Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 6, No. 2, pp. 263–279, May 2010

RADIANT AND CORADIANT DUALITIES

Jean-Paul Penot

Dedicated to the memory of Alexander M. Rubinov

Abstract: We associate a dual problem to a constrained optimization problem in which the objective is quasiconvex and either attains at 0 its global minimum or its global maximum. An attractive feature of this duality is the fact that the dual problem has the same form as the primal problem and that duality can be iterated. We present conditions ensuring strong duality. We relate our approach to the Lagrangian theory. Our results rely on classical separation properties.

Key words: *coradiant set, coradiant function, dual problem, even convexity, lagrangian, quasiconvexity, radiant function, radiant set*

Mathematics Subject Classification: *26B25, 49N15, 90C26*

1 Introduction

For several minimization problems over a feasible set F, the global minimizer of the objective function f over the whole space X containing F is known. For instance, if X is a normed vector space (n.v.s.) and if f is the norm of X, 0 is the global minimizer of f. Such a fact does not yield any information about the location of the solutions of the constrained problem over F, nor about the value $\inf f(F)$. However, a knowledge of the value $\inf f(F)$ can be used to get duality relationships under weak convexity assumptions. The needs of relaxed convexity assumptions in several fields, in particular in mathematical economics, incite to push further the results obtained so far in this direction (see [1], [3], [5], [6], for instance). Quasiconcavity is often considered as an admissible assumption when dealing with an utility function u because the preference sets which are its superlevel sets have a concrete content while u itself is usually out of reach.

In [40], [41], [42], P.T. Thach gets optimality conditions for constrained problems under even convexity assumptions. The concept of even convexity introduced by Fenchel ([7]) has been studied by several authors ([4], [12], [18], [29], [30]....). A recent comprehensive study has appeared in [9]. Here, we rather focus on more classical topological assumptions such as closedness and semicontinuity. Thus, our results rely on classical separation theorems and complete the ones in [42]. We take advantage of the viewpoint of abstract convexity (see [25], [34], [37] for example), but our methods are close to the ones of familiar convex programming, albeit the functions we deal with are quasiconvex and not convex. Bringing the viewpoint of polarities enables to consider a whole range of possible dual problems; we give a short account of these possibilities. We also reveal the case of radiant functions and sets which is not treated in [42]. We endeavour to give a unified presentation.

Recall that a subset C of X is said to be *radiant* if it is starshaped and convex, i.e. if it is convex and $tx \in C$ for all $t \in [0,1]$ and $x \in C$; in particular, the empty set is radiant. Thus, our definition differs from the ones in [15], [34], [35] and [46], [47], [48] in which C is just starshaped; it also differs from the terminology used in [38], [40], [41]; here we incorporate convexity as it is a crucial assumption for our methods which essentially rely on separation properties. A nonconvex duality theory could be obtained in the line of [46], [47], but the conjugate would not be defined in the usual dual space. A subset C of X is *evenly radiant* if it is the intersection of a family of open half-spaces containing 0 or if $C = X$. A subset C of X is said to be *coradiant* if either $C = X$ or it is convex and *costarshaped* (or starshaped at infinity) with $0 \notin C$, i.e. if it is convex, if $0 \notin C$ and if for all $x \in C$, $t \in (1, +\infty)$ one has $tx \in C$. It is *evenly coradiant* if either $C = X$ or it is the intersection of a family of open half-spaces whose closures do not contain 0; such a set is also called R-evenly convex (e.g. in Thach [42]). Such a set is coradiant and evenly convex but the converse is known to be untrue ([42, p. 724]), as reminded to us by A. Zaffaroni. A function $f : X \to \overline{\mathbb{R}}$ is said to be (quasi-)*radiant* if its sublevel sets are radiant, i.e. if for every $x, x' \in X$, $t \in [0,1]$, one has $f(tx) \leq f(x)$, $f(tx + (1-t)x') \leq \max(f(x), f(x'))$. Equivalently, f is radiant if its strict sublevel sets are radiant. The function f is said to be (quasi)*coradiant* if its sublevel sets are coradiant, i.e. if $f(0) = \sup f(X)$ and for every $x, x' \in X$, $t \in [1, +\infty)$ one has $f(tx) \leq f(x)$, $f((1/t)x + (1 - (1/t))x') \leq \max(f(x), f(x'))$. Equivalently, f is coradiant if its strict sublevel sets are coradiant. It is *evenly (quasi-)coradiant* if its strict sublevel sets are evenly coradiant. Such functions are useful in mathematical economics (see [5], [6], [42, Section 4] and the references in [3], [24], [27]). Here, for the sake of brevity, we omit "quasi" because there is no risk of confusion with the corresponding concepts of radiant (or coradiant) functions which involve the epigraphs (resp. the hypographs) of the functions. (see [15], [20], [23]).

A remarkable fact about these classes of functions is that a conjugate can be defined on the dual space. For general quasiconvex functions, conjugacies are not as simple, since they involve an extra parameter (see [1], [13], [18], [21], [27], [30], [32] for instance).

In the following two sections we focus the attention on two dual problems. In section 4 we relate these problems to the general scheme of dual problems associated with polarities. We devote section 5 to some remarks about variants of the two problems we considered in sections 2 and 3. We conclude with a study of the relationships with Lagrangian theory.

$\boxed{2}$ An Adapted Framework

Given a function $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and a nonempty subset F of X, let us consider the constrained optimization problem

$$(P) \qquad \text{minimize } f(x) \qquad x \in F.$$

In the sequel, for $r \in \mathbb{R}$, we set $S_f^<(r) = \{x \in X : f(x) < r\}$ and $\varpi := \inf(P)$. We also write $[f < r]$ for $S_f^<(r)$ and $[f = r]$ for $\{x \in X : f(x) = r\}$. Throughout we suppose f assumes at least one finite value on the feasible set F, so that $\varpi < +\infty$.

We introduce the dual problems

$$(D_\triangledown) \qquad \text{maximize } -f_\triangledown(x^*) \qquad x^* \in F^\triangledown,$$
$$(D_\vartriangle) \qquad \text{maximize } -f_\vartriangle(x^*) \qquad x^* \in F^\vartriangle$$

where

$$f_{\nabla}(x^*) := -\inf\{f(x) : x \in X,\ \langle x, x^* \rangle \geq 1\}, \qquad F^{\nabla} := \{x^* \in X^* : \forall x \in F\ \langle x, x^* \rangle \geq 1\},$$
$$f_{\Delta}(x^*) := -\inf\{f(x) : x \in X,\ \langle x, x^* \rangle \leq 1\}, \qquad F^{\Delta} := \{x^* \in X^* : \forall x \in F\ \langle x, x^* \rangle \leq 1\}.$$

The conjugate f_{Δ} is related to the conjugate f^R considered in [42] through the equality $f_{\Delta}(x^*) = f^R(-x^*)$ for all $x^* \in X^*$. The conjugate f_{∇} is not considered in [42], but it is used in [20], [21], [24], [25], [38], [39], [43], [44] and elsewhere under different guises. In particular, on $X^*\backslash\{0\}$ it coincides with the conjugate f^H used in [38], [39] for the maximization of f on F. Note that setting $f^H(0) = \inf f(X)$ instead of $f_{\nabla}(0) := -\infty$ is also natural if one considers that f takes its values in some interval $[\alpha, \omega]$ with $\alpha := \inf f(X)$. In the mentioned references, some properties of these conjugates are established; see also Section 4 below in which the relationships with polarities are studied.

