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SEPARATION PROPERTIES VIA CONNECTEDNESS OF TOPOLOGICAL CONVEXITY SPACES*

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Dedicated to the memory of Alexander M. Rubinov

Abstract: For a given collection $\mathcal{H}$ of subsets of a set X we examine the convexity on X generated by $\mathcal{H}$. We use a special type of connectedness of $\mathcal{H}$ and X for investigation of separation of convex sets by elements of $\mathcal{H}$. In particular, we give a description of convex sets, which can be represented as the intersection of a subfamily of $\mathcal{H}$. As an application, we give a description of abstract convex functions and sets. We also describe the abstract convex hull of a finite union of abstract convex sets.

Key words: *convexity space, closure space, separation, abstract convex function, abstract convex set.*

Mathematics Subject Classification: *52A01, 54A05*

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1 Introduction

Recall (see, for example, [11]) that a collection $\mathcal{G}$ of subsets of a set X is called a convexity on X if
(1) $\emptyset, X \in \mathcal{G}$
(2) $\bigcap \mathcal{A} \in \mathcal{G}$ for every $\mathcal{A} \subset \mathcal{G}$
(3) $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A} \subset \mathcal{G}$ is a chain with respect to the inclusion.
Members of $\mathcal{G}$ are called convex sets and the pair $(X, \mathcal{G})$ is called a convexity space. For any subset $A \subset X$ its convex hull $\mathrm{conv}_{\mathcal{G}} A$ is defined by $\mathrm{conv}_{\mathcal{G}} A = \bigcap\{G \in \mathcal{G} : \ A \subset G\}$.

Along with convexity spaces consider also the closure spaces ([11], p. 4). A collection $\mathcal{P}$ of subsets of X is called a protopology (Moore family) on X provided that $\emptyset, X \in \mathcal{P}$ and $\mathcal{P}$ is stable with respect to intersections, that is, $\bigcap \mathcal{A} \in \mathcal{P}$ for every $\mathcal{A} \subset \mathcal{P}$. If $\mathcal{P}$ is a protopology on X then the pair $(X, \mathcal{P})$ is called a closure space. Closure spaces go back to Moore [6].

Note that each convexity space is also a closure space. In other words, protopology is a more general notion than convexity. However, in practice we are usually interested in cases, when protopology is a subset of convexity. For example, in the classical convex case it is natural to assume that the convexity consists of all convex sets and the protopology consists of all closed convex sets.

It is easy to see that convexities and protopologies enjoy the property: intersection of any family of convexities (protopologies) on a set X is also a convexity (protopology) on X.

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Thus, a natural way to introduce a convexity and a protopology is the following. Let $\mathcal{H}$ be a collection of subsets of a set X. We say that the convexity (protopology) $\mathcal{G}$ is generated by $\mathcal{H}$ if $\mathcal{G}$ is the smallest convexity (protopology) on X, which contains $\mathcal{H}$. In the case of convexity we say also that $\mathcal{H}$ is a subbase for $\mathcal{G}$. Note that topologies enjoy the same property: intersection of any family of topologies on a given set X is also a topology on X. So we can consider subbases for topologies as well.

Proposition 1.1. *Let $\mathcal{G}$, $\mathcal{P}$ and $\mathcal{T}$ be the convexity, the protopology and the topology on X respectively, which are generated by $\mathcal{H}$. Then*

(i) *$\mathcal{P}$ consists of the empty set, whole X and all intersections of subfamilies of $\mathcal{H}$.*

(ii) *$\mathcal{T}$ consists of the empty set, whole X and all sets $A \subset X$ such that for each point $x \in A$ a finite collection $\{H_1, \dots, H_n\} \subset \mathcal{H}$ exists with $x \in \bigcap_i H_i \subset A$.*

(iii) *A set $A \subset X$ belongs to the convexity $\mathcal{G}$ if and only if for every finite subset $F \subset A$ and for every point $y \notin A$ a set $H \in \mathcal{H}$ exists such that $F \subset H$ and $y \notin H$.*

Statements (i) and (ii) are obvious. The characterization of convex sets via elements of a subbase stated in (iii) follows from Propositions 1.2 and 1.3 below.

For any set A let $[A]^{<\omega}$ denote the collection of all finite subsets of A.

