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A NOTE ON THE DUAL OF THE MINKOWSKI-RÅDSTRÖM-HÖRMANDER LATTICE

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Dedicated to Alex Rubinov

Abstract: For a locally convex vector space (X,τ) let $\mathcal{P}(X) := \{p : X \to \mathbb{R} \mid p \text{ is sublinear and continuous}\}$ be the convex cone of all real valued continuous sublinear functions defined on X and $\mathcal{D}(X) = \{\varphi = p - q \mid p, q \in \mathcal{P}(X)\}$ the real vector space of differences of continuous sublinear functions. With respect to the pointwise ordering of functions given by $\varphi \le \psi$ if and only if $\varphi(x) \le \psi(x)$ holds for every $x \in X$, the space $(\mathcal{D}(X), \le)$ is a vector lattice. If $(X, \| \ \|)$ is a Banach space then $\mathcal{D}(X)$ endowed with the norm

$$\|\varphi\|_\Delta = \inf_{\substack{p,q \\ \varphi = p-q}} \left\{ \max \left\{ \sup_{\|x\|\le 1} p(x), \sup_{\|x\|\le 1} q(x) \right\} \right\},$$

where the infimum is taken over all continuous sublinear functions p, q such that $\varphi = p - q$, is also a Banach space. On $\mathcal{D}(X)$ we characterize all continuous linear functionals of $(\mathcal{D}(X), \| \ \|_\Delta)$ which are order bounded and vanish on linear functions for finite dimensional X.

Key words: *Minkowski-Rådström-Hörmander lattice, mixed volumes, pairs of compact convex sets*

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1 Introduction

In 1954 L. Hörmander [5] investigated the equivalence classes of pairs of nonempty compact convex sets for a locally convex space in terms of their support functions. We summarize his results for the case of a Banach space.

For a Banach space $X = (X, \| \ \|)$ let $\mathcal{D}(X) = \{\varphi = p - q \mid p, q \in \mathcal{P}(X)\}$ be the real vector space of differences of continuous sublinear functions. With respect to the pointwise ordering of functions given by $\varphi \le \psi$ if and only if $\varphi(x) \le \psi(x)$ holds for every $x \in X$, the space $(\mathcal{D}(X), \le)$ is a vector lattice. Let us denote for a sublinear function $p : X \longrightarrow \mathbb{R}$ by $\partial\, p\Big|_0 \subset X'$ the subdifferential of p at the origin, which is a nonempty compact convex set in the weak topology of the dual X' and let us assign to $\varphi \in \mathcal{D}(X)$ the set $[\varphi] = \{(\partial p|_0, \partial q|_0) \mid \text{ with } \varphi = p - q,\ p, q \in \mathcal{P}(X)\}$.

To formalize this assignment more precisely, let us denote by $\mathcal{K}(X')$ the set of all nonempty weakly compact convex subsets of X'. Then introduce on $\mathcal{K}^2(X') = \mathcal{K}(X') \times \mathcal{K}(X')$ the equivalence relation $(A, B) \sim (C, D)$ if and only if $A + D = B + C$ holds and denote by $[A,B] \in \mathcal{K}^2(X')/_{\sim}$ denotes the equivalence class which contains $(A,B) \in \mathcal{K}^2(X')$.

In 1966 A. G. Pinsker [8] introduced the following ordering on $\mathcal{K}^2(X')/_{\sim}$, namely: $[A,B] \preceq [C,D] \iff A + D \subseteq B + C$, which is independent of the special choice of representatives, because of the order cancellation law (see [12] and [7], Theorem 3.2.1). The space $\left(\mathcal{K}^2(X')/_{\sim}, \preceq\right)$ is called the Minkowski-Rådström-Hörmander lattice. It is a complete vector lattice and a direct calculation shows that the assignment:

$$\mathcal{D}(X) \longrightarrow \mathcal{K}^2(X')/_{\sim} \quad \text{with} \quad \varphi \mapsto [\varphi] = \{(\partial p|_0, \partial q|_0) \mid \text{ with } \varphi = p - q,\ p, q \in \mathcal{P}(X)\}$$

is a lattice isomorphism, called *Minkowski duality* (see [7], Theorem 3.4.3).

