0$ is only possible for those x such that $\mu_C(x) = 0$, while $p(x) \geq 0$ yields $p(x) = \mu_C(x)$, so that $\mu_C = \max(p, 0)$.

By standard results about support functions, we know that, given two sets $A, B \in \mathcal{C}^*(X^*)$, it holds

$$\max(\sigma_A, \sigma_B) = \sigma_D,$$

where $D = \operatorname{cl}\operatorname{conv}(A \cup B)$. Hence

$$\mu_C(x) = \sigma_{C^\circ}(x) = \sigma_D(x), \qquad \forall x \in X,$$

with $D = \operatorname{cl}\operatorname{conv}(\partial p \cup \{0\})$ and the two support sets coincide. □

Various consequences of Theorem 3.6 should be underlined. The first is stated in the next result, whose proof is immediate.

Corollary 3.7. *Let X be a normed space and $C \in \mathcal{C}_0(X)$. Suppose that $SF(C^\circ)$ is dense in $b(C^\circ)$. Then the least gauge of C is the support function of the set* $\operatorname{ill}_K(C^\circ) + \operatorname{Rec}(C^\circ)$. *If moreover $0 \in \operatorname{int} C$ then the support set of the least gauge of C is given by*

$$M_K(C^\circ) := \operatorname{cl}\operatorname{conv}(\operatorname{ill}_K(C^\circ)).$$

The following example is useful to understand the content of the above result.

Example 3.8. Let $X = \mathbb{R}^2$ and $C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq -1, x_2 \geq 0, x_1 + x_2 \geq 0\}$. It holds

$$\mu_C(x_1, x_2) = \begin{cases} 0 & x_1 \geq 0,\, x_2 \geq 0 \\ -x_1 & x_1 \leq 0,\, x_1 + x_2 \geq 0 \\ +\infty & \text{else.} \end{cases}$$

To find the least gauge of C, we need to evaluate the following sets:

$$\begin{aligned} K &= X \setminus \operatorname{Rec} C = \{(x_1, x_2) : \min(x_1, x_2) < 0\}, \\ C^\circ &= \{(v_1, v_2) \in \mathbb{R}^2 : v_1 \leq 0,\, v_2 \leq 0,\, v_1 - v_2 \geq -1\}, \\ \operatorname{ill}_K (C^\circ) &= \{(v_1, v_2) : v_1 - v_2 = -1,\, v_1 \leq -1\}, \\ \operatorname{Rec} (C^\circ) &= \{(v_1, v_2) : v_1 - v_2 \geq 0,\, v_1 \leq 0\}, \end{aligned}$$

so that $\partial p = \operatorname{ill}_K (C^\circ) + \operatorname{Rec} (C^\circ) = \{(v_1, v_2) : v_1 \leq -1, v_1 - v_2 \geq -1\}$ and

$$p(x_1, x_2) = \begin{cases} -x_1 & x_2 \geq 0,\, x_1 + x_2 \geq 0 \\ +\infty & \text{else.} \end{cases}$$

Remark 3.9. Consider again the case when C is a halfspace, as in Example 3.5. Say $C = H^-[\ell, 1]$ for some nonzero $\ell \in X^*$. Then $C^\circ = \{\alpha\ell : \alpha \in [0, 1]\}$ and $\operatorname{ill}_K (C^\circ) = \{\ell\}$, so that the least gauge is linear.

An application of Theorem 3.6 (c) allows to characterize the cases in which μ_C is minimal. Since the Minkowski gauge is always the greatest gauge of C, if there exists no gauge lower than μ_C, then μ_C is the only sublinear gauge of C, hence also the least one.

Proposition 3.10. *Given $C \in \mathcal{C}_0(X)$, suppose that $SF(C^\circ)$ is dense in $b(C^\circ)$. Then the Minkowski gauge μ_C is the least sublinear gauge of C if and only if*

$$0 \in \operatorname{cl} \operatorname{conv} [\operatorname{ill}_K (C^\circ)] =: M_K(C^\circ).$$

Proof. It holds $0 \in M_K(C^\circ)$ if and only if

$$\operatorname{cl} \operatorname{conv} (M_K(C^\circ) \cup \{0\}) = M_K(C^\circ),$$

whence σ_M, for $M = M_K(C^\circ)$, is the minimal gauge of C. Applying Theorem 3.6 (c) we have $\sigma_M = \sigma_{C^\circ} = \mu_C$ and the result is proved. □

The following result allows to understand more clearly what type of convex sets have no sublinear gauges other than μ_C. We need to recall some properties of the barrier cone of a convex set C: it holds $b(C) = \operatorname{cone} (C^\circ)$ and moreover $\operatorname{cl} b(C) = (\operatorname{Rec} C)^-$, but $b(C)$ needs not be closed.