In order to deal simultaneously with the two dual problems (D_{∇}) and (D_{Δ}), we introduce the following notation. For $\Diamond =: \nabla$, we set $\varepsilon_{\Diamond} := 1$, while for $\Diamond =: \Delta$, we set $\varepsilon_{\Diamond} := -1$. Then, we can gather the two dual problems into the single one

$$(D_{\Diamond}) \qquad \text{maximize } -f_{\Diamond}(x^*) \qquad x^* \in F^{\Diamond},$$

where

$$F^{\Diamond} := \{x^* \in X^* : \forall x \in F\ \langle x, \varepsilon_{\Diamond} x^* \rangle \geq \varepsilon_{\Diamond}\},$$
$$f_{\Diamond}(x^*) := -\inf\{f(x) : x \in X,\ \langle x, \varepsilon_{\Diamond} x^* \rangle \geq \varepsilon_{\Diamond}\}.$$

Note that we can rewrite $(D_{\Diamond})$ as the equivalent adjoint problem

$$(P_{\Diamond}) \qquad \text{minimize } f_{\Diamond}(x^*) \qquad x^* \in F^{\Diamond}$$

which has a form similar to the one of (P). The adjoint problem of $(P_{\Diamond})$ is

$$(P_{\Diamond\Diamond}) \qquad \text{minimize } f_{\Diamond\Diamond}(x^{**}) \qquad x^{**} \in F^{\Diamond\Diamond}.$$

Its restriction to $X \subset X^{**}$ coincides with (P) when $\Diamond = \Delta$, F is closed and radiant and f is evenly coradiant by [42, Thm 2.3] and the bipolar theorem. When $\Diamond = \nabla$, F is closed and coradiant and f is evenly radiant with $f(0) = -\infty$, one can also show that the restriction of $(P_{\Diamond\Diamond})$ to $X \subset X^{**}$ coincides with (P); see Lemmas 4.3, 4.4 below.

We first observe that we have the weak duality inequality

$$-\inf(P_{\Diamond}) = \sup(D_{\Diamond}) \leq \inf(P), \tag{2.1}$$

since for all $x^* \in F^{\Diamond}$, we have $F \subset [\varepsilon_{\Diamond} x^* \geq \varepsilon_{\Diamond}] := \{x \in X : \langle \varepsilon_{\Diamond} x^*, x \rangle \geq \varepsilon_{\Diamond}\}$, hence $-f_{\Diamond}(x^*) = \inf f([\varepsilon_{\Diamond} x^* \geq \varepsilon_{\Diamond}]) \leq \inf f(F)$.

Let us give some examples. The first one illustrates the fact that the dual problem may be potentially simpler.

Example 1. Let $X := \mathbb{R}$, $F := [-a, a]$ for some $a > 0$ and let f be given by $f(x) := 0$ for $x \leq 0$, $f(x) := -x^2$ for $x > 0$. Then $F^{\Delta} = [-a^{-1}, a^{-1}]$ and $f_{\Delta}(y) = +\infty$ for $y \leq 0$, $f_{\Delta}(y) = y^{-2}$ for $y > 0$. Clearly (P) has a unique solution a and (D_{Δ}) has a unique solution a^{-1}. Both values are $-a^2$. Note that (P) is a nonconvex minimization problem while (P_{Δ}) is a convex minimization problem.

Example 2. Let $X := \mathbb{R}^d$, endowed with its Euclidean scalar product $(.|.)$, $X_+ := \mathbb{R}^d_+$, $c \in X_+\backslash\{0\}$ and let f and F be given by $f(x) := -\|x\|^{-2}$ for $x \in X\backslash\{0\}$, $f(0) = -\infty$ and

$$F := \{x \in X_+ : (c \mid x) \geq 1\}.$$

Then one has $f_\nabla(y) = \|y\|^2$ for $y \in X\backslash\{0\}$, $f_\nabla(0) = -\infty$ and $F^\nabla = c + X_+$. Thus the dual problem has a (unique) solution $\overline{y} := c$ and its value is $-\|c\|^2$. Taking $\overline{x} := \|c\|^{-2} c$, one has $\overline{x} \in F$, $f(\overline{x}) = -\|c\|^2 = f_\nabla(\overline{y})$, so that $\overline{x}$ is a solution to (P) and there is no duality gap. Note that (P) is a quasiconvex mathematical programming problem while (P_∇) is a convex problem.

Example 3. Let $X := \mathbb{R}^d$, $X_+ := \mathbb{R}^d_+$, $c, c' \in X_+\backslash\{0\}$. Consider problem (P) with f and F given by $f(x) := -\|x\|^{-2}$ for $x \in X\backslash\{0\}$, $f(0) = -\infty$ and

$$F := \{x \in X_+ : (c \mid x) \geq 1\} \cup \{x \in X_+ : (c' \mid x) \geq 1\}.$$

Then one has $f_\nabla(y) = \|y\|^2$ for $y \in X\backslash\{0\}$, $f_\nabla(0) = -\infty$ and $F^\nabla = (c + X_+) \cap (c' + X_+) = c'' + X_+$, where $c'' := \max(c, c')$ (componentwise). Thus the dual problem has a (unique) solution $\overline{y} := c''$ and its value is $-\|c''\|^2$. Taking $d = 2$, $c := (a, 0)$, $c' := (0, b')$ with $a > b' > 0$, we see that $\overline{x} := (a^{-1}, 0) \in F$ is a solution to (P) $f(\overline{x}) = -a^2$ while $f_\nabla(\overline{y}) = a^2 + b'^2$ and there is a duality gap. Note that F is nonconvex and that the estimate we get for the value ϖ of (P) corresponds to the change of F into $F^{\nabla\nabla} := (F^\nabla)^\nabla$.

Example 4. Let $X := \mathbb{R}^d$, $X_+ := \mathbb{R}^d_+$. Let A be a linear symmetric definite positive operator on X. Consider problem (P) with f and F given by $f(x) := (x_1 + ... + x_d)^{-1}$ for $x \in \mathrm{int}X_+$, $f(x) = +\infty$ otherwise and

$$F := \{x \in X : (Ax \mid x) \leq 1\}.$$

Then $F^\Delta = \{y \in X : (A^{-1}y \mid y) \leq 1\}$ and $f_\Delta(y) = -1/\min(y_1, ..., y_d)$ for $y := (y_1, ..., y_d)$ with $y_i > 0$ for $i \in \mathbb{N}_d := \{1, ..., d\}$ (as seen by picking some $K \subset \{k \in \mathbb{N}_d : y_k \leq y_i \ \forall i \in \mathbb{N}_d\}$ and taking $x = (x_1, ..., x_d)$ with $x_k = (1/\sharp K)(1/y_k)$ for $k \in K$, $x_j = 0$ for $j \notin K$), $f_\Delta(y) = 0$ otherwise. Corollary 3.5 below ensures that there is no duality gap, f being u.s.c. and coradiant and F being convex and absorbent.

$\boxed{3}$ Criteria for Strong Duality

Let us give conditions ensuring strong duality, i.e. that there is no duality gap and that $(D_\diamondsuit)$ has solutions when the value ϖ of (P) is finite. As in [42, Thm 3.2] for the case of problem (D_Δ), we shall show that a separation property entails such a result. We first observe that, conversely, strong duality implies a separation property.

Proposition 3.1. *Suppose ϖ is finite, there is no duality gap between (P) and $(D_\diamondsuit)$ and $(D_\diamondsuit)$ has a solution $\overline{x}^*$. Then the hyperplane $[\varepsilon_\diamondsuit \overline{x}^* = \varepsilon_\diamondsuit]$ separates F and $S_f^<(\varpi)$:*

$$\forall u \in F,\ x \in S_f^<(\varpi) \qquad \langle x, \varepsilon_\diamondsuit \overline{x}^*\rangle < \varepsilon_\diamondsuit \leq \langle u, \varepsilon_\diamondsuit \overline{x}^*\rangle.$$

Proof. Since $\overline{x}^* \in F^\diamondsuit$, we have $\varepsilon_\diamondsuit \leq \langle u, \varepsilon_\diamondsuit \overline{x}^*\rangle$ for all $u \in F$. Now $\varpi = -f_\diamondsuit(\overline{x}^*)$ since there is no duality gap, so that $\varpi \leq f(x)$ for all $x \in X$ satisfying $\langle x, \varepsilon_\diamondsuit \overline{x}^*\rangle \geq \varepsilon_\diamondsuit$. Equivalently, for all $x \in S_f^<(\varpi)$ we have $\langle x, \varepsilon_\diamondsuit \overline{x}^*\rangle < \varepsilon_\diamondsuit$. □

Proposition 3.2. *Suppose that for some $\overline{x}^* \in X^*$ and some $\rho \in \mathbb{R}$, the hyperplane $[\varepsilon_\diamondsuit \overline{x}^* = \varepsilon_\diamondsuit]$ separates F and $S_f^<(\rho)$ in the sense that*

$$\forall u \in F,\ x \in S_f^<(\rho) \qquad \langle x, \varepsilon_\diamondsuit \overline{x}^*\rangle < \varepsilon_\diamondsuit \leq \langle u, \varepsilon_\diamondsuit \overline{x}^*\rangle. \tag{3.1}$$

Then $\rho \leq \varpi$, the value of (P), and $\rho \leq -f_\diamondsuit(\overline{x}^*) \leq \sup(D_\diamondsuit)$. If $\rho = \varpi$ there is no duality gap and $\overline{x}^*$ is a solution to $(D_\diamondsuit)$.