Proposition 1.2 ([11], p. 31, Proposition 2.1). *Let $(X, \mathcal{G})$ be a convexity space. Then for every subset $A \subset X$*

$$\mathrm{conv}_{\mathcal{G}} A = \bigcup_{F \in [A]^{<\omega}} \mathrm{conv}_{\mathcal{G}} F. \tag{1.1}$$

Proposition 1.3 ([11], p. 10, Proposition 1.7.3). *Let $(X, \mathcal{G})$ be a convexity space. If $\mathcal{H}$ is a subbase for the convexity $\mathcal{G}$ then for every finite subset $F \subset X$*

$$\mathrm{conv}_{\mathcal{G}} F = \bigcap \{H \in \mathcal{H} : \ F \subset H\}. \tag{1.2}$$

In the right-hand side of (1.2) it is assumed that the intersection over the empty set is equal to X. In other words, if $F \not\subset H$ for any $H \in \mathcal{H}$ then we set $\mathrm{conv}_{\mathcal{G}} F = X$.

Separation properties of convexity and closure spaces are based on separation of complicated sets by sufficiently simple sets. Here we consider a strong version of separability for disjoint sets. Let $A, B, H \subset X$. We say that H separates A from B provided that $A \subset H$ and $B \subset X \backslash H$. We are mainly interested in separation of two sets and separation of a set from a point in its complement. Of course, it is intended that A, B and H belong to a convexity (protopology). It is important that the set H, which separates A from B, is simple enough.

First, consider separation properties of convexity spaces. In the classical convex case (see [2]) two disjoint convex sets in a real vector space can be separated by a halfspace (i.e, a convex set with the convex complement). In particular, each convex set and each point in its complement can be separated by a halfspace.

We can generalize the notion of halfspace in the following natural way: a subset $H \subset X$ of a convexity space $(X, \mathcal{G})$ is called a halfspace provided $H \in \mathcal{G}$ and $(X \backslash H) \in \mathcal{G}$. There are some results related to separation of convex sets by halfspaces. The *Polytope Screening Characterization* (see Theorem 3.8 in [11]) states, in particular, that separability of arbitrary convex sets is equivalent to separability of all polytopes (i.e., convex hulls of finite sets). This means the following: if any two disjoint polytopes can be separated by a halfspace, then the same is valid for any two disjoint convex sets. Unfortunately, this general result is not suitable for use in practice, because verification of the separability of all polytopes is not simple. Moreover, it does not imply any description of convex sets. The situation becomes easier if the convexity is of finite arity. Let N be a positive integer and $(X, \mathcal{G})$ be a convexity space. Then $\mathcal{G}$ is called N-ary (or of arity N) (see [11]) provided that a set $A \subset X$ is convex if and only if $\text{conv}_{\mathcal{G}}\{a_1, \ldots, a_N\} \subset A$ for all $a_1, \ldots, a_N \in A$. So, if the number N is not large, there is a sufficiently simple description of convex sets. As it follows from ([5], Theorem 4.2), if the convexity is of arity N, then separability of arbitrary convex sets is equivalent to separability of all N-polytopes (convex hulls of N-point sets). However, this result (as well as the Polytope Screening Characterization) does not imply clear description of the collection of all halfspaces. Thus, there are two main problems concerning separation of convex sets by halfspaces, namely the description of convex sets and the description of the collection of all halfspaces.

Consider an interesting example of convexity on $I \times \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval.

Example 1.4. Let $\mathcal{F}$ be a family of continuous functions $\varphi : I \to \mathbb{R}$. Assume that $\mathcal{F}$ is a two-parameter family (see [1]). It means that for any two points $(x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$ with $x_1 \neq x_2$ there exists exactly one $\varphi \in \mathcal{F}$ such that $\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$. Let $\varphi_{(x_1,y_1)(x_2,y_2)}$ be the function determined by (x_1, y_1) and (x_2, y_2).