Theorem 3.11. *Given $C \in \mathcal{C}_0(X)$, suppose that $SF(C^\circ)$ is dense in $b(C^\circ)$. If $b(C)$ is not closed, then there exists no sublinear gauge of C lower than μ_C.*

Proof. We first check the equality

$$\operatorname{cl} \operatorname{conv} [\operatorname{ill}_K (C^\circ) \cup \{0\}] = C^\circ. \tag{3.7}$$

Indeed it is

$$C^\circ = \operatorname{cl} \operatorname{conv} [M_K(C^\circ) \cup \{0\}] = \operatorname{cl} \operatorname{conv} [\operatorname{cl} \operatorname{conv} (\operatorname{ill}_K (C^\circ)) \cup \{0\}] \tag{3.8}$$

and, calling C_1 the left hand side in (3.7) and C_2 the right hand side in (3.8), we obviously have $C_1 \subseteq C_2$. On the other hand it holds $\operatorname{cl}\operatorname{conv}(\operatorname{ill}_K(C^\circ)) \subseteq C_1$, $0 \in C_1$ and, since C_1 is closed and convex, we have $C_2 \subseteq C_1$, so that (3.7) holds.

As $b(C)$ is not closed, there exists $\ell \in (\operatorname{Rec} C)^- \setminus b(C)$. Since $b(C) = \operatorname{cone} C^\circ$, we have that $R_\ell \cap C^\circ = \emptyset$.

Since $(\operatorname{Rec} C)^- = \operatorname{cl}\operatorname{cone} C^\circ$, and recalling (3.7), there exists a net $\{\ell_\alpha\} \subseteq \operatorname{ill}_K(C^\circ)$ and a net $\{t_\alpha\}$ of positive real numbers, such that $t_\alpha \ell_\alpha$ converges to ℓ.

If t_α converges to $\bar{t} > 0$, then ℓ_α converges to $\bar{\ell} = \ell/\bar{t} \in C^\circ \subseteq b(C)$, which is not possible. If t_α converges to 0, then ℓ_α is unbounded and

$$\ell \in \operatorname{As}(C^\circ) = \operatorname{Rec}(C^\circ) \subseteq C^\circ, \tag{3.9}$$

where $\operatorname{As}(A) = \{\ell \in X^* : \exists \ell_\alpha \in A, \exists t_\alpha \to 0^+, \text{ with } \ell = \lim t_\alpha \ell_\alpha\}$ is the asymptotic cone of the set A and the last inclusion in (3.9) stems from the definition of recession cone, since $0 \in C^\circ$. Hence we have again a contradiction.

Thus we have $t_\alpha \to +\infty$ and ℓ_α converges to 0 in X^*. It is enough to apply Proposition 3.10 to conclude. □

Theorem 3.11 can be applied in particular if C is a continuous convex set. These sets were originally introduced by Gale and Klee [12], as those convex sets for which the support function is continuous on $X \setminus \{0\}$, and were more recently studied for instance in [1, 6, 10]. Among the many useful characterizations (in finite and infinite dimensional spaces), we have that C is continuous if and only if $b(C) \setminus \{0\} = \operatorname{int} b(C)$. Hence Theorem 3.11 applies to convex continuous sets.

We conclude this section with an example which shows that the converse to Theorem 3.11 does not hold, i.e. not all sets C for which the Minkowski gauge is minimal, have a barrier cone which is not closed.

Example 3.12. Consider the set $C \subseteq \mathbb{R}^2$ given by

$$C = \left\{(x_1, x_2) : -1 \le x_1 \le 1, \, x_2 \ge -\sqrt{1 - x_1^2}\right\}.$$

It holds $b(C) = \{(v_1, v_2) : v_2 \le 0\}$, which is closed. On the other hand we have $C^\circ = B_{\mathbb{R}^2} \cap \{(v_1, v_2) : x_2 \le 0\}$ and, since $\operatorname{Rec} C = \{(0, x_2) : x_2 \ge 0\}$ and $K = \{(x_1, x_2) : x_2 < 0\} \cup \{(x_1, x_2) : x_1 \ne 0, \, x_2 \ge 0\}$, we have that

$$\operatorname{ill}_K(C^\circ) = \{v = (v_1, v_2) : \|v\| = 1, \, v_2 \le 0\}$$

and $0 \in \operatorname{conv}\operatorname{ill}_K(C^\circ) = M_K(C^\circ)$, so that μ_C is minimal.

