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AN EXTENSION OF FKKM LEMMA WITH AN APPLICATION TO GENERALIZED EQUILIBRIUM PROBLEMS

Susana Scheimberg and Flávia M. Jacinto

Dedicated to the memory of Alexander M. Rubinov

Abstract: In this paper, we present a new generalization of the Ky Fan KKM Lemma. We also introduce an extension of the KKM Principle related to the finite intersection property. Finallly, we apply these results to establish the existence of solutions for the generalized equilibrium problem. As a consequence, we improve existence results for the equilibrium problem and for the Fan's minimax inequality given in the literature.

Key words: *FKKM Lemma, generalized equilibrium problems, minimax inequalities, equilibrium problems, KKM Principle*

Mathematics Subject Classification: *90C48*

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1 Introduction

The FKKM Lemma given in [16] is extensively used to obtain fixed points theorems and results on the existence of solutions for several problems like variational and quasivariational inequalities, and nonlinear complementarity problems. These problems belong to a unified framework given by the generalized equilibrium problem (GEP) considered in [21].

In this work we present a new version of the FKKM Lemma by relaxing closedness and compactness conditions. There have been numerous generalizations of the Ky Fan property (see for example, [7, 11, 13, 15, 22, 29, 31] and references therein). We use the new version of the Ky Fan property to study the existence of solutions for the generalized equilibrium problem. To illustrate the usefulness of this result, we deduce an existence theorem for the classical equilibrium problem ([5, 6, 14, 15, 19] and references therein) and we establish the Fan's minimax inequality [17].

The outline of this work is as follows. In section 2 we present most of the material used in this paper. In section 3 we derive a characterization of the finite intersection property which extends the KKM Principle [16] and establish the generalization of the Fan KKM Lemma. Several results given in the literature can be considered as particular cases of this extension. In section 4 we use this version to obtain an existence result for the generalized equilibrium problem. Finally, as a consequence of this result, we improve some theorems of existence of solutions for the classical equilibrium problem (EP) and we also obtain the minimax inequality under weaker conditions showing the power of the formulation (GEP).

2 Preliminaries

In this section we recall some of the concepts we will use. The first part of this section is devoted to the KKM theory. In the second part we present the generalized equilibrium probem (GEP). Denote by $\mathcal{P}(C)$ the set of subsets of C.

2.1 KKM Theory

We begin with the well known result of Ky Fan that generalizes the classic theorem of Knaster-Kuratowski-Mazurkiewicz given in finite dimension, called KKM Theorem. [1, 6, 7, 8, 11, 16, 29].

Lemma 2.1 ([16, Fan KKM Lemma]). *Let A be a nonempty subset of a real Hausdorff topological vector space X and let $G : A \to \mathcal{P}(X)$ be a multivalued mapping such that the following conditions are satisfied:*

1. *$G(a)$ is a closed subset in X for all $a \in A$;*
2. *$G(a_0)$ is compact for at least one $a_0 \in A$;*
3. *the convex hull of any finte subset $\{a_1, \dots, a_n\}$ of A is contained in $\bigcup_{i=1}^n G(a_i)$.*

Then, $\bigcap_{a\in A} G(a) \neq \emptyset$

The following notions and properties can be seen with slight differences in [1], [11] and [31]. The next concept corresponds to condition 3 of the lemma above.

Definition 2.2. Let A be a nonempty subset of a real Hausdorff topological vector space X. A multivalued mapping $F : A \to \mathcal{P}(X)$ is said to be a *KKM* mapping if, for any finite subset $\{x_1, \dots, x_n\} \subset A$ it holds

$$co\{x_i, \dots, x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$$

where $co\{x_i, \dots, x_n\}$ denotes the convex hull of the set $\{x_1, \dots, x_n\}$.
Next we give the classical characterization of the finite intersection property.

Lemma 2.3 ([3, KKM Principle]). *Let A be a nonempty subset in a locally convex space E and let $T : A \to \mathcal{P}(E)$ be a multivalued application such that:*

(i) *$T(a)$ is a closed subset in E for all $a \in A$;*

(ii) *T is a KKM mapping.*

Then, $\{T(a) : a \in A\}$ has the finite intersection property, that is, any finite intersection of subsets of the family is nonempty.