By $\|\varphi\|_\infty = \sup_{\|x\|\le 1} |\varphi(x)|$ we denote the *supremum norm* for $\mathcal{D}(X)$. It is shown in [2] (see the remark after [2] Lemma 6.1), that the normed space $(\mathcal{D}(X), \|\ \|_\infty)$ is not complete. A more detailed investigation of $(\mathcal{D}(X), \|\ \|_\infty)$ can be found in the recent paper of J. Grzybowski and R. Urbański [4].

A norm under which the linear space $\mathcal{D}(X)$ is complete is given in [7]. Some preliminary results in this direction are proved in [1] and [2]. Now we state (see [7], Theorem 8.1.26):

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a Banach space. Then the space*

$$\mathcal{D}(X) = \{\varphi = p - q \mid p, q \text{ are sublinear and continuous}\}$$

endowed with the norm $\|\cdot\|_\Delta$ given by

$$\|\varphi\|_\Delta = \inf_{\substack{p,q \\ \varphi = p-q}} \left\{ \max \left\{ \sup_{\|x\|\le 1} p(x), \sup_{\|x\|\le 1} q(x) \right\} \right\},$$

where the infimum is taken over all continuous sublinear functions p, q such that $\varphi = p - q$, is a Banach space.

2 Some Basic Facts

In this section we recall some basic facts about vector lattices which will be used in this paper.

Let $(E, \le)$ be a vector lattice and let us denote by E^* the *algebraic dual,* that is the set of all linear mappings from E to $\mathbb{R}$. For two elements $x, y \in E$ we denote by $[x,y] = \{z \in E \mid x \le z \le y\}$ the *order interval* and we call a linear functional $f \in E^*$ *order bounded* if the set $f([x,y]) \subset \mathbb{R}$ is bounded for every order interval $[x,y]$. Moreover we call a linear functional $f \in E^*$ *positive* if for every element $x \in E_+ = \{z \in E \mid z \ge 0\}$ of the *positive cone* we have $f(x) \ge 0$.

Note that every order bounded linear functional on the vector lattice $(\mathcal{D}(X), \le)$ is continuous with respect to the supremum norm $\|\ \|_\infty$. Now the following result holds (see [11] Chapt. II, §4, Corollary 2):

Proposition 2.1. *Let $(E, \le)$ a vector lattice. Then $f \in E^*$ is order bounded if and only if it is the difference of two positive linear functionals.*

Now we prove the following auxiliary statements:

Proposition 2.2. *For a real Banach space* $(X, \| \, \|)$ *let* $\mathcal{P}(X) := \{p : X \to \mathbb{R} \mid p$ is sublinear and continuous$\}$ *be the convex cone of all real valued continuous sublinear functions defined on X. If* $\mathcal{D}(X) = \{\varphi = p - q \mid p, q \in \mathcal{P}(X)\}$ *is endowed with the norm* $\|\varphi\|_\Delta = \inf_{\substack{p,q \\ \varphi = p-q}} \Big\{ \max\Big\{ \sup_{\|x\|\le 1} p(x), \sup_{\|x\|\le 1} q(x) \Big\}\Big\}$, *then for every* $\varphi \in \mathcal{D}(X)$ *holds*

$$\|\varphi\|_\infty = \sup_{\|x\|\le 1} |\varphi(x)| \le 2\|\varphi\|_\Delta.$$

Proof. First note, that for every $\varphi \in \mathcal{D}(X)$ there exists a representation $\varphi = p - q$ with $\min\{p(x), q(x)\} \ge 0$ for all $x \in X$ (see [7], Proposition 10.2.3). Now the assertion follows immediately from:

$$\|\varphi\|_\infty = \sup_{\|x\|\le 1} |\varphi(x)| \le \sup_{\|x\|\le 1} p(x) + \sup_{\|x\|\le 1} q(x) \le 2 \left\{ \sup_{\|x\|\le 1} p(x), \sup_{\|x\|\le 1} q(x) \right\}.$$

□

Proposition 2.3. *Let* $(X, \| \, \|)$ *be a real Banach space and* $C \subset X$ *a generating closed convex cone, i.e.* $X = C - C$. *Moreover let us assume that there exists a real* $K > 0$ *such that* $\bar{\mathbb{B}}(0,1) \subseteq K\big(\bar{\mathbb{B}}(0,1) \cap C - \bar{\mathbb{B}}(0,1) \cap C\big)$ *holds, where* $\bar{\mathbb{B}}(0,1)$ *denotes the closed unit ball. If* $f \in E^*$ *and if for every* $x \in C$ *holds* $f(x) \ge 0$, *then* f *is continuous.*