If the Minkowski gauge μ_C is not minimal for the set $C \in \mathcal{C}_0(X)$, then there exists a gauge p of C with $p(x) < 0$ for at least one $x \in \operatorname{Rec} C$. In this case 0 is not the minimal value of p and $0 \notin \partial p$. The existence of such a gauge of C can be characterized by reverting Proposition 3.10 and proving that the origin can be separated from the set of illuminated points. The study of those sets which admit a 'negative' gauge is carried out in [24].

4 Cogauges of Convex Coradiant Sets

A question very similar to the one treated in the previous section can be raised in connection to convex coradiant sets.

The conditions under which a convex coradiant set C admits a *continuous* sublinear cogauge were studied in [23]. In this section we wish to show how the results discussed in Section 2 can be used in order to obtain in a different way, and somehow extend, some results presented in [23], to which we refer for further details on the topics treated in this section.

In [23] the main attention was devoted to the functional characterization of a convex coradiant set C, which we also call *shady*, as in [17], and mainly to the possibility to define a *superlinear* continuous function $\varphi : X \to \mathbb{R}$ such that $C = [\varphi \geq 1]$. In order to make the discussion comparable to the present setting, in which sublinear functions are considered, we slightly modify our approach. We will say that a positively homogeneous function $p : X \to \overline{\mathbb{R}} = [-\infty, +\infty]$ is a *cogauge* of the coradiant set $A \subseteq X$ if $A = [p \leq -1]$.

The application of the Minkowski idea to a closed coradiant set $A \subseteq X$, yields the notion of Minkowski cogauge (see e.g. [20] for details):

$$\nu_A(x) := -\sup\{\lambda > 0 : \, x \in \lambda A\},$$

which is a real valued, positively homogeneous function, with $\nu_A < 0$ for all $x \in \operatorname{cone} A$ and $\nu_A(x) = 0$ otherwise. Notice that $-\nu_A$, rather than ν_A, was named Minkowski cogauge in [20].

A different functional description of a shady set C, was given by Barbara and Crouzeix [2], and relies on the concept of reverse polarity. Given a nonempty set $C \subseteq X$ we call *reverse polar* of C the set

$$C^{\ominus} := \{\ell \in X^* : \, \langle c, \ell \rangle \leq -1, \, \forall c \in C\}.$$

The name reverse polar is sometimes used (and this happens for instance in [23]) for the set $-C^{\ominus}$. We adopt the convention that $C^{\ominus} = X^*$ if $C = \emptyset$. It is easy to see that $C^{\ominus}$ is always closed, convex and coradiant in X^* and that $C \subseteq X$ is closed and shady if and only if it satisfies $C^{\ominus\ominus} = C$.

We are interested in the support function of $C^{\ominus}$, which turns out to be a sublinear cogauge of C. Assume that $C^{\ominus} \neq \emptyset$, i.e. C is contained in some closed halfspace disjoint from the origin, and let $\varphi_C : X \to \mathbb{R}_\infty$ be the function

$$\varphi_C(x) = \sup\{\langle x, \ell \rangle : \, \ell \in C^{\ominus}\} = \sigma_{C^{\ominus}}(x).$$

This is obviously a l.s.c. sublinear function. It was proved in [2] that the equality $\varphi_C(x) = \nu_C(x)$ holds for all $x \in \operatorname{cl} \operatorname{cone} C$, while $\varphi_C(x) = +\infty$ otherwise. Thus both ν_C and φ_C are cogauges of C and actually they are, respectively, the least and the greatest among all possible cogauges of C, that is if $p : X \to \mathbb{R}_\infty$ is a positively homogeneous cogauge of C, then it holds