We consider below an extension of the concept of the KKM mapping that we will use in our extension of the Fan KKM Lemma.

Definition 2.4. Let A and B be nonempty subsets of two real Hausdorff topological vector spaces X and Y respectively such that B is convex. A multivalued mapping $F : A \to \mathcal{P}(B)$ is said to be a *generalized KKM* mapping if, for any finite subset $\{x_1, \dots, x_n\}$ of A, there is a finite subset $\{y_1, \ldots, y_n\}$ of B such that, for any subset $\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \{y_1, \ldots, y_n\}$ it holds

$$co\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \bigcup_{j=1}^{k} F(x_{i_j}).$$

It is easy to see that every KKM mapping is a generalized KKM mapping. In [11] there is a counterexample to illustrate that the converse does not hold. Let us note that, if F is a multivalued KKM mapping then it must be $x \in F(x)$ for all $x \in A$. If F is a generalized multivalued mapping then $F(x) \neq \emptyset$ for all $x \in A$.
We will also consider the folowing notion.

Definition 2.5. Let A and B be nonempty subsets of two real Hausdorff topological vector spaces X and Y respectively. The multivalued mapping $F : A \to \mathcal{P}(B)$ is said to be *transfer closed-valued* if for every $x \in A$, $y \notin F(x)$, there exists an element $x' \in A$ such that $y \notin cl_B F(x')$, where cl_B denotes the closure relative to B, $cl_B(.) = cl_Y(.) \cap B$.

The next characterization of the definition above will be useful.

Lemma 2.6 ([29]). *A multivalued mapping $G : A \to \mathcal{P}(B)$ is transfer closed-valued if, and only if,*

$$\bigcap_{a\in A} G(a) = \bigcap_{a\in A} \mathrm{cl_B} G(a) \tag{2.1}$$

Let us observe that a multivalued application $G : A \to \mathcal{P}(B)$ with closed values is transfer closed-valued.
In proving our main results we need the following two theorems.

Theorem 2.7 ([4, VII,Theorem 6]). *A topological space X is compact if and only if the finite intersection axiom is verified, that is, if $\{F_i : i \in I\}$ is a family of closed sets in X for which any finite intersection is non empty, then $\bigcap_{i\in I} F_i \neq \emptyset$.*

Theorem 2.8 ([28, Theorem 1]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space X. Let $F : K \to \mathcal{P}(X)$ be a multivalued mapping such that:*

(a) *for each $x \in K$, $F(x)$ is a nonempty convex subset of K;*

(b) *for each $y \in K$, the set $F^{-1}(y) = \{x \in K : y \in F(x)\}$ contains an open subset O_y of K (that may be empty);*

(c) $\bigcup_{y\in K} O_y = K$.

Then, there exists a point $x_0 \in K$ such that $x_0 \in F(x_0)$.

2.2 The Generalized Equilibrium Problem

We consider the following Generalized Equilibrium Problem (GEP) introduced in [21]:

$$(\mathrm{GEP}) \begin{cases} \text{Find } \bar{x} \in D \text{ such that} \\ f(\bar{x}, y) + \varphi(\bar{x}, y) + h(y) \geq \varphi(\bar{x}, \bar{x}) + h(\bar{x}) \text{ for all } y \in X, \end{cases} \tag{2.2}$$

where X is a real Hausdorff topological vector space, D is a nonempty subset of X and $f, \varphi : X \times X \to (-\infty, +\infty]$ and $h : X \to (-\infty, +\infty]$ are functions satisfying:

1. $f(x,x) = 0$ for all $x \in D$;

2. h is a convex function;

3. $\operatorname{dom} f(x,\cdot) \cap \operatorname{dom} \varphi(x,\cdot) \cap \operatorname{dom} h \neq \emptyset$ for all $x \in D$.