Proof. Let us assume that the assertion is not true. Then there exists a noncontinuous linear functional $f \in X^*$, which assumes only nonnegative values on the closed cone $C \subset X$. Hence there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the closed unit ball $\bar{\mathbb{B}}(0,1)$ such that the sequence $(f(x_n))_{n\in\mathbb{N}}$ is unbounded. Now we proceed as follows. Since $\bar{\mathbb{B}}(0,1) \subseteq K\big(\bar{\mathbb{B}}(0,1) \cap C - \bar{\mathbb{B}}(0,1) \cap C\big)$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of the closed set $\bar{\mathbb{B}}(0,1) \cap C$ with $f(x_n) \ge n^3$. Let us now consider the sequence $\big(\frac{1}{n^2} x_n\big)_{n\in\mathbb{N}}$. Since $\displaystyle\sum_{n=1}^{\infty} \|\frac{1}{n^2} x_n\|$ is convergent and $\bar{\mathbb{B}}(0,1) \cap C$ is closed there exists an element $z_0 \in \bar{\mathbb{B}}(0,1) \cap C$ with $z_0 = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$. Analogously we see, that for every $k \in \mathbb{N}$ holds $\displaystyle\sum_{n=k+1}^{\infty} \frac{1}{n^2} x_n \in \bar{\mathbb{B}}(0,1) \cap C \subset C$. Hence $z_0 - \sum_{n=1}^{k} \frac{1}{n^2} x_n \in C$ for every $k \in \mathbb{N}$, which implies $f(z_0) \ge \displaystyle\sum_{n=1}^{k} \frac{1}{n^2} f(x_n) \ge \frac{k(k+1)}{2}$, and this leads to a contradiction. □

3 Representation Theorems

From now on we consider only the finite-dimensional case. Note that $\mathcal{D}(X) = \mathcal{P}(X) - \mathcal{P}(X)$ and that $\| \, \|_\Delta$ satisfies the assumption of Proposition 2.3. Furthermore note that in the finite dimensional case the norm $\| \, \|_\Delta$ does not depend on the particular choice of the norm for $X = \mathbb{R}^n$.

Theorem 3.1. *Let* $(\mathcal{D}(\mathbb{R}^n), \| \, \|_\Delta)$ *be given. Then every linear functional* $f \in \mathcal{D}(\mathbb{R}^n)'$ *which is order bounded with respect to the pointwise ordering of functions and vanishes on the linear functions is continuous with respect to the norm* $\| \, \|_\Delta$ *and is the difference of two continuous linear functionals* $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)'$ *which vanish also on the linear functions and* *are nonnegative on the convex cone $\mathcal{P}(\mathbb{R}^n)$ of all continuous sublinear functions.*

Moreover both functionals $f_1, f_2 \in \mathcal{D}(\mathbb{R}^n)'$ are also monotone on the convex cone $\mathcal{P}(\mathbb{R}^n)$ of all continuous sublinear functions,i.e. for every $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $p \geq q$ holds $f_1(p) \geq f_1(q)$ and $f_2(p) \geq f_2(q)$.

Proof. First let us endow the space $\mathcal{D}(\mathbb{R}^n)$ with the pointwise ordering of functions, i.e. $\varphi \leq \psi$ if and only if for every $x \in \mathbb{R}^n$ holds $\varphi(x) \leq \psi(x)$. Now choose a continuous linear functional $f \in \mathcal{D}(\mathbb{R}^n)'$. Since by assumption f is order bounded it is the difference of two positive linear functionals $\mu_1, \mu_2 \in \mathcal{D}(\mathbb{R}^n)^*$ with respect to the pointwise ordering.

Now we define the following $2n$ linear functional $\delta_i^+, \delta_i^- \in \mathcal{D}(\mathbb{R}^n)^*,\ i \in \{1, ..., n\}$ by:

$$\delta_i^-(\varphi) \ = \ -\varphi(-e_i) \qquad \text{and} \qquad \delta_i^+(\varphi) \ = \ \varphi(e_i),$$

where $e_1, ..., e_n \in \mathbb{R}^n$ are the unit vectors.