$$\nu_C(x) \leq p(x) \leq \varphi_C(x), \qquad \forall x \in X.$$

Notice that ν_C is not sublinear, as it takes the value $0 = \inf(0, +\infty)$ outside the set $K = \operatorname{cone} C$. The main aim in [23] is to describe those shady sets C for which there exists a sublinear cogauge which is continuous and characterize the least sublinear cogauge. Since all cogauges p of C satisfy $p(x) = \nu_C(x) = \varphi_C(x)$ for all $x \in \operatorname{cone} C$, we can reformulate the same question in a different way. Given a closed, convex, coradiant set C, its reverse polar $C^{\ominus}$ and the cogauge $\varphi_C = \sigma_{C^{\ominus}}$, how can we describe the sets in $\mathcal{C}^*_\infty(X^*)$ which are equivalent to $C^{\ominus}$ with respect to $K = \operatorname{cone} C$? And how can we characterize the minimal set in $\mathcal{E}_K(C^{\ominus})$? In what cases can we find bounded sets in $\mathcal{E}_K(C^{\ominus})$ (so that their support functions are continuous)?

Obviously we have $C^{\ominus} = C^{\ominus} - K^{+} = G_K(C^{\ominus})$, and $\sigma_{C^{\ominus}} = \varphi_C$ is the maximal cogauge of C.

The following definition, which introduces some particular classes of shady sets, helps us to give an answer.

Definition 4.1. Let $C \subseteq X$ be a proper coradiant set. We will say that C is:

a) *coradiative* [20] if every ray from the origin has at most one intersection with the boundary of C;

b) *strongly shady* if $C \in \mathcal{C}_\infty(X)$ and $0 \in \operatorname{int} \operatorname{oker} C$;

c) *reducible* if $C \in \mathcal{C}_\infty(X)$ and there exists some $M > 0$ such that $C = \operatorname{shw}(C \cap B_M(0))$.

It is proved in [20] that a set $A \subseteq X$ is coradiative if and only if its Minkowski cogauge ν_A is continuous on X. Moreover, for a coradiative set A, it holds $\operatorname{bd} A = [\nu_A = -1]$ and hence, for a coradiative set $C \in \mathcal{C}_\infty(X)$, it holds

$$\operatorname{bd} C = [\nu_C = -1] = [\varphi_C = -1] \qquad \text{and} \qquad \operatorname{bd} \operatorname{cone} C = [\varphi_C = 0]. \tag{4.1}$$

It is possible to prove that every strongly shady set is coradiative (see [23]), while, for a convex coradiant set C, the specifications that C is coradiative and that C is reducible are mutually exclusive. The following result, which was proved in [23], explains why those two classes are important in order to find a continuous sublinear cogauge. It also shows that strongly shady sets and reducible sets are dual to each other.

Proposition 4.2. *[23] For a set $C \in \mathcal{C}_\infty(X)$ the following are equivalent:*

(a) *There exists a continuous sublinear function $p : X \to \mathbb{R}$ such that $[p \leq -1] = C$;*

(b) *C is strongly shady;*

(c) *$C^{\ominus}$ is reducible.*

This result underlines that the continuity of ν_C on X, which is guaranteed if the shady set C is coradiative, and its convexity on $K = \operatorname{cone} C$, do not imply that C admits a sublinear cogauge p which is continuous on X. For instance the set

$$C = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0,\, x_1 x_2 \geq 1\},$$

yields $\varphi_C(x_1, x_2) = -\sqrt{x_1 \cdot x_2}$ for $(x_1, x_2) \in \mathbb{R}^2_+$ and $+\infty$ elsewhere. This function cannot be extended to a continuous sublinear function defined on $\mathbb{R}^2$ since its subdifferential is empty at points $(0, x_2)$, with $x_2 \geq 0$ or $(x_1, 0)$, with $x_1 \geq 0$.

If we want to use the results of Section 2 to answer the questions raised above, we need to check whether the main assumptions are satisfied. The results of this verification are gathered together in the following proposition, which also contains a useful characterization of illuminated points of $C^{\ominus}$.

Proposition 4.3. *Let X be a normed space and $C \in \mathcal{C}_\infty(X)$ be coradiative. Then the following hold:*

a) *$K = \operatorname{cone} C$ is open;*

b) $b(C^{\ominus}) = \operatorname{cl}\operatorname{cone} C$;

c) *the set* $SF(C^{\ominus})$ *of* w^**-supporting functionals for* $C^{\ominus}$ *satisfies*

$$K = \operatorname{cone} C \subseteq SF(C^{\ominus}).$$

Hence $SF(C^{\ominus})$ *is dense in* $b(C^{\ominus})$;

d) *for* $K = \operatorname{cone} C$, *it holds*

$$\operatorname{ill}_K(C^{\ominus}) = \Lambda_C := \{\ell \in C^{\ominus},\ \ell(c) = -1 \text{ for some } c \in C\}. \tag{4.2}$$