Let us observe that our scheme contains other equilibrium problems given in the literature. For example, if we consider in (2.2), $h = \delta_K$ the indicator function of a nonempty, convex and closed subset K of X, then we get the scheme given in [18]. If $h = \delta_K$ and $\varphi(x,y) = \phi(y)$ is a real function on K, for all $x \in X$, the scheme (GEP) becomes the mixed equilibrium problem ([9] and reference therein) . If we take in (GEP) $\varphi \equiv 0$ and $h = \delta_K$, we obtain the classical equilibrium problem (see for example [5, 6, 13, 15, 19, 22] and references therein):

$$\text{(EP)} \left\{ \begin{array}{c} \text{Find } \bar{x} \in K \text{ such that} \\ f(\bar{x}, y) \geq 0 \text{ for all } y \in K, \end{array} \right. \tag{2.3}$$

We illustrate the flexibility of formulation (2.2) with the following examples. When $f \equiv 0$, $\varphi(x,y) = \phi(y)$ for all $x \in X$ and $h = \delta_K$, (2.2) becomes a convex nonlinear programming. Also, the generalized quasivariational inequality considered in [24] and in [25] defined by multivalued applications $A : X \to \mathcal{P}(X^*)$ and $G : X \to \mathcal{P}(X^)$, where X^* is the dual of a Banach space X such that $G(x)$ is a convex set for all $x \in X$, follows the scheme (2.2). In fact, take $\Omega = X \times X^*$, $D = \{(x,\xi) \in \Omega : x \in G(x), \xi \in A(x)\}$, $f((x,\xi),(y,\rho)) = < \xi, y - x >$, $\varphi((x,\xi),(y,\rho)) = \delta_{G(x)}(y)$ and $h \equiv 0$.
Finally, we note that numerical methods for solving equilibrium problems have been extensively studied recently (see for example, [9, 10, 20, 26, 30] and references therein).

3 Contributions to the KKM Theory

We begin with a generalization of the KKM Principle given in [16]. We use this result to obtain a new version of the FKKM Lemma. We relax the closedness condition and we consider a generalized KKM mapping instead of a KKM mapping.

Theorem 3.1. *Let A and B be two nonempty subsets of a real Hausdorff topological vector space X such that B is convex and let $G : A \to \mathcal{P}(B)$ be a multivalued application. Then, $\operatorname{cl}_B G : A \to \mathcal{P}(B)$ is a generalized KKM mapping if, and only if, the family of sets $\{\operatorname{cl}_B G(a) : a \in A\}$ has the finite intersection property.*

Proof. Assume that $\operatorname{cl}_B G : A \to \mathcal{P}(B)$ is a generalized KKM mapping. For purpose of contradiction we suppose that there is a finite subset $\mathcal{A} = \{a_1, \ldots, a_n\}$ of A verifying

$$\bigcap_{i=1}^{n} \operatorname{cl}_B G(a_i) = \emptyset. \tag{3.1}$$

Since $\operatorname{cl}_B G(.)$ is a generalized KKM multivalued mapping, there is a finite subset $B_{\mathcal{A}} = \{b_1, \ldots, b_n\}$ of B such that for any subset $\{b_{i_1}, \ldots, b_{i_k}\}$ of $\{b_1, \ldots, b_n\}$, with $k \in I := \{1, \ldots, n\}$ it holds

$$\operatorname{co}\{b_{i_1}, \ldots, b_{i_k}\} \subseteq \bigcup_{j=1}^{k} \operatorname{cl}_B G(a_{i_j}). \tag{3.2}$$

From (3.1) we have that for each point $b \in \operatorname{co} B_{\mathcal{A}}$ there is $i \in I$ such that $b \notin \operatorname{cl}_B G(a_i)$. So, for each $b \in \operatorname{co} B_{\mathcal{A}}$ we consider the following nonempty set:

$$I_b := \{i \in I : b \notin \operatorname{cl}_B G(a_i)\}. \tag{3.3}$$

Let us define $T : \operatorname{co} B_{\mathcal{A}} \to \mathcal{P}(\operatorname{co} B_{\mathcal{A}})$ by

$$T(b) := \operatorname{co}\{b_i \in B_{\mathcal{A}} : i \in I_b\}. \tag{3.4}$$

Since $B_{\mathcal{A}}$ is a nonempty finite set we have that $\operatorname{co} B_{\mathcal{A}}$ is nonempty and compact. Furthermore, it is convex. We will show that T verifies all the conditions of Theorem 2.8.