Proof. By assumption (3.1), for all $u \in F$ we have $\langle u, \varepsilon_{\Diamond}\overline{x}^* \rangle \geq \varepsilon_{\Diamond}$, so that $\overline{x}^* \in F^{\Diamond}$. Let us show that $f(x) \geq \rho$ for every $x \in [\varepsilon_{\Diamond}\overline{x}^* \geq \varepsilon_{\Diamond}]$; that will ensure that $\rho \leq -f_{\Diamond}(\overline{x}^*) \leq \sup(D_{\Diamond}) \leq \varpi$ and that $\overline{x}^*$ is a solution to $(D_{\Diamond})$ when $\rho = \varpi$. Suppose on the contrary that there exists some $x \in [\varepsilon_{\Diamond}\overline{x}^* \geq \varepsilon_{\Diamond}]$ such that $f(x) < \rho$. Then $x \in S_f^<(\rho)$, hence, by the first inequality in (3.1), $\langle x, \varepsilon_{\Diamond}\overline{x}^* \rangle < \varepsilon_{\Diamond}$, a contradiction. □

Let us observe that the preceding result does not require the knowledge of ϖ but, on the contrary, that it provides an estimate of it. Moreover, it shows that ϖ is a threshold for the considered separation property. For the next corollaries, it suffices to know a minorant of $f(F)$ and a value of f less than this minorant.

Corollary 3.3. *Suppose f is radiant and upper semicontinuous, F is convex and $\varpi > \inf f(X)$. Then there is no duality gap between (P) and (D_{∇}) and (D_{∇}) has a solution.*

Proof. As we suppose $\varpi > \inf f(X)$ and $f(F) \cap \mathbb{R} \neq \varnothing$, ϖ is finite. Since $S_f^<(\varpi)$ is a nonempty open convex subset disjoint from F, the Hahn-Banach separation theorem ensures that one can find $x_0^* \in X^*$, $r \in \mathbb{R}$ such that $S_f^<(\varpi) \subset [x_0^* < r]$ and $F \subset [x_0^* \geq r]$. As f is radiant and $S_f^<(\varpi)$ is nonempty, $S_f^<(\varpi)$ contains 0, so that we have $r > 0$ and (3.1) holds with $\varepsilon_{\Diamond} = 1$, $\overline{x}^* := r^{-1}x_0^*$, so that the preceding proposition applies. □

A variant of this result can be given when X is finite dimensional. Here we say that f is *upper semicontinuous along rays* if its restriction to any line passing through 0 is upper semicontinuous. This mild condition can be further weakened. Let us say that a function $g : \mathbb{R}_+ \to \overline{\mathbb{R}}$ is quasi-nonincreasing (resp. quasi-nondecreasing) if for all $r \in \mathbb{R}_+$ one has $g(r) \geq \inf_{s>r} g(s)$ (resp. $g(r) \geq \inf_{s<r} g(s)$). We say that $f : X \to \overline{\mathbb{R}}$ is *quasi-nonincreasing (resp. quasi-nondecreasing) along rays* if for all $x \in X\backslash\{0\}$ the function $t \mapsto f(tx)$ on $\mathbb{R}_+$ is quasi-nonincreasing (resp. quasi-nondecreasing). Let us note that $g : \mathbb{R}_+ \to \overline{\mathbb{R}}$ is quasi-nonincreasing whenever g is right upper regular in the sense that $g(r) \geq \liminf_{s \to r_+} g(s)$ for all $r \in \mathbb{R}_+$, in particular whenever g is right upper semicontinuous. Thus, if f is upper semicontinuous along rays, then f is quasi-nonincreasing along rays. If f is radiant (or just nondecreasing along rays), the converse holds. Similar assertions hold for quasi-nondecreasing functions in terms of left lower regularity.

Corollary 3.4. *Suppose X is finite dimensional, F is convex, $\varpi > \inf f(X)$ and f is radiant and upper semicontinuous along rays or just radiant and quasi-nonincreasing along rays. Then, there is no duality gap between (P) and (D_{∇}) and (D_{∇}) has a solution.*

Proof. Again, ϖ is finite since $\varpi > \inf f(X)$ and we have $0 \in S_f^<(\varpi)$ since $S_f^<(\varpi)$ is radiant and nonempty. Since f is quasi-nonincreasing along rays, $S_f^<(\varpi)$ is absorbent: for all $x \in X$ we have $\varpi > f(0) \geq \inf_{t>0} f(tx)$, so that there exists some $t > 0$ such that $tx \in S_f^<(\varpi)$. Now $0 \notin S_f^<(\varpi) - F$ which is convex. Since X is finite dimensional, we can find some $x_0^* \in X^*\backslash\{0\}$ such that $r := \sup x_0^*(S_f^<(\varpi)) \leq \inf x_0^*(F)$. Taking $x_0 \in X$ such that $x_0^*(x_0) > 0$ and using the fact that $S_f^<(\varpi)$ is absorbent, we get $r > 0$. Now, for all $x \in S_f^<(\varpi)$, since f is quasi-nonincreasing along rays, we have $f(tx) < \varpi$ for some $t > 1$. Thus $x_0^*(tx) \leq r$ and $x_0^*(x) < r$ and we can apply the proposition with $\varepsilon_{\Diamond} = 1$, $\overline{x}^* := r^{-1}x_0^*$. □

Corollary 3.5. *Suppose f is coradiant (or just quasiconvex) and upper semicontinuous, and F is convex and absorbent. Then there is no duality gap and, if $\varpi > \inf f(X)$, the dual problem (D_{Δ}) has a solution.*

Proof. By weak duality, the first assertion is obvious when $\varpi = -\infty$. Thus we may suppose ϖ is finite and we apply the Hahn-Banach separation theorem since $S_f^<(\varpi)$ is a nonempty open convex subset disjoint from F, so that one can find $x_0^* \in X^*\backslash\{0\}$, $r \in \mathbb{R}$ such that $S_f^<(\varpi) \subset [x_0^* > r]$ and $F \subset [x_0^* \leq r]$. As F is absorbent, we have $r > 0$ and we can apply the proposition with $\varepsilon_\Diamond = -1$, $\overline{x}^* := r^{-1}x_0^*$. □

Note that since $0 \in F$ which is disjoint from $S_f^<(\varpi)$, we necessarily have $0 \notin S_f^<(\varpi)$, so that the result does not apply to radiant functions.

Combining the techniques of the proofs of the two preceding corollaries we get the following variant.

Corollary 3.6. *Suppose X is finite dimensional, f is quasiconvex, quasi-nondecreasing along rays, and F is convex and absorbent. Then there is no duality gap and if $\varpi > \inf f(X)$, the dual problem(D_Δ) has a solution.*

Let us also show that a sufficient optimality condition ensures strong duality. For such a purpose, we introduce the subdifferential $\partial_\Diamond$ by

$$\partial_\Diamond f(\overline{x}) := \{x^* \in X^* : \langle \overline{x}, \varepsilon_\Diamond x^*\rangle \geq \varepsilon_\Diamond,\ f(x) \geq f(\overline{x}) \ \ \forall x \in [\varepsilon_\Diamond x^* \geq \varepsilon_\Diamond]\}$$

which encompasses the subdifferentials $\partial^\wedge$ and $\partial^\vee$ defined in [21] along a general line introduced by Martínez-Legaz and Singer [16] (see also [42] for a related definition) in view of the following equivalence akin to the Young-Fenchel equality: for $\overline{x} \in f^{-1}(\mathbb{R})$ one has

$$\overline{x}^* \in \partial_\Diamond f(\overline{x}) \Longleftrightarrow f_\Diamond(\overline{x}^*) + f(\overline{x}) = c_\Diamond(\overline{x}, \overline{x}^*) > -\infty,$$

where $c_\Diamond(x, x^*) = 0$ if $\langle x, \varepsilon_\Diamond x^*\rangle \geq \varepsilon_\Diamond$, $c_\Diamond(x, x^*) = -\infty$ otherwise. Denoting by $\partial_\Diamond^= f(\overline{x})$ the set of $x^* \in \partial_\Diamond f(\overline{x})$ such that $\langle \overline{x}, x^*\rangle = 1$, one observes that $\overline{x}^* \in \partial_\Diamond^= f(\overline{x})$ if, and only if, $\varepsilon_\Diamond \overline{x}^*$ belongs to the Greenberg-Pierskalla subdifferential $\partial^{GP} f(\overline{x}) := \{x^* \in X^* : \forall x \in S_f(f(\overline{x}))\ \ \langle x - \overline{x}, x^*\rangle < 0\}$ [8] and $\langle \overline{x}, \varepsilon_\Diamond \overline{x}^*\rangle = \varepsilon_\Diamond$. Moreover,

$$\partial_\nabla^= f(\overline{x}) \subset \partial_\nabla f(\overline{x}) \subset [1, +\infty)\partial_\nabla^= f(\overline{x}).$$