Proof. a) Since the Minkowski cogauge ν_C of a coradiative set C is continuous (see [20]), then $K = \operatorname{cone} C = [\nu_C < 0]$ is open;

b) for every $x \in K = \operatorname{cone} C$ and $\ell \in C^{\ominus}$ it holds $\langle x, \ell\rangle < 0$; hence it holds $\langle x, \ell\rangle \le 0$ for all $x \in \operatorname{cl}\operatorname{cone} C$ and $\operatorname{cl}\operatorname{cone} C \subseteq b(C^{\ominus})$. If $w \notin \operatorname{cl}\operatorname{cone} C$, there exists $\ell \in X^*$ such that $\langle w, \ell\rangle > 0 \ge \langle x, \ell\rangle$ for all $x \in \operatorname{cl}\operatorname{cone} C$. As $\operatorname{cone} C$ is open, it holds $\langle x, \ell\rangle < 0$ for all $x \in \operatorname{cone} C$ and there exists $\alpha > 0$ such that $\bar\ell = \alpha\ell \in C^{\ominus}$. Since $\langle w, \bar\ell\rangle > 0$ and $C^{\ominus}$ is coradiative, then the linear functional $\langle w, \cdot\rangle$ is unbounded above on $C^{\ominus}$ so that $w \notin b(C^{\ominus})$;

c) since C is coradiative, if $x \in \operatorname{cone} C$ there exists (unique) $\alpha > 0$ such that $y = \alpha x \in \operatorname{bd} C$. As $\operatorname{int} C \neq \emptyset$ (recall that $C = [\nu_C \le -1]$ and ν_C is continuous), there exists $\ell \in C^{\ominus}$ such that $\langle y, \ell\rangle = -1$, hence (y, ℓ) is a supporting pair for C. This implies that (ℓ, y) is a w^*-supporting pair for $C^{\ominus}$ and $x \in SF(C^{\ominus})$.

d) take $\ell \in C^{\ominus}$ and $c \in C$ such that $\ell(c) = -1$. If $l \in C^{\ominus}$ then $l(c) \le -1$ and consequently

$$\ell(c) \ge l(c), \qquad \forall l \in C^{\ominus},$$

whence

$$\ell(c) = \max\{l(c) : l \in C^{\ominus}\}$$

which yields $\ell \in \operatorname{ill}_K(C^{\ominus})$.

If, conversely, $\ell \in \operatorname{ill}_K(C^{\ominus})$, there exists $k \in K = \operatorname{cone} C$ such that

$$\ell(k) = \max\{l(k) : l \in C^{\ominus}\}.$$

This yields $\ell(k) = \varphi_C(k)$.

Since C is coradiant and closed, the set $L_k = \{\alpha > 0 : \alpha k \in C\}$ is a nonempty interval of the type $[\bar\alpha, +\infty)$, with $\bar\alpha > 0$. The point $\bar c = \bar\alpha k$ satisfies $\varphi_C(\bar c) = -1$ and $\ell(\bar c) = \varphi_C(\bar c) = -1$ which implies $\ell \in \Lambda_C$.

□

Observe that Λ_C is nonempty whenever C has a nonempty interior, hence in particular when C is coradiative.

Part (d), which does not actually depend on $C \in \mathcal{C}_\infty(X)$ being coradiative, says that the set of illuminated points of $C^{\ominus}$ coincides with the radial boundary of $C^{\ominus}$, i.e. those points ℓ of the coradiant set $C^{\ominus}$ such that $\alpha\ell \notin C^{\ominus}$ for $\alpha < 1$.

The following result characterizes those sublinear functions which are cogauges of a closed shady set.

Corollary 4.4. *Let $C \in \mathcal{C}_\infty(X)$ be coradiative. Then the l.s.c. sublinear function $p : X \to \mathbb{R}_\infty$ is a cogauge of C if and only if*

$$\Lambda_C \subseteq \partial p \subseteq C^\ominus. \tag{4.3}$$

Moreover

$$M_K(C^\ominus) = \operatorname{cl}\operatorname{conv}\operatorname{ill}_K(C^\ominus),$$

with $K = \operatorname{cone} C$, is the least element in $\mathcal{E}_K(C^\ominus)$.