(a) From (3.4) and (3.3) we obtain that $T(b)$ is a nonempty and convex subset of $\operatorname{co} B_{\mathcal{A}}$ for all $b \in \operatorname{co} B_{\mathcal{A}}$.

(b) We claim that for each $w \in \operatorname{co} B_{\mathcal{A}}$, there is a relative open subset $\mathcal{O}_w$ of $\operatorname{co} B_{\mathcal{A}}$ such that $\mathcal{O}_w \subseteq T^{-1}(w)$. Indeed, if $T^{-1}(w) = \emptyset$, we take $\mathcal{O}_w = \emptyset$. Otherwise, let $v \in T^{-1}(w)$. We consider

$$O_v := \operatorname{co} B_{\mathcal{A}} \setminus \left(\bigcup_{i \in I_v} \operatorname{cl}_B G(a_i) \right) \tag{3.5}$$

By the definition of $T(v)$ we have that $v \in O_v$. Furthermore, O_v is relative open in $\operatorname{co} B_{\mathcal{A}}$. In fact, since $\operatorname{co} B_{\mathcal{A}} \subseteq B$ we can rewrite O_v as

$$O_v = \operatorname{co} B_{\mathcal{A}} \bigcap \left(B \setminus \left(\bigcup_{i \in I_b} \operatorname{cl}_B G(a_i) \right) \right). \tag{3.6}$$

The set $B \setminus \left(\bigcup_{i \in I_b} \operatorname{cl}_B G(a_i) \right)$ is relative open in B, that is, there is an open set V of X such that:

$$B \setminus \left(\bigcup_{i \in I_b} \operatorname{cl}_B G(a_i) \right) = V \bigcap B \tag{3.7}$$

Therefore, from (3.5)-(3.7) we deduce that O_v is relative open in $\operatorname{co} B_{\mathcal{A}}$. Finally, we define

$$\mathcal{O}_w := \bigcup_{v \in T^{-1}(w)} O_v \tag{3.8}$$

which is relative open in $\operatorname{co} B_{\mathcal{A}}$. Now, we prove that $\mathcal{O}_w \subseteq T^{-1}(w)$. Indeed, if $z \in \mathcal{O}_w$ then $z \in O_v$ for some $v \in T^{-1}(w)$. Hence, by (3.5) it results that $z \in \operatorname{co} B_{\mathcal{A}}$ and $z \notin \operatorname{cl}_B G(a_i)$ for all $i \in I_v$. So, $I_v \subseteq I_z$, that is, $T(v) \subseteq T(z)$ and together with $v \in T^{-1}(w)$ we deduce that $z \in T^{-1}(w)$. So, it follows that $\mathcal{O}_w \subseteq T^{-1}(w)$.

(c) We show that $\operatorname{co} B_{\mathcal{A}} = \bigcup_{w \in \operatorname{co} B_{\mathcal{A}}} \mathcal{O}_w$. Indeed, from (3.5) and (3.8) we have that $\bigcup_{w \in \operatorname{co} B_{\mathcal{A}}} \mathcal{O}_w \subseteq \operatorname{co} B_{\mathcal{A}}$. On the other hand, let $z \in \operatorname{co} B_{\mathcal{A}}$. So, by (a) there is $w \in T(z) \subseteq \operatorname{co} B_{\mathcal{A}}$, that is, $z \in T^{-1}(w)$. Replacing v by z in (3.5) and (3.8) we deduce that $z \in O_z \subseteq \mathcal{O}_w \subseteq \bigcup_{u \in \operatorname{co} B_{\mathcal{A}}} \mathcal{O}_u$. Then, we obtain that $\operatorname{co} B_{\mathcal{A}} \subseteq \bigcup_{w \in \operatorname{co} B_{\mathcal{A}}} \mathcal{O}_w$.