Now let $p \in \mathcal{P}(\mathbb{R}^n)$ be any continuous sublinear function. Then it follows from a direct calculation, that for every linear function $l(x) = \sum_{i=1}^n a_i x_i$ which supports p at the origin, i,e. $p(x) - l(x) \geq 0$ for all $x \in \mathbb{R}^n$, the following inequality holds:

$$\delta_i^-(p) \leq a_i \leq \delta_i^+(p), \quad i \in \{1, ..., n\}. \tag{3.1}$$

Now we put:

$$\nu_i \ = \ \begin{cases} \delta_i^- & \text{if} \quad \mu_1(x_i) \geq 0 \\ \\ \delta_i^+ & \text{if} \quad \mu_1(x_i) < 0 \end{cases}$$

and put $\nu = \displaystyle\sum_{i=1}^n \mu_1(x_i)\nu_i$.

Since $f = \mu_1 - \mu_2$ vanishes on the linear functions, we have $\mu_1(x_i) = \mu_2(x_i)$ for all $i \in \{1, ..., n\}$. Hence we can represent the functional $f \in \mathcal{D}(\mathbb{R}^n)'$ in the form $f = (\mu_1 - \nu) - (\mu_2 - \nu)$.

Now we show that $\min\{(\mu_1 - \nu)(p), (\mu_2 - \nu)(p)\} \geq 0$ for every $p \in \mathcal{P}(\mathbb{R}^n)$. Since $\mu_1(x_i) = \mu_2(x_i)$ for all $i \in \{1, ..., n\}$, we have to give the proof only for $\mu_1 - \nu$.

Therefore let $p \in \mathcal{P}(\mathbb{R}^n)$ be given and choose a linear function $l(x) = \sum_{i=1}^n a_i x_i$ which supports p at the origin, i,e. $p(x) - l(x) \geq 0$ for all $x \in \mathbb{R}^n$. Since μ_1 is nonnegative one has $\mu_1(p - l) \geq 0$ which gives $\mu_1(p) \geq \mu_1(l)$. Now let us put $I_+ = \{i \in \{1, ..., n\} \mid \mu_1(x_i) \ \geq \ 0 \ \}$ and $I_- = \{i \in \{1, ..., n\} \mid \mu_1(x_i) \ < \ 0 \ \}$. Then:

$$\begin{aligned} (\mu_1 \ - \ \nu)(p) \ &= \ \mu_1(p) \ - \ \nu(p) \\ &\geq \ \mu_1(l) - \nu(p) = \sum_{i=1}^n \left(a_i \mu_1(x_i) \ - \ \mu_1(x_i)\nu_i(p)\right) \\ &= \ \sum_{i \in I_+} \mu_1(x_i)\left(a_i \ - \ \delta_i^-(p)\right) + \sum_{i \in I_-} \mu_1(x_i)\left(a_i \ + \ \delta_i^+(p)\right) \geq 0, \end{aligned}$$

because of equation (3.1).

Analogously it follows that $(\mu_2 \ - \ \nu)(p) \ \geq \ 0$ holds for all $p \in \mathcal{P}(\mathbb{R}^n)$. Hence both functional $\mu_1 - \nu$ and $\mu_2 - \nu$ are nonnegative on $\mathcal{P}(\mathbb{R}^n)$ and therefore by Proposition 2.3

also continuous.

Now we show that $\mu_1 - \nu$ and $\mu_2 - \nu$ vanish on linear functions. Therefore let a linear function $l(x) = \sum_{i=1}^n a_i x_i$ be given. Then $\mu_1(l) = \sum_{i=1}^n a_i \mu_1(x_i) = \nu(l)$ and $\mu_2(l) = \sum_{i=1}^n a_i \mu_2(x_i) = \sum_{i=1}^n a_i \mu_1(x_i) = \nu(l)$, because of $\mu_1(x_i) = \mu_2(x_i)$ which implies that both functionals $\mu_1 - \nu$ and $\mu_2 - \nu$ vanish on the linear functions.

Finally we have to prove the monotonicity:

Let us only consider the functional $\mu_1 - \nu$. First let us prove that for every nonnegative function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ the inequality $\nu(\varphi) \leq 0$ holds. This can be seen as follows: We have:

$$\nu(\varphi) \;=\; \sum_{i\in A} \mu_1(x_i)\delta_i^r(\varphi) + \sum_{i\in B} \mu_1(x_i)(-\delta_i^l)(\varphi).$$

Since $\varphi \geq 0$ one has for $i \in A$ that $\mu_1(x_i)\delta_i^r(\varphi) \leq 0$ and for $i \in B$ that $\mu_1(x_i)\delta_i^l(\varphi) \geq 0$. Hence $\nu(\varphi) \leq 0$.