Proof. As we observed above, the assumption that C be coradiative is (necessary and) sufficient for $K = \operatorname{cone} C$ to be open. Moreover $K \subseteq b(C^\ominus)$, so that $b(C^\ominus) \cap K = b(\partial p) \cap K = K$ and (4.3) follows from (3.5). The proof of the last statement is immediate, since $M_K(C^\ominus)$ is contained in every other equivalent cogauge of C. □

Proposition 4.5. *Let C be strongly shady and $K = \operatorname{cone} C$. Then $M_K(C^\ominus)$ is bounded, and hence C admits a continuous sublinear cogauge.*

Proof. Let the outer kernel of C contain the ball $B(0,\delta) = B_\delta$, with $\delta > 0$. Then, for all $\ell \in \Lambda_C$, we have that $\ell(B_\delta) \geq -1$ and $\ell(B) \geq -1/\delta$. As B is symmetric, we have $|\ell(B)| \leq 1/\delta$ and $\|\ell\| \leq 1/\delta$, so that $\Lambda_C = \operatorname{ill}_K(C^\ominus)$ is bounded in X^*. □

The assumption that C be coradiative cannot be dispensed with. It is needed to guarentee that every cogauge be continuous on the boundary of K and this implies that every sublinear function which coincides with φ_C on K has positive values on its complement. If C were not coradiative this would not necessarily be true and we could find a sublinear function p which coincides with φ_C on K, but such that the sublevel set $[p \leq -1]$ does not coincide with C. For more details on the extension of convex functions from a convex domain to $X = \mathbb{R}^n$, see [21].

5 Minimality of a Second Order Hypo-differential

Let us recall the definition of a twice hypodifferentiable function, introduced by Demyanov and Rubinov in [7].

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called twice hypodifferentiable at the point $x \in \mathbb{R}^n$ if there exists a convex and compact set $d^2 f(x) \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ such that, for all $u \in \mathbb{R}^n$,

$$f(x+u) = f(x) + \max_{[a,l,A] \in d^2 f(x)} \left[a + \langle l, u \rangle + \frac{1}{2} \langle Au, u \rangle \right] + o_x(u), \tag{5.1}$$

with

$$\lim_{u \to 0} \|u\|^{-2} o_x(u) = 0, \tag{5.2}$$

being $\mathbb{R}^{n\times n}$ the space of all square matrices of order n. The set $d^2 f(x)$ is called a *second order hypodifferential* of the function f at the point x.

A particularly important instance is given by marginal functions. Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be

$$f(x) = \max_{y \in G} \varphi(x, y),$$

where $x \in S$, $y \in G$, S is an open set in $\mathbb{R}^n$, G is a compact set in $\mathbb{R}^m$ and the function $\varphi : S \times G \to \mathbb{R}$ is continuous on $S \times G$ and twice continuously differentiable as a function of x on the open set $S \subset \mathbb{R}^n$. Then we can write

$$f(x+u) = \max_{y \in G} \left[\varphi(x,y) + \langle \varphi'_x(x,y), u \rangle + \frac{1}{2} \langle \varphi''_{xx}(x,y) u, u \rangle + o_x(u,y) \right], \tag{5.3}$$

where $o_x(\cdot, y)$ satisfies (5.2) for all $y \in G$.

Let us also suppose that the gradient φ'_x and the Hessian matrix φ''_{xx}, are continuous with respect to the variable y on G; this implies that condition (5.2) holds, for the function $o_x\left(u, y\right)$ in (5.3), uniformly on G.

Under these assumptions (see [7]) the function f can be represented on S in the form

$$f\left(x+u\right) = f\left(x\right) + \max_{y \in G}\left[\varphi\left(x,y\right) - f\left(x\right) + \langle \varphi'_x\left(x,y\right), u\rangle + \frac{1}{2}\langle \varphi''_{xx}\left(x,y\right)u, u\rangle\right] + o_x\left(u\right),$$

with the remainder o_x satisfying condition (5.2). In this case the function f is twice hypodifferentiable at the point x, taking for example the second order hypodifferential given by

$$\begin{aligned} d^2 f(x) \quad = \quad & \operatorname{conv}\Big\{z = [a, l, A] \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} : \ a = \varphi(x,y) - f(x) \\ & l = \varphi'_x(x,y), \quad A = \frac{1}{2}\varphi''_{xx}(x,y), \quad y \in G\Big\}. \end{aligned}$$