The following result completes [42, Thm 3.3], as it also deals with the case of problem (D_∇); moreover, here $\partial_\Diamond^= f(\overline{x})$ is replaced with the larger set $\partial_\Diamond f(\overline{x})$. Since the Plastria subdifferential

$$\partial^< f(\overline{x}) := \{\overline{x}^* \in X^* : \forall x \in S_f^<(f(\overline{x}))\ \ f(x) - f(\overline{x}) \geq \langle x - \overline{x}, \overline{x}^*\rangle\}$$

is contained in $\partial^{GP} f(\overline{x})$, this result also implies [10, Prop. 5] by taking $\varepsilon_\Diamond = 1$. Here, even if F in nonconvex, we set

$$N(F, \overline{x}) := \{\overline{x}^* \in X^* : \forall x \in F\ \langle x - \overline{x}, \overline{x}^*\rangle \leq 0\}.$$

Proposition 3.7. *Let $\overline{x} \in F$ and $\overline{x}^* \in X^*$ be such that $\varepsilon_\Diamond \overline{x}^* \in \partial_\Diamond f(\overline{x}) \cap (-N(F, \overline{x}))$. Then the hyperplane $\{x : \langle x, \varepsilon_\Diamond \overline{x}^*\rangle = \varepsilon_\Diamond\}$ separates F and $S_f^<(\varpi)$, $\overline{x}$ is a solution of (P), $\overline{x}^*$ is a solution of $(D_\Diamond)$ and there is no duality gap.*

Proof. Since $-\varepsilon_\Diamond \overline{x}^* \in N(F, \overline{x})$, for all $x \in F$ we have $\langle x - \overline{x}, \varepsilon_\Diamond \overline{x}^*\rangle \geq 0$, hence $\langle x, \varepsilon_\Diamond \overline{x}^*\rangle \geq \langle \overline{x}, \varepsilon_\Diamond \overline{x}^*\rangle \geq \varepsilon_\Diamond$ by the first condition in the definition of $\partial_\Diamond f(\overline{x})$. Thus $F \subset [\varepsilon_\Diamond \overline{x}^* \geq \varepsilon_\Diamond]$ and we have $\overline{x}^* \in F^\Diamond$. The second condition in the definition of $\partial_\Diamond f(\overline{x})$ yields $f(x) \geq f(\overline{x})$ for all $x \in [\varepsilon_\Diamond \overline{x}^* \geq \varepsilon_\Diamond]$, hence for all $x \in F$ and $\overline{x}$ is a solution of (P). Moreover, since $f(x) \geq f(\overline{x})$ for all $x \in [\varepsilon_\Diamond \overline{x}^* \geq \varepsilon_\Diamond]$ and $\overline{x} \in [\varepsilon_\Diamond \overline{x}^* \geq \varepsilon_\Diamond]$, we get $-f_\Diamond(\overline{x}^*) = f(\overline{x})$. It follows that $\overline{x}^*$ is a solution of $(D_\Diamond)$ and $\sup(D_\Diamond) = \inf(P)$. The separation property stems from Proposition 3.1. □

One may wonder whether conversely the separation property assumed in Proposition 3.2 implies the condition $\varepsilon_\diamond \overline{x}^* \in \partial_\diamond f(\overline{x}) \cap (-N(F,\overline{x}))$ assumed in the preceding result. The next proposition gives a partial answer to that question.

Proposition 3.8. *Suppose that for some $\overline{x}^* \in X^*\backslash\{0\}$ the hyperplane $[\varepsilon_\diamond \overline{x}^* = \varepsilon_\diamond]$ separates F and $S_f^<(\varpi)$ in the sense of relation (3.1). If a solution $\overline{x}$ of (P) is not a local minimizer of f, then $\overline{x}^*$ is a solution to $(D_\diamond)$ and there is no duality gap.*

Proof. For $\overline{x}^* := r^{-1}x_0^*$, we have $\langle \overline{x}, \varepsilon_\diamond \overline{x}^*\rangle \geq \varepsilon_\diamond$ as $\overline{x} \in F$. Since $\overline{x}$ is not a local minimizer of f, $\overline{x}$ belongs to the closure of $S_f^<(\varpi)$, hence $\langle \overline{x}, \varepsilon_\diamond \overline{x}^*\rangle = \varepsilon_\diamond$. Since $\langle u, \varepsilon_\diamond \overline{x}^*\rangle \geq \varepsilon_\diamond$ for all $u \in F$, we get $-\varepsilon_\diamond \overline{x}^* \in N(F,\overline{x})$. It remains to show that $f(x) \geq f(\overline{x})$ for every $x \in [\varepsilon_\diamond x^* \geq \varepsilon_\diamond]$ to ensure that $\varepsilon_\diamond \overline{x}^* \in \partial_\diamond f(\overline{x})$. Suppose there exists some $x \in [\varepsilon_\diamond \overline{x}^* \geq \varepsilon_\diamond]$ such that $f(x) < f(\overline{x})$. Then, since $f(\overline{x}) = \varpi$, we would have $x \in S_f^<(\varpi)$, hence $\langle x, \varepsilon_\diamond \overline{x}^*\rangle < \varepsilon_\diamond$, a contradiction. Thus $\varepsilon_\diamond \overline{x}^* \in \partial_\diamond f(\overline{x})$ and the conclusion follows from the preceding proposition. □

Corollary 3.9. *Suppose f is upper semicontinuous and radiant and F is convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of (D_∇) which satisfies the optimality conditions $\overline{x}^* \in \partial_{\nabla}^{=} f(\overline{x}) \cap (-N(F,\overline{x}))$ and there is no duality gap.*

Proof. This follows from the preceding proposition and from the Hahn-Banach separation theorem since $S_f^<(\varpi)$ is an open convex subset disjoint from F. □

A variant of this result can be given when X is finite dimensional.

Corollary 3.10. *Suppose X is finite dimensional, f is radiant and upper semicontinuous along rays and F is convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of (D_∇) which satisfies the optimality conditions $\overline{x}^* \in \partial_\nabla f(\overline{x}) \cap (-N(F,\overline{x}))$, $\langle \overline{x}, \overline{x}^*\rangle = 1$ and there is no duality gap.*

Proof. Again, we have $0 \in S_f^<(\varpi)$ and $f(0) < \varpi$ since $f(0) = \inf f(X) < f(\overline{x})$, as $\overline{x}$ is not a minimizer of f. Since f is upper semicontinuous along rays at 0, $S_f^<(\varpi)$ is absorbent. Now $0 \notin S_f^<(\varpi) - F$ which is convex. Since X is finite dimensional, one can find some $x_0^* \in X^*\backslash\{0\}$ such that $r := \sup x_0^*(S_f^<(\varpi)) \leq \inf x_0^*(F)$. Taking $x_0 \in X$ such that $x_0^*(x_0) > 0$ and using the fact that $S_f^<(\varpi)$ is absorbing, we get $r > 0$. Since $\overline{x}$ is not a local minimizer of f, $\overline{x}$ is in the closure of $S_f^<(\varpi)$, so that $r := \langle \overline{x}, x_0^*\rangle$. Let $\overline{x}^* := r^{-1}x_0^*$, so that $\langle \overline{x}, \overline{x}^*\rangle = 1$ and $-\overline{x}^* \in N(F,\overline{x})$. Since f is upper semicontinuous along rays, for every $x \in S_f^<(\varpi)$ we have $tx \in S_f^<(\varpi)$ for some $t > 1$, hence $\langle x, \overline{x}^*\rangle < 1$. Thus we can apply the proposition. □

The assumption that X is finite dimensional is eliminated in the next corollary, but the substituted assumption is more difficult to check. We keep the preceding notation and we say that a subset C of X is *evenly convex* if it is the intersection of a family of open half-spaces. Obviously, open or closed convex subsets of a normed vector space are evenly convex, but the class of evenly convex subsets is larger than the union of these two subclasses.