Now let $p, q \in \mathcal{P}(\mathbb{R}^n)$ with $p \geq q$ be given. Then $\varphi = p - q \geq 0$ and $\nu(p) - \nu(q) = \nu(\varphi) \leq 0$. Now choose a linear function $\tilde{l} \in \mathcal{D}(\mathbb{R}^n)$ with $q \geq \tilde{l}$. Then $p - \tilde{l} \geq q - \tilde{l} \geq 0$ and therefore $\mu_1(p - \tilde{l}) \geq \mu_1(q - \tilde{l})$ which implies $\mu_1(p) - \mu_1(q) \geq 0$. Hence we have

$$\mu_1(p) - \mu_1(q) \geq 0 \geq \nu(p) \;-\; \nu(q),$$

which gives the monotonicity:

$$(\mu_1 \;-\; \nu)(p) \geq (\mu_1 \;-\; \nu)(p).$$

□

Remark. Let us note that Theorem 3.1 was first proved by Urban Cegrell (see [1]), except of the monotony.

Corollary 3.2. *Every continuous linear functional $f \in \mathcal{D}(\mathbb{R}^n)'$ is the sum of two linear functionals which are nonnegative on the convex cone $\mathcal{P}(\mathbb{R}^n)$ and a linear combination of the functionals $\delta_i^r(\varphi) = \varphi(e_i), \quad i \in \{1, ..., n\}$, where $e_1, ..., e_n$ are the unit vectors of $\mathbb{R}^n$.*

Proof. This follows immediately form Theorem 3.1 and the fact that

$$\delta_i^r(e_j) \;=\; \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

□

Using a result of W. Firey [3] there exists for every continuous linear functional $\eta \in \mathcal{D}(\mathbb{R}^n)'$ which is nonnegative and monotone on the convex cone $\mathcal{P}(\mathbb{R}^n)$ of all continuous sublinear functions and which vanishes on the linear functions a unique compact convex set $\bar{A} \in \mathcal{K}(\mathbb{R}^n)$, such that η can be represented as a mixed volume, which uniquely depends on $\bar{A} \in \mathcal{K}(\mathbb{R}^n)$ as $\eta(p_A) = V(A, \bar{A}; p-1; \sigma_1, ..., \sigma_{n-p})$, where $\sigma_1, ..., \sigma_{n-p}$ are segments of unit length, whose directions are mutually orthogonal and span the orthogonal complement of the affine hull of $\bar{A}$.

4 Non Order Bounded Linear Functionals

Now we give a negative answer to the question whether every continuous linear functional $f \in \mathcal{D}(\mathbb{R}^n)'$ with respect to $\| \ \|_\Delta$ is also order bounded.

Therefore let us recall the following formula about the addition of faces:

Let (X, τ) be a topological vector space and X' its dual space. Then we denote for $A \in \mathcal{K}(X)$ and $f \in X'$ by

$$H_f(A) = \left\{ z \in A \mid f(z) \ = \ \max_{y \in A} f(y) \right\}$$

the (maximal) *face* of A with respect to f.

For the sum of the faces of $A, B \in \mathcal{K}(X)$ holds:

Proposition 4.1. *Let X be a topological vector space, $f \in X'$ and $A, B \in \mathcal{K}(X)$. Then*

$$H_f(A + B) = H_f(A) + H_f(B).$$

This result was first proved by W. Weil [13], see also [7], Proposition 3.3.1.

We will construct an example for $\mathcal{D}(\mathbb{R}^2)$. Therefore let $\eta \in (R^2)' \setminus \{0\}$ be a nontrivial linear functional. Then we define

$$f : \mathcal{D}(\mathbb{R}^2) \longrightarrow \mathbb{R} \ \text{ by } \ f(\varphi) \ = \ \text{length}(H_\eta(\partial p|_0)) - \text{length}(H_\eta(\partial q|_0)), \qquad (4.1)$$

where $\varphi = p - q \in \mathcal{D}(\mathbb{R}^2)$ and $\text{length}(H_\eta(\partial p|_0))$ denotes the length of the interval $H_\eta(\partial p|_0)$. It follows from Proposition 4.1 that the linear functional f is well defined, i.e. independent of the special choice of p and q for $\varphi = p - q$ and that f is linear.