For a fixed $x \in S$, let us take the function

$$h\left(u\right) = \max_{y \in G}\left[\varphi\left(x,y\right) - f\left(x\right) + \langle \varphi'_x\left(x,y\right), u\rangle + \frac{1}{2}\langle \varphi''_{xx}\left(x,y\right)u, u\rangle\right]$$

and let us consider the second order hypodifferential $d^2 f\left(x\right)$ of f at the point x. Such a set is not unique. For some $\bar{y} \in G$ and some given $[a', l, A] \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ such that

$$a' < \varphi\left(x, \bar{y}\right) - f\left(x\right), \qquad l = \varphi'_x\left(x, \bar{y}\right), \qquad A = \frac{1}{2}\varphi''_{xx}\left(x, \bar{y}\right),$$

we can take for instance the set

$$D = \operatorname{conv}\left\{d^2 f\left(x\right), [a', l, A]\right\}.$$

This is also a second order hypodifferential of the function f at the point x.

We can consider the problem of finding a minimal second order hypodifferential of f at the point x , i.e. the problem of finding a minimal convex compact set $L \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} = \mathbb{R}^{n^2+n+1}$ such that for all $u \in \mathbb{R}^n$, it holds

$$h\left(u\right) = \max_{\ell \in L}\langle \ell, (1, u, uu^*)\rangle, \tag{5.4}$$

where

$$u = \left(u_1, u_2, ..., u_n\right),$$

$$\begin{aligned} uu^* \quad &= \quad \left(u_1^2, u_1u_2, ..., u_1u_n, u_2u_1, u_2^2, u_2u_{3...}, u_2u_n, u_3u_1, u_3u_2, u_3^2, ..., u_{(n-1)}u_n, u_n^2\right) \\ &= \quad \operatorname{vec}\left(uu^T\right), \end{aligned}$$

and, for any matrix M, $\operatorname{vec}(M)$ is the (row) vector obtained by putting together all the rows of M and uu^T is the (rank 1) matrix obtained by multiplying the (column) vector u by its transpose u^T. Note that the inner product in (5.4) coincides with the one which can be written with the help of the Frobenius product of square matrices, $\langle A, B\rangle_F = \operatorname{tr}(A \cdot B)$. Indeed, with $\ell = (a, l, A) \in \mathbb{R}^{1+n+n^2}$, it holds

$$\langle \ell, (1, u, uu^*)\rangle = a + \langle l, u\rangle + \langle \operatorname{vec} A, uu^*\rangle = a + \langle l, u\rangle + \langle A, uu^T\rangle_F.$$

To treat this problem with the tools developed in Section 2, we must express h as a function of a new variable, so that it becomes sublinear. Let $q : \mathbb{R}^{1+n+n^2} \to \mathbb{R}_\infty$ be the support function of the set L. It holds

$$q(g) = \max_{\ell \in L} \langle \ell, (1, u, uu^*) \rangle = h(u),$$

with $g = [1, u, uu^*]$, $u \in \mathbb{R}^n$. If we set

$$K = \left\{ g \in \mathbb{R}^{n^2+n+1} : \, g = \lambda \left[1, u, uu^*\right], \quad u \in \mathbb{R}^n, \quad \lambda > 0 \right\},$$

such a cone does not satisfy condition (2.6) and then the results developed in Section 2 cannot be used to find a set L which is minimal with respect to K.

We have to give up our previous aim of finding a unique minimal second order hypodifferential of f and consider a more modest problem: given the second order hypodifferential $d^2 f(x)$, is it possible to reduce its size computing another second order hypodifferential of f that is included in the previous one? We will show how to perform this reduction in two steps.

Let us consider all the matrices A that appear as the third component of the second order hypodifferential $d^2 f(x)$. We can suppose without loss of generality that they are symmetric, and each of them can be represented by elements of the space $\mathbb{R}^{n(n+1)/2}$ in the following way:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{12} & a_{22} & & & \\ a_{13} & & & & \\ & & & & \\ a_{1n} & & & & a_{nn} \end{bmatrix}$$

$$\to \quad \widetilde{A} = \left(a_{11}, 2a_{12}, ..., 2a_{1n}, a_{22}, 2a_{23} ..., 2a_{2n}, a_{33}, ..., 2a_{(n-1)(n-1)}, 2a_{(n-1)n}, a_{nn}\right),$$

in which all off-diagonal elements of the matrix are multiplied by two. Then, for all w in the set