Corollary 3.11. *Suppose f is radiant and $S_f^<(\varpi) - F$ is evenly convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of (D_∇) which satisfies the optimality conditions $\overline{x}^* \in \partial_\nabla f(\overline{x}) \cap (-N(F,\overline{x}))$ and $\langle \overline{x}, \overline{x}^*\rangle = 1$.*

The proof is similar to the preceding one after using the fact that, since $0 \notin S_f^<(\varpi) - F$, one can find some $x_0^* \in X^*$ such that $\langle x - u, x_0^* \rangle < 0$ for all $x \in S_f^<(\varpi)$, $u \in F$. Since $0 \in S_f^<(\varpi)$, we have $r := \langle \overline{x}, x_0^* \rangle > 0$ and, for $\overline{x}^* := r^{-1} x_0^*$, we get $\langle x, \overline{x}^* \rangle < \langle \overline{x}, \overline{x}^* \rangle = 1$ for all $x \in S_f^<(\varpi)$. □

Now let us turn to the coradiant case. In such a case we write $\partial_\Delta f(\overline{x})$ instead of $\partial_\Diamond f(\overline{x})$ with $\varepsilon_\Diamond = -1$.

Corollary 3.12. *Suppose f is coradiant and upper semicontinuous and F is convex and absorbent. Then, there is no duality gap and (D_Δ) has a solution. Moreover, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of (D_Δ) which satisfies the optimality conditions $-\overline{x}^* \in \partial_\Delta f(\overline{x}) \cap (-N(F, \overline{x}))$ and $\langle \overline{x}, \overline{x}^* \rangle = 1$.*

Proof. Since F and $S_f^<(\varpi)$ are convex and disjoint, and $S_f^<(\varpi)$ is open while F is absorbent, there exist $x_0^* \in X^*$ and $r > 0$ such that

$$\forall u \in F,\ \forall x \in S_f^<(\varpi) \qquad \langle x, x_0^* \rangle > r \geq \langle u, x_0^* \rangle.$$

Taking $\varepsilon_\Diamond = -1$ in Proposition 3.2 we get that relation (3.1) is satisfied, hence the first assertion holds. Let $\overline{x}$ be a solution of (P) which is not a local minimizer of f. Then, by Proposition 3.8, $\varepsilon_\Diamond \overline{x}^* := r^{-1} \varepsilon_\Diamond x_0^* \in \partial_\Delta f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \varepsilon_\Diamond \overline{x}^* \rangle = \varepsilon_\Diamond$, and $\overline{x}^*$ is a solution of (D_Δ). □

Corollary 3.13. *Suppose f is coradiant and upper semicontinuous along rays and F is convex and contains 0 in its interior. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of (D_Δ) which satisfies the optimality conditions $-\overline{x}^* \in \partial_\Delta f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \overline{x}^* \rangle = 1$ and there is no duality gap.*

Proof. Since $\mathrm{int}F$ and $S_f^<(\varpi)$ are convex and disjoint, there exist $x_0^* \in X^*$ and $r \in \mathbb{R}$ such that

$$\forall u \in \mathrm{int}F,\ x \in S_f^<(\varpi) \qquad \langle x, x_0^* \rangle \geq r > \langle u, x_0^* \rangle.$$

Since $0 \in \mathrm{int}F$, we have $r > 0$. Since F is contained in the closure of $\mathrm{int}F$, for all $u \in F$, we have $\langle u, x_0^* \rangle \leq r$. Since $\overline{x}$ is not a local minimizer of f, $\overline{x}$ is in the closure of $S_f^<(\varpi)$. Thus $\langle \overline{x}, x_0^* \rangle = r$. Let $\overline{x}^* := r^{-1} x_0^*$. Then $\langle \overline{x}, \overline{x}^* \rangle = 1 \geq \langle u, \overline{x}^* \rangle$ for all $u \in F$, hence $\overline{x}^* \in N(F, \overline{x})$. Moreover, if $x \in S_f^<(\varpi)$, we have $\langle x, \overline{x}^* \rangle \geq 1$. Since f is upper semicontinuous along rays, we have $tx \in S_f^<(\varpi)$ for $t < 1$ close enough to 1; thus we have $\langle x, \overline{x}^* \rangle > 1$. Therefore $-x^* \in \partial_\Delta f(\overline{x})$. □

4 Links with Polarities

The conjugates we considered are particular instances of conjugates associated with a polarity. Recall that a *polarity* between two sets X, Y is a map $P : 2^X \to 2^Y$ between the power sets of X and Y which satisfies the relation

$$P(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} P(A_i) \tag{4.1}$$

for every family $(A_i)_{i \in I}$ of subsets of X. We also denote $P(A)$ by A^P for $A \subset X$. The preceding relation yields, for any $A \subset X$

$$P(A) = \bigcap_{a \in A} P(\{a\}) = \{y \in Y : A \subset D(y)\}, \tag{4.2}$$

where $D(y) := P^{-1}(y) := \{x \in X : y \in P(\{x\})\}$. Conversely, given a family $(D(y))_{y \in Y}$ of subsets of X, one gets a polarity by setting, for $A \subset X$, $P(A) = \{y \in Y : A \subset D(y)\}$. When X and Y are topological vector spaces in duality, it is natural to take for family $(D(y))_{y \in Y}$ a family of half-spaces. In [20], [21], we detected four families of such half-spaces of special interest. They give rise to four polar sets:

$$A^{\Delta} := \{y \in Y : A \subset [y \leq 1]\}, \qquad A^{\wedge} := \{y \in Y : A \subset [y < 1]\},$$
$$A^{\nabla} := \{y \in Y : A \subset [y \geq 1]\}, \qquad A^{\vee} := \{y \in Y : A \subset [y > 1]\};$$

here we change the notation for the first one, which is the usual polar set often denoted by A^0; we do that in order to remind that one passes from $A^{\wedge}$ to A^{Δ} by adding a bar to the symbol $<$, changing it into $\leq$. In this way, we provide an unified notation for these four conjugacies.

We note the following observation which is an immediate consequence of the definitions.

Lemma 4.1. *For any subset A of X, the sets $A^{\wedge}$ and A^{Δ} are radiant; A^{Δ} is weak* closed and $A^{\wedge}$ is evenly convex. For any nonempty subset A of X, the sets $A^{\vee}$ and A^{∇} are coradiant; A^{∇} is weak* closed and $A^{\vee}$ is evenly coradiant.*

Now, for any function f on X, it is classical to define a conjugate function f^P associated with a polarity P by setting:

$$f^P(y) := \sup\{-f(x) : x \in X \backslash D(y)\},$$

where $D(y) := P^{-1}(y)$. Taking for P one of the preceding four polarities, besides the conjugates f_{Δ} and f_{∇} we have already used, we get two other conjugates:

$$f^{\wedge}(y) := -\inf\{f(x) : x \in X,\ \langle x, y\rangle \geq 1\}, \qquad f^{\vee}(y) := -\inf\{f(x) : x \in X,\ \langle x, y\rangle \leq 1\}$$
$$f^{\Delta}(y) := -\inf\{f(x) : x \in X,\ \langle x, y\rangle > 1\}, \qquad f^{\nabla}(y) := -\inf\{f(x) : x \in X,\ \langle x, y\rangle < 1\}.$$

We observe that $f^{\wedge} = f_{\nabla}$ and $f^{\vee} = f_{\Delta}$. In the sequel, we set similarly $f_{\vee} := f^{\Delta}$, $f_{\wedge} := f^{\nabla}$. Such a choice of notation is reminiscent to the notation f_* for the concave conjugate of a function f, in contrast with the notation for the convex conjugate f^*. An interpretation of these equalities can be given as follows. Whenever a polarity P is given, one can associate to it another polarity P' given by $P'(x) = (D')^{-1}(x)$ where $D'(y) := X \backslash D(y)$, $D : 2^Y \to 2^X$ being the inverse of P. In analogy with an usual notation for the concave conjugate of a function, we have chosen to write f_P instead of $f^{P'}$. With such a convention, the dual problems (D_{∇}) and (D_{Δ}) considered in the preceding sections can be given a simple notation instead of a notation which would reflect the composite character of these dual problems. Note that the conjugacy $f \mapsto f^P$ enters the general framework of the Fenchel-Moreau duality scheme (see [17], [21], [34] and [37]) associated with a coupling function c^P.