Hence, by Theorem 2.8 we conclude that the operator T has a fixed point b^*. Therefore, (3.4) and (3.2) imply

$$b^* \in T(b^*) = \operatorname{co}\{b_i \in B_{\mathcal{A}} : i \in I_{b^*}\} \subseteq \bigcup_{i \in I_{b^*}} \operatorname{cl}_B G(a_i) \tag{3.9}$$

Using the definition of I_{b^*} it results that $b^* \notin \operatorname{cl}_B G(a_i)$ for all $i \in I_{b^*}$. Then,

$$b^* \notin \bigcup_{i \in I_{b^*}} \operatorname{cl}_B G(a_i),$$

which is in contradiction with (3.9). Thus, for any finite subset $\mathcal{A} = \{a_1, \ldots, a_n\}$ of A it holds that $\bigcap_{i=1}^n \operatorname{cl}_B G(a_i) \neq \emptyset$, that is, the family $\{\operatorname{cl}_B G(a) : a \in A\}$ has the finite intersection property.
Conversely, assume that $\{\operatorname{cl}_B G(a) : a \in A\}$ has the finite intersection property. Consider a finite subset $\{a_1, \ldots, a_n\}$ of A. Therefore,

$$\bigcap_{i=1}^{n} \operatorname{cl}_B G(a_i) \neq \emptyset. \tag{3.10}$$

Hence, there exists $b \in \bigcap_{i=1}^n \operatorname{cl}_B G(a_i) \subseteq B$. Take $b_i = b$ for all $i = 1, 2, \ldots, n$. Then, for each $\{b_{i_1}, \ldots, b_{i_k}\} \subseteq \{b_1, \ldots, b_n\}$, with $k \in I$ it holds that

$$\operatorname{co}\{b_{i_1}, \ldots, b_{i_k}\} = \{b\} \subseteq \bigcap_{i=1}^{n} \operatorname{cl}_B G(a_i) \subseteq \bigcup_{j=1}^{k} \operatorname{cl}_B G(a_{i_j}). \tag{3.11}$$

Thus, we conclude that $\operatorname{cl}_B G : A \to \mathcal{P}(B)$ is a generalized KKM mapping. The proof is complete. □

Now we are able to give an extension of the FKKM Lemma.

Theorem 3.2. *Let A and B be nonempty subsets of a real Hausdorff topological vector space X such that B is convex. If $G : A \to \mathcal{P}(B)$ verifies the following conditions:*

(i) $\operatorname{cl}_B G : A \to \mathcal{P}(B)$ *is a generalized KKM-mapping,*

(ii) G *is a transfer closed-valued mapping,*

(iii) *there exists a finite subset A_0 of A such that $C = \bigcap_{z \in A_0} \operatorname{cl}_B G(z)$ is compact,*

then, $\bigcap_{a \in A} G(a) \neq \emptyset$.

Proof. Using conditions (i) and (ii) and Theorem 3.1 it follows that $\{\operatorname{cl}_B G(a) \cap C : a \in A\}$ is a family of nonempty sets and it verifies the finite intersection property. Moreover, for each $a \in A$ we have that $\operatorname{cl}_B G(a) \cap C = \operatorname{cl}_X G(a) \cap B \cap C = \operatorname{cl}_X G(a) \cap C$ is a closed subset of C which is compact. Hence, by considering on C the induced topology of X and Theorem 2.7 we get that

$$\bigcap_{a \in A} \left\{ \operatorname{cl}_B G(a) \bigcap C \right\} \neq \emptyset. \tag{3.12}$$

On the other hand, using the definition of C and Lemma 2.6 it results that:

$$\bigcap_{a \in A} \left\{ \operatorname{cl}_B G(a) \bigcap C \right\} = \bigcap_{a \in A} \operatorname{cl}_B G(a) = \bigcap_{a \in A} G(a). \tag{3.13}$$

Thus, the conclusion follows from(3.12) and (3.13). □

Let us note that Fan's Lemma [16] can be obtained from the property above. Furthermore, the following results given in the literature are also immediate consequences of Theorem 3.2.