$$\widehat{K} = \left\{ w \in \mathbb{R}^{(n+2)(n+1)/2} : \, w = \lambda \left(1, u, \widetilde{u}\right), \lambda > 0, \, u = (u_1, u_2, ..., u_n), \right.$$
$$\left. \widehat{uu} = \left(u_1^2, u_1 u_2, ..., u_1 u_n, u_2^2, u_2 u_3 ..., u_2 u_n, u_3^2, ..., u_{(n-1)}^2, u_{(n-1)} u_n, u_n^2\right) \right\},$$

where the vector $\widehat{uu}$ puts together the rows corresponding to the upper triangular part of the matrix uu^T, we can consider the function

$$p(w) = \max_{\ell \in D^2 f(x)} \langle \ell, w \rangle, \tag{5.5}$$

with

$$w = (1, u, \widehat{uu}), \qquad u \in \mathbb{R}^n,$$

and

$$D^2 f(x) = \left\{ \ell = (a, l, \widetilde{A}) \in \mathbb{R}^{(n+1)(n+2)/2} : a = \varphi(x, y) - f(x), \right.$$
$$\left. l = (l_1, l_2, ..., l_n) = \varphi'_x(x, y), \quad A = \frac{1}{2} \varphi''_{xx}(x, y) \right\}.$$

Thus the description of f given by $D^2f(x)$ coincides with the one given by $d^2f(x)$ in (5.1), because writing

$$p(w) = p(1, u, \widetilde{u}) = h_1(u),$$

it holds $h_1(u) = h(u)$ for all $u \in \mathbb{R}^n$, where h is defined in (5.4). But p is defined on a space of lower dimension than q. Unfortunately we still cannot analyze this problem with the tools developed in Section 2 because $\widehat{K}$, despite the reduced dimensionality, is not open.

In any case we can consider the cone (more precisely a halfspace)

$$\widetilde{K} = \left\{ w = \lambda\,(1, u, v) :\ \lambda > 0,\ u \in \mathbb{R}^n,\ v \in \mathbb{R}^{n(n+1)/2} \right\} \subset \mathbb{R}^{(n+2)(n+1)/2},$$

which verifies $\widehat{K} \subseteq \widetilde{K}$ and satisfies condition (2.6). Then we can apply the previous results in order to find the unique minimal set equivalent to $D^2f\,(x)$ with respect to $\widetilde{K}$, i.e. $M_{\widetilde{K}}\left(D^2f\,(x)\right)$, which is certainly included in $D^2f(x)$.

On the other hand, as $\widetilde{K}$ is quite a large set, it may be that the set $M_{\widetilde{K}}\left(D^2f\,(x)\right)$ is still too big. To operate a further reduction of this set, we observe that, by the usual meaning of the remainder function $o(u)$ in (5.1), the quality of the approximation of the difference $f(x+u) - f(x)$ offered by the hypodifferential $d^2f(x)$ (or, equivalently, $D^2f(x)$), depends on the norm of the increment u. For this reason, given any $\varepsilon > 0$, we can substitute the function $p(w)$ given by (5.5), by any other function which coincides with it on the cone

$$\widetilde{K}_\varepsilon = \left\{ w = \lambda\,(1, u, v) :\ \lambda > 0,\ u \in \mathbb{R}^n,\ v \in \mathbb{R}^{n(n+1)/2},\ \|(u, v)\| < \varepsilon \right\} \subset \widetilde{K}$$

and obtain a different hypodifferential for f. Indeed in this case equation (5.1) holds with a new remainder function $o_1(u)$ which is different than $o(u)$ and coincides with it for all u in an appropriate neighbourhood of the origin, and hence satisfies (5.2). Since the cone $\widetilde{K}_\varepsilon$ is open, we can find the least element in class $\mathcal{E}_{\widetilde{K}_\varepsilon}(D^2f(x))$, which is a second order hypodifferential for f smaller than the set $D^2f(x)$ obtained above.

Bibliographic Note: After the paper was submitted, we became aware of the article [16], which treats a topic very similar to ours, and gives further understanding and source of applications to the interested reader. The analysis carried out in [16] can be seen as a particular instance of our analysis in different respects. The authors study sublinear and increasing functions defined on the nonnegative orthant of L^∞, and the possibility to extend them to the space L^∞. The analysis is based on some features which are specific of the setting chosen for the problem, but nevertheless some of the concepts developed in the paper have close relation to ours, as for instance the notion of quasiextremal points for subsets of L^1_+, which takes the place of illuminated points.

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