Since we have four polars and conjugates, we can also introduce the dual problems (D_P) for $P = \wedge$ and $P = \vee$ in which the inequalities are replaced with strict inequalities. However, the separation techniques devised in section 3 do not seem to be as adapted to these two supplementary dual problems. We will make some further comments in the next section.

Let us first note a general weak duality inequality.

Proposition 4.2. *Given a polarity $P : 2^X \to 2^Y$ and the associated polarity $P' : 2^X \to 2^Y$ defined above, one has the weak duality relation*

$$\sup(D_P) \leq \inf(P) := \inf f(F),$$

where the dual problem (D_P) *is the maximization over the set* $F^P := P(F)$ *of the function* $-f_P$ *where* f_P *is given by*

$$f_P(y) := \sup\{-f(x) : x \in P^{-1}(y)\} := -\inf f(P^{-1}(y)).$$

Proof. For every $y \in F^P$ one has $F \subset P^{-1}(y)$ by (4.2) and the definition of $F^P := P(F)$. Thus $-f_P(y) = \inf f(P^{-1}(y)) \leq \inf f(F)$. It follows that $\sup(D_P) := \sup_{y \in P(F)} -f_P(y) \leq \inf f(F) = \inf(P)$. □

Some properties of these conjugates are presented in [20], [22], [24], [38], [39], [43]. Let us note some of them which have some bearing to our study.

Lemma 4.3. *For any function* f *on* X*, the functions* $f^{\wedge} = f_{\nabla}$ *and* $f^{\Delta} = f_{\vee}$ *are radiant and the functions* $f^{\vee} = f_{\Delta}$ *and* $f^{\nabla} = f_{\wedge}$ *are coradiant. Moreover,* f^{Δ} *and* f^{∇} *are lower semicontinuous while* $f^{\wedge}$ *and* $f^{\vee}$ *are evenly quasiconvex; in fact* $f^{\vee}$ *is evenly coradiant. Furthermore,* $f^{\wedge}(0) = f^{\Delta}(0) = -\infty$ *and* $f^{\vee}(0) = f^{\nabla}(0) = -\inf f(X)$.

Proof. These assertions are consequence of the preceding lemma and of the following relation, valid for every $r \in \mathbb{R}$, any function f and any polarity P :

$$\left[f^P \leq r\right] = [f < -r]^P. \tag{4.3}$$

This relation, established in [43], [24], follows from the equivalences:

$$\begin{aligned}\left(y \in \left[f^P \leq r\right]\right) &\Leftrightarrow (-r \leq f(x)\ \forall x \in X \backslash D(y)) \\ &\Leftrightarrow (-r > f(x) \Rightarrow x \in D(y)) \\ &\Leftrightarrow ([f < -r] \subset D(y)) \Leftrightarrow (y \in [f < -r]^P).\end{aligned}$$

□

The following result is useful when interchanging the role of the primal and dual problems.

Lemma 4.4. *([24, Cor. 7]) For* $\bowtie \in \{\wedge, \Delta, \vee, \nabla\}$*, a function* f *on* X *is the conjugate of some function* g *on* $Y := X^*$ *if, and only if for all* $r \in \mathbb{R}$ *the sublevel set* $[f \leq r]$ *is equal to its bipolar set for* $\bowtie$ *. In such a case on can take* $g = f^{\bowtie}$.

5 Some Variants

Since we have four different sorts of polar sets and four different sorts of conjugate functions, it is tempting to study other combinations. Such a temptation is increased by the fact that the combinations we have selected above are mixed, as we have just observed. Let us see whether it is sensible to resist to such a temptation.

Choosing the two symbols $\bowtie$, $\between$ among the sequence $(\wedge, \Delta, \vee, \nabla)$ identified with the sequence $(<, \leq, >, \geq)$, one gets 16 instances of dual problems by setting

$$(D_{\between}^{\bowtie}) \text{ maximize } - f_{\between}(x^*) := \inf f([x^* \between 1]) \text{ over } F^{\bowtie} := \{x^* \in X^* : F \subset [x^* \bowtie 1]\}.$$

As noted in the preceding section, we have weak duality when $\bowtie = \between$. When $\bowtie \in \{\wedge, \Delta\}$ and $\between \in \{\vee, \nabla\}$ we have $0 \in F^{\bowtie}$ and $-f_{\between}(0) = +\infty$, so that weak duality is excluded by our assumption that $\inf f(F) < +\infty$. When $\bowtie \in \{\vee, \nabla\} = \{>, \geq\}$, and $\between \in \{\wedge, \Delta\}$, $F^{\bowtie}$ is coradiant and $f_{\between}$ is coradiant, so that the set of solutions of $(D_{\between}^{\bowtie})$ is either empty or an union of half lines. These observations reduce the number of interesting dual problems.

Furthermore, the inequalities $f^\wedge \geq f^\Delta$, $f^\vee \geq f^\nabla$ and the inclusions $F^\wedge \subset F^\Delta$, $F^\vee \subset F^\nabla$ entail the obvious relationships

$$\begin{aligned}
\sup(D^\vee_\nabla) &:= \sup -f^\wedge(F^\vee) \leq \sup -f^\wedge(F^\nabla) =: \sup(D_\nabla) \leq \sup -f^\Delta(F^\nabla) =: \sup(D^\nabla_\vee), \\
\sup(D^\wedge_\Delta) &:= \sup -f^\vee(F^\wedge) \leq \sup -f^\vee(F^\Delta) =: \sup(D_\Delta) \leq \sup -f^\nabla(F^\Delta) =: \sup(D^\Delta_\wedge),
\end{aligned}$$

Although the estimates provided by the dual problems $(D^\vee_\nabla)$ and $(D^\wedge_\Delta)$ may be useful, the duality gaps between these problems and the primal one (P) are always larger than the duality gaps for the dual problems we have chosen. Thus, strong duality would be more difficult to get with the dual problems $(D^\vee_\nabla)$ and $(D^\wedge_\Delta)$. The following examples show that the problems $(D^\nabla_\vee)$ and $(D^\Delta_\wedge)$ do not satisfy the weak duality property and thus should be excluded.

Example 5. Let $X := \mathbb{R}$, $F := [1, +\infty)$, $f(r) = 0$ for $r \in (-\infty, 1]$, $f(r) := 1$ for $r \in (1, +\infty)$. Then $F^\nabla = [1, +\infty)$ and $-f_\vee(1) := -f^\Delta(1) = 1$. Thus $\sup(D^\nabla_\vee) \geq 1 > 0 = \inf(P)$.

Example 6. Let $X := \mathbb{R}$, $F := (-\infty, 1]$, $f(r) := 1$ for $r \in (-\infty, 1)$, $f(r) = 0$ for $r \in [1, +\infty)$. Then $F^\Delta = [0, 1]$ and $-f_\wedge(1) := -f^\nabla(1) = 1$. Thus $\sup(D^\Delta_\wedge) \geq 1 > 0 = \inf(P)$.

These facts explain why we focused our attention on the dual problems $(D_\nabla) := (D^\nabla_\nabla)$ and $(D_\Delta) := (D^\Delta_\Delta)$ rather than on $(D^\nabla_\vee)$, $(D^\Delta_\wedge)$, or, in view of the beginning of our discussion, on $(D^\wedge_\nabla)$, $(D^\wedge_\vee)$, $(D^\Delta_\vee)$, (D^Δ_∇). Under some semicontinuity assumptions, equalities hold in the inequalities $\sup(D_\nabla) \leq \sup(D^\nabla_\vee)$ and $\sup(D_\Delta) \leq \sup(D^\Delta_\wedge)$ in view of the following result. In fact a mild monotonicity assumption suffices.