Now the following statement holds:

Proposition 4.2. *Let $\eta \in (R^2)' \setminus \{0\}$ be a nontrivial linear functional and*

$$f : \mathcal{D}(\mathbb{R}^2) \longrightarrow \mathbb{R} \ \text{ with } \ f(\varphi) \ = \ \text{length}(H_\eta(\partial p|_0)) - \text{length}(H_\eta(\partial q|_0))$$

the linear functional defined by equation (4.1), where $\varphi = p - q \in \mathcal{D}(\mathbb{R}^2)$. Then f is a continuous linear functional on $\mathcal{D}(\mathbb{R}^n)$ with respect to $\| \ \|_\Delta$ which is not order bounded.

Proof. Let us first prove that

$$f : \mathcal{D}(\mathbb{R}^2) \longrightarrow \mathbb{R} \ \text{ with } \ f(\varphi) \ = \ \text{length}(H_\eta(\partial p|_0)) - \text{length}(H_\eta(\partial q|_0))$$

is continuous, where $\varphi = p - q \in \mathcal{D}(\mathbb{R}^2)$. Therefore we show that in the Euclidean norm for every $A \in \mathcal{K}(\mathbb{R}^n)$ holds:

$$\text{diam}(A) = \sup_{x,y \in A} \|x - y\| \ \leq 2 \sup_{\|x\| \leq 1} p_A(x)$$

with $p_A(x) = \sup_{v \in A} \langle v, x \rangle$.

This can be seen as follows:

$$\begin{aligned} \text{diam}(A) \ &= \ \sup_{x,y \in A} \|x - y\| = \sup_{w \in A - A} \|w\| \\ &= \ \sup_{\|x\| \leq 1} \sup_{w \in A - A} \langle w, x \rangle \ = \ \sup_{\|x\| \leq 1} p_{A-A}(x) \leq 2 \sup_{\|x\| \leq 1} p_A(x). \end{aligned}$$

The last inequality follows from the equation

$$p_{-A}(x) \;=\; \sup_{v\in -A}\langle v,x\rangle \;=\; \sup_{v\in A}\langle -v,x\rangle \;=\; p_A(-x).$$

Now let $\varphi \in \mathcal{D}(\mathbb{R}^2)$ and $\varepsilon > 0$ be given. Choose a representation $\varphi = p - q$ such that

$$\max\left(\sup_{\|x\|\le 1} p(x),\ \sup_{\|x\|\le 1} q(x)\right) \le \inf_{\substack{p,q\\ \varphi=p-q}}\left\{\max\left\{\sup_{\|x\|\le 1} p(x),\ \sup_{\|x\|\le 1} q(x)\right\}\right\} + \varepsilon = \|\varphi\|_\Delta + \varepsilon$$

holds. Then:

$$\begin{aligned} |f(\varphi)| &= |\mathrm{length}(H_\eta(\partial p|_0)) - \mathrm{length}(H_\eta(\partial q|_0))| \\ &\le |\mathrm{length}(H_\eta(\partial p|_0))| + |\mathrm{length}(H_\eta(\partial q|_0))| \\ &\le \mathrm{diam}(\partial p|_0) + \mathrm{diam}(\partial q|_0) \\ &\le 2\left(\sup_{\|x\|\le 1} p(x) + \sup_{\|x\|\le 1} q(x)\right) \le 4\left(\|\varphi\|_\Delta + \varepsilon\right), \end{aligned}$$

which gives the continuity of f.

Now we show that the functional is not order bounded. By a linear coordinate transformation in $\mathbb{R}^2$ we can assume that in suitable coordinates the functional η is of the type: $\eta(x_1,x_2) = x_2$. Now define for $a > 1$ the following family of functions:

$$\varphi_a : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

with

$$\varphi_a(x_1,x_2) = \max\{0, ax_1, (a-1)x_1 + x_2\} - \max\{0, ax_1\}.$$

Now it follows from a straightforward calculation that for all $(x_1,x_2) \in \mathbb{R}^2$ the following inequality holds:

$$0 \;\le\; \varphi_a(x_1,x_2) \;\le\; 2\|(x_1,x_2)\| \;=\; 2\sqrt{x_1^2 \;+\; x_2^2}.$$

Hence φ_a is contained in the order interval bounded by 0 and twice the Euclidean norm. But $f(\varphi_a) = -a$ and therefore f is not order bounded. □

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