Corollary 3.3 ([11, Theorem 3.2]). *Let X be a nonempty convex subset of a real Hausdorff topological vector space E. Let $F : X \to \mathcal{P}(E)$ be a multivalued application where $F(x)$ is closed for all $x \in X$ and such that $F(x_0)$ is compact for at least one $x_0 \in X$. Then, $\bigcap_{x\in X} F(x) \neq \emptyset$ if and only if F is a generalized KKM application.*

Corollary 3.4 ([1, Theorem 2.1]). *Let X be a nonempty convex subset of a Hausdorff topological vector space E. Let $F : X \to \mathcal{P}(E)$ be a transfer closed-valued mapping such that $\mathrm{cl}_E F(x_0) = K$ is compact for at least one $x_0 \in X$ and let $\mathrm{cl}_E F$ be a generalized KKM mapping. Then, $\bigcap_{x\in X} F(x) \neq \emptyset$.*

4 Generalized Equilibrium Problems

We begin this section with an existence theorem for solutions to problem (GEP).

Theorem 4.1. *Let D be a nonempty convex subset of a real Hausdorff topological vector space X, and let $f, \varphi : X \times X \to (-\infty, +\infty]$ and $h : X \to (-\infty, +\infty]$ be functions such that they verify all requirements of problem (GEP). Assume that the conditions below hold:*

(A1) *For any finite subset $\{y_1, \dots, y_m\} \subset X$ there is a finite subset $\{x_1, \dots, x_m\}$ of D such that, for any subset $\{x_{i_1}, \dots, x_{i_k}\}$ of $\{x_1, \dots, x_m\}$ and for any point $x \in co\{x_{i_1}, \dots, x_{i_k}\}$ there are $j \in \{1, \dots, k\}$ and a filtered family $\{z_\lambda : \lambda \in \Lambda\} \subseteq D$ having x as a limit point such that*

$$f(z_\lambda, y_{i_j}) + \varphi(z_\lambda, y_{i_j}) + h(y_{i_j}) \ \geq \ \varphi(z_\lambda, z_\lambda) + h(z_\lambda) \ \forall \lambda \in \Lambda. \tag{4.1}$$

(A2) *If, for some $x \in D$ and $y \in X$ it holds that*

$$f(x, y) + \varphi(x, y) \ + \ h(y) \ < \ \varphi(x, x) + h(x) \tag{4.2}$$

then, there are a point $y' \in X$ and a neighbourhood $U(x)$ of x such that:

$$f(z, y') + \varphi(z, y') + h(y') \ < \ \varphi(z, z) + h(z) \quad \forall z \in U(x) \cap D. \tag{4.3}$$

(A3) *There exist a nonempty and compact subset B of D and a finite subset $\{y_1, \dots, y_l\}$ of X such that for each $w \in D \setminus B$ there are a neighbourhood $V(w)$ of w and $j \in \{1, 2, \dots, l\}$ satisfying:*

$$f(z, y_j) + \varphi(z, y_j) + h(y_j) \ < \ \varphi(z, z) + h(z) \quad \forall z \in V(w) \cap (D \setminus B). \tag{4.4}$$

Then, there exists a solution $\bar{x}$ to problem (GEP). In addition, the following minimax inequality holds:

$$sup_{x\in D} inf_{y\in X}\{f(x, y) + \varphi(x, y) + h(y)\} \geq inf_{x\in D}\{f(x, x) + \varphi(x, x) + h(x)\} \tag{4.5}$$

Proof. We define the multivalued mapping $T : X \to \mathcal{P}(D)$ by

$$T(y) := \{x \in D : f(x, y) + \varphi(x, y) + h(y) \geq \varphi(x, x) + h(x)\ \} \tag{4.6}$$

Observe that x is a solution of problem (GEP) if and only if $x \in \bigcap_{y\in X} T(y)$. So, we will deduce that this last intersection is nonempty by proving that T satisfies all conditions of Theorem 3.2.