Proposition 5.1. *If f is upper semicontinuous along rays, or, more generally, if f is quasi-nonincreasing along rays, then $f^\Delta = f^\wedge$. Then, for all subsets F, one has* $\sup(D^\nabla_\vee) = \sup(D_\nabla)$, $\sup(D^\vee_\vee) = \sup(D^\vee_\nabla)$, $\sup(D^\wedge_\vee) = \sup(D^\wedge_\nabla)$ *(and* $+\infty = \sup(D^\Delta_\vee) = \sup(D^\Delta_\nabla) = \sup(D^\wedge_\vee) = \sup(D^\wedge_\nabla)$*).*

If f is lower semicontinuous along rays, or more generally if f is quasi-nondecreasing along rays, then $f^\nabla = f^\vee$. Then, for all subsets F, one has $\sup(D^\Delta_\wedge) = \sup(D_\Delta)$ *and* $\sup(D^\nabla_\wedge) = \sup(D^\nabla_\Delta)$, $\sup(D^\wedge_\wedge) = \sup(D^\wedge_\Delta)$, $\sup(D^\vee_\wedge) = \sup(D^\vee_\Delta)$.

Proof. Since $f^\Delta \leq f^\wedge$, to prove that $f^\Delta = f^\wedge$, it suffices to show that for every $x^* \in X^*$ and every $r \in \mathbb{R}$ satisfying $r \geq f^\Delta(x^*)$ we have $r \geq -f(x)$ for all $x \in [x^* \geq 1]$. The inequality $r \geq -f(x)$ holding when $x \in [x^* > 1]$, we may suppose $\langle x, x^* \rangle = 1$. Then, for all $t > 1$, we have $tx \in [x^* > 1]$, hence $-f(tx) \leq r$. Since $f(x) \geq \inf_{t>1} f(tx)$ we get $f(x) \geq -r$.The proof of the equality $f^\nabla = f^\vee$ is similar. $\square$

Proposition 5.2. *Let F be an arbitrary nonempty subset of X. If $f^\wedge$ is upper semicontinuous along rays, or even quasi-nonincreasing along rays, then* $\sup(D_\nabla) = \sup(D^\vee_\nabla)$. *If $f^\vee$ is lower semicontinuous along rays, or even quasi-nondecreasing along rays, then* $\sup(D_\Delta) = \sup(D^\wedge_\Delta)$.

Proof. To prove the first assertion, it suffices to show that $\sup -f^\wedge(F^\vee) \geq \sup -f^\wedge(F^\nabla)$. Given $r < \sup -f^\wedge(F^\nabla)$ one can find some $x^* \in F^\nabla$ such that $-r > f^\wedge(x^*)$. For $t \in (1, +\infty)$ one has $tx^* \in F^\vee$ and since $f^\wedge$ is quasi-nonincreasing along rays, there exists $t > 1$ such that $-r > f^\wedge(tx^*)$. Thus $\sup -f^\wedge(F^\vee) \geq -f^\wedge(tx^*) > r$ and, as r can be arbitrarily close to $\sup -f^\wedge(F^\nabla)$, we get the expected inequality. The proof of the second assertion is similar, using the fact that $tF^\Delta \subset F^\wedge$ for $t \in (0, 1)$. $\square$

The following criteria are taken from [38, Thm 3.3], [20, Prop. 4.8] in the case of $f^\wedge$; the proof for $f^\vee$ is similar. Note that for a lower semicontinuous function f on a finite dimensional space, weak inf-compactness (i.e. weak compactness of sublevel sets) is equivalent to coercivity (i.e. $\lim_{\|x\| \to \infty} f(x) = +\infty$).

Proposition 5.3. *If f is inf-compact for the weak topology, then $f^{\wedge}$ and $f^{\vee}$ are upper semicontinuous.*

If f is such that $\lim_{x\to 0} f(x) = -\infty$, then f^{Δ} is inf-compact for the weak* topology on X^*.

Corollary 5.4. *If f is such that $\lim_{x\to 0} f(x) = -\infty$, then $(D_{\vee}^{\Delta})$ (resp. $(D_{\vee}^{\nabla})$) has a solution. If moreover f is quasi-nonincreasing along rays, then (D_{∇}) has a solution.*

Proof. The first assertion is consequence of the fact that the inf-compact function f^{Δ} attains its infimum over the weak* closed convex set F^{Δ} (resp. F^{∇}). The second assertion is a consequence of Proposition 5.1 which guarantees that $f_{\nabla} = f^{\Delta}$ whenever f is quasi-nonincreasing along rays. □

Now let us tackle the question of existence of solutions for the dual problems $(D_{\wedge}^{\Delta})$ and (D_{Δ}). As noted above, such a question is more interesting for the second problem than for the first one. Since the sublevel sets of $f_{\wedge} := f^{\nabla}$ are coradiant, $f_{\wedge}$ cannot be inf-compact unless it is identically $+\infty$. The first assertion of the next statement is a partial generalization of [38, Thm 3.4] since f is not assumed to be continuous and F is not supposed to be compact. The second assertion is new. Note that considering (P) as a dual problem (under appropriate assumptions, c.f. Lemma 4.4) one would deduce existence results for (P) itself.

Proposition 5.5. *Under each of the following assumptions, the problem $(D_{\wedge}^{\Delta})$ has a solution:*

(a) 0 *is in the interior of* F;

(b) *X is finite dimensional, $\mathbb{R}_+F - \mathbb{R}_+S_f^{<}(\varpi)$ is dense in X and $S_f^{<}(\varpi)^{\nabla} \cap F^{\Delta}$ is nonempty.*

If moreover f is quasi-nondecreasing along rays, then (D_{Δ}) has a solution.

Proof. (a) The sublevel sets of the function $f_{\wedge}$ being weak* closed convex, $f_{\wedge}$ is weak* lower semicontinuous. Now, since F contains the closed ball B with center 0 and radius r for some $r > 0$, F^{Δ} is contained in B^{Δ} which is the closed ball with center 0 and radius $1/r$. Thus F^{Δ} is weak* compact, nonempty (as $0 \in F^{\Delta}$) and $f_{\wedge}$ attains its infimum on F^{Δ}.

(b) Let us first observe that if A and B are two closed convex subsets of a finite dimensional Banach space X and if $0^+A \cap 0^+B = \{0\}$, then $A \cap B$ is bounded because $0^+(A\cap B) \subset 0^+A \cap 0^+B$; here, for a subset C of X, we denote by $0^+C := \{v \in X : v + C \subset C\}$ the recession cone of C. Now we have

$$0^+(F^{\Delta}) = F^{\ominus} := \{y \in Y : \forall x \in F\ \langle x, y\rangle \le 0\},$$
$$0^+(F^{\nabla}) = (-F)^{\ominus} := \{y \in Y : \forall x \in F\ \langle x, y\rangle \ge 0\},$$

as easily checked. Let $G := S_f^{<}(\varpi)$. Since $G^{\nabla} = [f^{\nabla} \le -\varpi]$ by relation (4.3), we have

$$0^+([f^{\nabla} \le -\varpi]) = 0^+(G^{\nabla}) = (-G)^{\ominus}.$$

Since $H := \mathbb{R}_+F - \mathbb{R}_+G$ is dense in X, its polar cone $H^{\ominus}$ is $\{0\}$, hence

$$0^+(F^{\Delta}) \cap 0^+([f^{\nabla} \le -\varpi]) = F^{\ominus} \cap (-G)^{\ominus} = (\mathbb{R}_+F - \mathbb{R}_+G)^{\ominus} = \{0\}.$$

Thus $[f^{\nabla} \le -\varpi] \cap F^{\Delta}$ is nonempty, bounded, hence weak* compact and since f^{∇} is weak* lower semicontinuous, f^{∇} attains its infimum over F^{Δ} : $(D_{\wedge}^{\Delta})$ has a solution. The last assertion is a consequence of the equality $f^{\nabla} = f^{\vee}$ when f is quasi-nondecreasing along rays. □

Note that the assumption $S_f^<(\varpi)^\nabla \cap F^\Delta \neq \varnothing$ amounts to the existence of some $y \in Y$ such that

$$\forall u \in F,\ x \in S_f^<(\varpi) \quad \langle x, y\rangle \geq 1 \geq \langle u, y\rangle,$$

a weak separation property.