For each $a = (x_1, y_1), b = (x_2, y_2) \in I \times \mathbb{R}$ define the generalized segment $[a, b] \subset I \times \mathbb{R}$:

$$[a, b] = \{(x, \varphi_{(x_1,y_1)(x_2,y_2)}(x)) : \min\{x_1, x_2\} \leq x \leq \max\{x_1, x_2\}\}, \qquad \text{if} \quad x_1 \neq x_2$$

and

$$[a, b] = \{(x_1, y) : \min\{y_1, y_2\} \leq y \leq \max\{y_1, y_2\}\}, \qquad \text{if} \quad x_1 = x_2.$$

Then a set $A \subset I \times \mathbb{R}$ is said to be $\mathcal{F}$-convex (see [4]) if for any $a, b \in A$ we have $[a, b] \subset A$. It is easy to check that the collection of all $\mathcal{F}$-convex sets is a convexity on $I \times \mathbb{R}$.
Such type of generalized convex sets possesses strong separation properties. The following result was proved in [7]. Let $A, B \subset I \times \mathbb{R}$ be disjoint $\mathcal{F}$-convex sets. Then there exists an $\mathcal{F}$-convex set H which separates A from B and such that its complement $(I \times \mathbb{R}) \backslash H$ is also $\mathcal{F}$-convex.

In this paper we use elements of a subbase instead of halfspaces. More precisely, for a given collection $\mathcal{H}$ of subsets of a set X we introduce the convexity on X generated by $\mathcal{H}$ and investigate separation of convex sets by elements of $\mathcal{H}$. Unlike the separation by halfspaces, this approach, as a rule, cannot give separability for all convex sets. Hence we need to describe convex sets, which can be separated by elements of $\mathcal{H}$. Nevertheless, the collection $\mathcal{H}$ in this case, unlike the collection of all halfspaces, is initially given.

Similar problems arise for closure spaces. Here we are interested in the cases, when the protopology is generated by a given collection of sets. Namely, we consider the case, when the protopology and the convexity on X are generated by the same collection $\mathcal{H}$. Then the main problem is the description of elements of protopology. In other words, we need to describe the subsets of X, which can be represented as the intersection of a subfamily of $\mathcal{H}$.

Obviously, there is no solution of these problems in the general case. Hence we need to apply a restriction on the choice of the collection $\mathcal{H}$. Here, as a sort of such restriction, we choose a special type of connectedness of a topological space with respect to a convexity on this space. This type of connectedness was used in [10] in conditions, which guarantee that the convexity generated by $\mathcal{H}$ is of arity N. The present paper is a continuation of [10] and aims mainly at applications in abstract convex analysis.

Recall some definitions related to abstract convexity (see, for example, [8]). Let L be a set of functions $l : Y \to \mathbb{R}$ defined on a set Y. A function $f : Y \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ is called abstract convex with respect to L (shortly L-convex) if there exists a set $U \subset L$ such that f is the upper envelope of this set:

$$f(y) = \sup\{l(y) : \ l \in U\} \quad \text{ for all } y \in Y.$$

The set

$$\mathrm{supp}\,(f, L) = \{l \in L : \ l(y) \le f(y) \ \forall\, y \in Y\}$$

of all L-minorants of f is called the support set of the function f with respect to L.
A set $U \subset L$ is called abstract convex with respect to Y (or (L, Y)-convex) if $l \in U$ whenever $l \in L$ and $l(y) \le \sup_{u \in U} u(y)$ for all $y \in Y$. The intersection of all (L, Y)-convex sets containing a set $U \subset L$ is called the abstract convex hull or (L, Y)-convex hull of the set U. This set is denoted by $\mathrm{co}_{L,Y}U$ (or shortly $\mathrm{co}_L U$). We have

$$\mathrm{co}_L U = \left\{ l \in L : \ l(y) \le \sup_{u \in U} u(y) \ \forall\, y \in Y \right\}.$$

It is easy to see that abstract convex analysis deals with closure spaces. Indeed, the family of all epigraphs $\mathrm{epi}\, f = \{(y, c) \in Y \times \mathbb{R} : \ f(y) \le c\}$ of L-convex functions f is stable with respect to intersections. So, this family is a protopology on $Y \times \mathbb{R}$ generated by the collection of all epigraphs $\mathrm{epi}\, l$ of functions $l \in L$. The family of all (L, Y)-convex sets is also a protopology on L. It is generated by the collection of all sets $H = \{l \in L : \ l(y) \le c\}$ with $(y, c) \in Y \times \mathbb{R}$.
Thus, if we know how to describe elements of protopologies, then the results can be applied to the description of abstract convex functions and sets.

Section 2 contains main contributions of the paper. We investigate separability of a convex set from a point in its complement, and also separability of two disjoint convex sets. In Section 3 we describe convex hull of a finite union of convex sets. In Sections 4 and 5 we examine description of abstract convex functions and sets. An important issue is the description of the abstract convex hull of a finite union of abstract convex sets (see Proposition 5.4).