(i) $\mathrm{cl}_D\, T$ is a generalized KKM mapping. Indeed, consider a finite set $\{y_1, \ldots, y_m\}$ of X. By assumption (A1) there is a finite subset $\{x_1, \ldots, x_m\} \subset D$ such that, for any subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_m\}$ and for any point $x \in co\{x_{i_1}, \ldots, x_{i_k}\}$ there are $j \in \{1, \ldots, k\}$ and a filtered family $\{z_\lambda : \lambda \in \Lambda\} \subseteq D$ having x as a limit point such that (4.1) is verified. Therefore, $z_\lambda \in T(y_{i_j})$ for all $\lambda \in \Lambda$. Since x is a limit point of this filtered family it follows that $x \in \mathrm{cl}_D\, T(y_{i_j}) \subseteq \bigcup_{s=1}^{k} \mathrm{cl}_D\, T(y_{i_s})$. Hence, $\mathrm{cl}_D\, T$ is a generalized KKM mapping.

(ii) T is a transfer closed-valued mapping. In fact, consider $y \in X$ and $x \in D$ such that $x \notin T(y)$. So, by assumptiom (A2) there are $y' \in X$ and a neighbourhood $U(x)$ of x such that (4.3) is verified. Thus, $x \notin \mathrm{cl}_D\, T(y')$. Our claim is valid.

(iii) There is a finite subset A_0 of X such that $\bigcap_{y \in A_0} \mathrm{cl}_D\, T(y)$ is compact. In fact, we consider the compact subset B of D and the finite subset $A_0 = \{y_1, y_2, \ldots, y_l\}$ of X of assumption (A3). We will prove that this A_0 satisfies our assertion. First, we show that $\bigcap_{y \in A_0} \mathrm{cl}_D\, T(y) \subseteq B$. In fact, assume that there is $w \in \bigcap_{y \in A_0} \mathrm{cl}_D\, T(y)$ such that $w \notin B$. Since, $w \in D \setminus B$ by condition (A3) there exist $j \in \{1, 2, \ldots, l\}$ and a neighbourhood $V(w)$ of w such that (4.4) holds. Therefore $z \notin T(y_j)$ for all $z \in V(w) \cap (D \setminus B)$. Since B is a compact set it follows that its complement $W = X \setminus B$ is an open set. So, $V(w) \cap (D \setminus B) = (V(w) \cap W) \cap D$. Then we get that $w \notin \mathrm{cl}_D\, T(y_j)$ which is in contradiction with our assumption. Thus, we deduce that $\bigcap_{y \in A_0} \mathrm{cl}_D\, T(y) \subseteq B$. Finally, we prove that this intersection is a compact set. In fact, we have that

$$\bigcap_{y \in A_0} \mathrm{cl}_D\, T(y) = \left(\bigcap_{y \in A_0} \mathrm{cl}_D\, T(y) \right) \cap B = \left(\bigcap_{y \in A_0} (\mathrm{cl}\, T(y) \cap D) \right) \cap B =$$

$$= \bigcap (\mathrm{cl}_X\, T(y) \cap B))$$

which is a closed set. Hence, the assertion is valid since a closed subset of a compact set is also a compact set (see [4]). Therefore, by applying Theorem 3.2 to $A = X$, $B = D$ and $G = T$ we conclude that $\bigcap_{y \in X} T(y) \neq \emptyset$. Hence, there is $\bar{x} \in T(y)$ for all $y \in X$. So, $\bar{x}$ is a solution of problem (GEP). Moreover, using (2.2) and the notions of infimum and supremum we derive inequality (4.5). The proof is complete. □

As a consequence of this theorem we establish an existence result for the equilibrium problem (EP) and we obtain the Fan's minimax inequality.