$\boxed{6}$ Mathematical Programming Problems

Now, let us try to answer to the natural question: does the preceding results enter a general theory of duality? In fact, we even consider a more general problem written under the form of a mathematical programming problem.

Suppose Z is another Banach space, $g : X \to Z$ is a map, C is a closed convex subset of Z, and Y is the dual space of Z. Let $F := \{x \in X : g(x) \in C\}$. Given $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$, problem (P) turns into the problem

$$(M) \qquad \text{minimize } f(x) :\ x \in X,\ g(x) \in C$$

which can be rewritten as the minimization of $f(\cdot) + \iota_C(g(\cdot))$, where ι_C is the *indicator function* of C given by $\iota_C(z) := 0$ if $z \in C$, $\iota_C(z) = +\infty$ otherwise. Again, we denote by ϖ the value of this problem.

We can introduce the *perturbation* $P : X \times Z \to \mathbb{R}_\infty$ given by

$$P(x, z) := f(x) + \iota_C(g(x) + z)$$

and its associated *performance function* p given by

$$p(z) := \inf\{f(x) : x \in X,\ g(x) + z \in C\}.$$

We observe that Y and Z can be coupled with the coupling function $c_\vee : Y \times Z \to \overline{\mathbb{R}}$ given by

$$c_\vee(y, z) = -\iota_{[y \leq 1]}(z).$$

Since $c_\vee(y, 0) = 0$ for all $y \in Y$, the perturbational dual problem of (M) (see [25], [28]) is the problem

$$(D^\vee) \qquad \text{maximize } -p^\vee(y) \qquad y \in Y,$$

of computing $p^{\vee\vee}(0) := (p^\vee)^\vee(0)$, where $p^\vee$ is the conjugate of p for the coupling function $c_\vee$:

$$\begin{aligned}
p^\vee(y) &:= -\inf\{p(z) - c_\vee(y, z) : z \in Z\} \\
&:= -\inf\{f(x) : z \in Z,\ \langle y, z\rangle \leq 1, g(x) + z \in C\} \\
&= -\inf\{f_g(w - z) : z \in Z,\ \langle y, z\rangle \leq 1, w \in C\} \\
&= -\inf\{f_{g,C}(z) : z \in Z,\ \langle y, z\rangle \leq 1\} = (f_{g,C})^\vee(y),
\end{aligned}$$

where

$$f_g(v) := \inf\{f(x) : x \in g^{-1}(v)\}, \qquad f_{g,C}(z) := \inf\{f_g(w - z) : w \in C\}.$$

Similarly, we can use the coupling function c_∇ given by $c_\nabla(y, z) = -\iota_{[y<1]}(z)$. However, this process cannot be applied with the coupling function $c_\wedge$ given by $c_\wedge(y, z) = -\iota_{[y\geq 1]}(z)$ nor with the coupling function c_Δ given by $c_\Delta(y, z) = -\iota_{[y>1]}(z)$. Thus, we take a direct Lagrangian approach rather than a perturbational approach.

We use the simple observation that since C is included in $C^{\Delta\Delta} := (C^{\Delta})^{\Delta}$ (and in the three other bipolar sets of C), we have $\iota_C \geq \iota_{C^{\Delta\Delta}}$. Now

$$\iota_{C^{\Delta\Delta}} = \sup_{y \in C^{\Delta}} \iota_{[y \leq 1]},$$

as easily checked. It follows that we can introduce the *sub-Lagrangian function* L_{Δ} given by

$$L_{\Delta}(x, y) := f(x) + \iota_{[y \leq 1]}(g(x)) \qquad \text{for } (x, y) \in X \times C^{\Delta},$$
$$L_{\Delta}(x, y) := -\infty \qquad \text{for } (x, y) \in X \times (Y \backslash C^{\Delta}).$$

Here we use the terminology of [25]: L_{Δ} is a sub-Lagrangian means that

$$f(x) + \iota_C(g(x)) \geq \sup_{y \in Y} L_{\Delta}(x, y) \qquad \text{for all } x \in X,$$

L_{Δ} being called a *Lagrangian* when equality holds for all $x \in X$ (which is the case when C is closed and radiant). The Lagrangian *dual function* d_{Δ} is given by

$$d_{\Delta}(y) := \inf_{x \in X} L_{\Delta}(x, y).$$

In order to express it, we use again the function $f_g : Z \to \overline{\mathbb{R}}$ given by $f_g(z) := \inf\{f(x) : x \in g^{-1}(z)\}$, with the usual convention that $f_g(z) := +\infty$ when $g^{-1}(z)$ is empty. Then

$$d_{\Delta}(y) = \inf_{z \in Z}(f_g(z) + \iota_{[y \leq 1]}(z)) = -(f_g)_{\Delta}(y) \quad \text{ for } y \in C^{\Delta}$$
$$d_{\Delta}(y) = -\infty \quad \text{ for } y \in Y \backslash C^{\Delta}.$$

Thus, the dual problem can be written

$$(D^{\Delta}) \text{ maximize } -(f_g)_{\Delta}(y) \quad \text{ for } y \in C^{\Delta}.$$

When $Z = X$ and g is the identity mapping I_X, we recover the dual problem we have considered in Section 2. A similar approach can be given for the dual problem (D_{∇}). We also notice that using the inclusions $C \subset C^{\wedge\wedge}$ and $C \subset C^{\vee\vee}$ we can obtain new dual problems. These problems provide new estimates as weak duality holds; but strong duality results are not at hand.

When C is a convex cone or is costarshaped, in particular when C is coradiant, we have $C^{\Delta} = C^{\ominus} := \{y \in Y : \forall z \in C \ \langle y, z \rangle \leq 0\}$, the usual polar cone. Then, for $y \in C^{\Delta}$, we have

$$L_{\Delta}(x, y) := f(x) + \iota_{[y \leq 1]}(g(x)) \leq L_{<}(x, y) := f(x) + \iota_{[y \leq 0]}(g(x))$$

where $L_{<}$ is the surrogate Lagrangian considered in [10], [31]. Thus, if $\overline{y} \in C^{\Delta}$ is a multiplier for the Lagrangian L_{Δ}, it is also a multiplier for the Lagrangian $L_{<}$. Recall that $\overline{y}$ is a *multiplier* for a sub-Lagrangian L if $\inf_{x \in X} L(x, \overline{y}) = \inf_{x \in F} f(x)$.

The advantage of strong duality is reminded in the following statement which relies on [25, Prop. 1.2] and uses the fact that $L_{\Delta}(x, y) = f(x)$ when $\langle g(x), y \rangle \leq 1$ and $L_{\Delta}(x, y) = +\infty$ otherwise.

Proposition 6.1. *Let $\overline{y}$ be a multiplier for the Lagrangian L_{Δ}, i.e. a solution to the dual problem (D^{Δ}) such that $(f_g)_{\Delta}(\overline{y}) = \varpi$. Then $\overline{x}$ is a solution to (P) if, and only if, $\overline{x}$ is a solution to the simplified problem*

$$(Q_{\overline{y}}) \qquad \textit{minimize } f(x) \textit{ subject to the constraint } \langle g(x), \overline{y} \rangle \leq 1.$$

When g is a continuous affine map, in particular, when $Z = X$ and $g = I_X$, the feasible set of $(Q_{\overline{y}})$ is simply a half-space. Clearly, one has a similar result for the Lagrangian L_{∇}. When g is an affine map from $\mathbb{R}^n$ to $\mathbb{R}^m$, with $m < n$, the dual problem is set in a lower dimensional space and may be more tractable.

Acknowledgements

The author is grateful to C. Zalinescu and an anonymous referee for a particularly careful reading of the manuscript and numerous criticisms.

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Manuscript received 30 May 2008
revised 19 December 2008, 24 April 2009, 15 May 2009
accepted for publication 26 May 2009

Jean-Paul Penot
Laboratoire de Mathématiques appliquées, CNRS UMR 5142,
Faculté des Sciences, Université de Pau, BP 1155 64013 PAU cedex France
E-mail address: `jean-paul.penot@univ-pau.fr`