Corollary 4.2. *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, and let $f : K \times K \to (-\infty, +\infty]$ be a function such that $f(x, x) = 0$ for all $x \in K$. Assume that the conditions below hold:*

(E1) *For any finite subset $\{y_1, \ldots, y_m\} \subset K$ there is a finite subset $\{x_1, \ldots, x_m\}$ of K such that, for any subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_m\}$ and for any point $x \in co\{x_{i_1}, \ldots, x_{i_k}\}$ there are $j \in \{1, \ldots, k\}$ and a filtered family $\{z_\lambda : \lambda \in \Lambda\} \subseteq K$ having x as a limit point such that $\varphi(z_\lambda, y_{i_j})) \geq 0$ for all $\lambda \in \Lambda$.*

(E2) *For every $y \in K$, the function $f(., y)$ is transfer upper semicontinuous in $x \in K$, that is, for all $x, y \in K$ with $f(x, y) < 0$ there are a point $y' \in K$ and a neighbourhood $U(x)$ of x such that $f(z, y') < 0$ for all $z \in U(x) \cap K$.*

(E3) *There exist a nonempty compact subset B of K and a finite subset W of K such that for each $u \in K \setminus B$ there are a point $w \in W$ and a neighbourhood $V(u)$ of u such that $f(z,w) < 0$ for all $z \in V(u) \cap (K \setminus B)$.*

Then, there exists $\bar{x} \in K$ such that $f(\bar{x}, y) \geq 0$ for all $y \in K$.

Corollary 4.3. *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X and let $\varphi : K \times K \to (-\infty, +\infty]$ be a function such that the conditions below hold:*

(M1) *For any finite subset $\{y_1, \ldots, y_m\} \subset K$ exists a finite subset $\{x_1, \ldots, x_m\}$ of K such that, for any subset $\{x_{i_1}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_m\}$ and for any point $x \in co\{x_{i_1}, \ldots, x_{i_k}\}$ there are $j \in \{1, \ldots, k\}$ and a filtered family $\{z_\lambda : \lambda \in \Lambda\} \subseteq K$ having x as a limit point such that $\varphi(z_\lambda, y_{i_j}) \geq \varphi(z_\lambda, z_\lambda)$ for all $\lambda \in \Lambda$.*

(M2) *For every $x, y \in K$, such that $\varphi(x,y) < \varphi(x,x)$ then there are a point $y' \in K$ and a neighbourhood $U(x)$ of x such that $\varphi(z, y') < \varphi(z,z)$ for all $z \in U(x) \cap K$.*

(M3) *There exist a nonempty compact subset B of K and a finite subset W of K such that for each $u \in K \setminus B$ there are a point $w \in W$ and a neighbourhood $V(u)$ of u such that $\varphi(z,w) < \varphi(z,z)$ for all $z \in V(u) \cap (K \setminus B)$.*

Then, it holds

$$sup_{x \in K}\, inf_{y \in K}\, \varphi(x,y) \geq inf_{x \in K}\, \varphi(x,x).$$

Remark 4.4. 1. The following conditions imply (E1)or (M1): $f(.,y)$ is quasiconvex ([2, 18, 27]), 0-diagonally quasiconvex ([5, 13, 15]), 0-generalized quasiconvex ([11]). In [29] is assumed that $\mathrm{cl}_X \{x \in K : f(x,y) \geq 0\}$ is a generalized KKM mapping.

2. Condition E2 or M2 is considered for example, in [1, 2, 22, 29]. In [5, 11, 13, 18, 27] is assumed that the function is semicontinuous in x which implies E2.

3. More restrictive conditions than E3 and M3 are given in [15] and [22]. A similar condition to E3 is used in [23] for Minty variational inequalities. In [11] it is assumed that there is y_0 such that $T(y) = \{x \in K : \varphi(x,y) \geq inf_{x \in k}\, \varphi(x,x)\}$ is compact.

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Susana Scheimberg
Instituto de Matemática, COPPE/Sistemas
Universidade Federal do Rio de Janeiro - UFRJ
E-mail address: susana@cos.ufrj.br

Flávia M. Jacinto
Departamento de Matemática
Universidade Federal do Amazonas-UFA
E-mail address: morgana@cos.ufrj.br