2 Separation Theorems

Let us consider the notion of N-connectedness of a topological space with respect to a convexity on this space (see [10]). Let $(X, \mathcal{T})$ be a topological space and $\mathcal{G}$ be a convexity on X. For any points $x, y \in X$ denote by $[x, y]_{\mathcal{G}}$ their convex hull $\mathrm{conv}_{\mathcal{G}}\{x, y\}$.

Definition 2.1. A set $H \subset X$ is called 1-connected (briefly, connected) with respect to $\mathcal{G}$ if, for all $x, y \in H$, the interval $[x, y]_{\mathcal{G}}$ is connected in topology $\mathcal{T}$. The topological space $(X, \mathcal{T})$ is called N-connected with respect to $\mathcal{G}$ if X is the union of N 1-connected sets.

Remark 2.2. N-connectedness in the sense of Definition 2.1 remains valid if the topology decreases. This means that $(X, \mathcal{T}_1)$ is N-connected with respect to $\mathcal{G}$ whenever $\mathcal{T}_1 \subset \mathcal{T}$ and $(X, \mathcal{T})$ is N-connected with respect to $\mathcal{G}$.

Let $\mathcal{H}$ be a collection of subsets of a set X. We need the following notations:

- $\mathcal{H}' = \{X \backslash H : \ H \in \mathcal{H}\}$ is the collection of all complements of sets $H \in \mathcal{H}$;

- $\mathcal{H}_x = \{H \in \mathcal{H} : \ x \in H\}$ for every $x \in X$;
- $\mathcal{H}^*$ is the collection of all sets $\mathcal{H}_x$ with $x \in X$;
- $\mathcal{H}^{*\prime} = \{\mathcal{H} \backslash \mathcal{H}_x : \ x \in X\}$ is the collection of all complements of sets $\mathcal{H}_x \in \mathcal{H}^*$;
- $\mathcal{G}$ is the convexity on X generated by $\mathcal{H}$;
- $\bar{\mathcal{G}}$ is the convexity on X generated by the union $\mathcal{H} \cup \mathcal{H}'$;
- $\bar{\mathcal{G}}^*$ is the convexity on $\mathcal{H}$ generated by the union $\mathcal{H}^* \cup \mathcal{H}^{*\prime}$;
- $\mathcal{T}_X$ is the topology on X generated by $\mathcal{H}$;
- $\mathcal{T}'_X$ is the topology on X generated by $\mathcal{H}'$;
- $\mathcal{T}_\mathcal{H}$ is the topology on $\mathcal{H}$ generated by $\mathcal{H}^*$;
- $\mathcal{T}'_\mathcal{H}$ is the topology on $\mathcal{H}$ generated by $\mathcal{H}^{*\prime}$.

Consider a description of convex hulls $\mathrm{conv}_{\bar{\mathcal{G}}}$ and $\mathrm{conv}_{\bar{\mathcal{G}}^*}$ of finite subsets of X and $\mathcal{H}$ respectively (see [10]).

Proposition 2.3. *Let $F \in [X]^{<\omega}$ and $\mathcal{E} \in [\mathcal{H}]^{<\omega}$. Then*

(i) *The set $\mathrm{conv}_{\bar{\mathcal{G}}} F$ consists of all points $x \in X$ such that for every $H \in \mathcal{H}$ the following implications hold*

$$\begin{aligned} F \subset H \quad &\Longrightarrow \quad x \in H, \\ x \in H \quad &\Longrightarrow \quad F \cap H \neq \emptyset. \end{aligned}$$

(ii)

$$\mathrm{conv}_{\bar{\mathcal{G}}^*} \mathcal{E} = \left\{ H \in \mathcal{H} : \ \bigcap_{E \in \mathcal{E}} E \subset H \subset \bigcup_{E \in \mathcal{E}} E \right\}.$$

Let us begin with the following lemma.

Lemma 2.4. *Assume that $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Let $x, y \in X$ and $H_1, H_2 \in \mathcal{H}$.*

(i) *If $x \in H_1$, $y \in H_2$ and $[x,y]_{\bar{\mathcal{G}}} \subset H_1 \cup H_2$ then $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 \neq \emptyset$.*

(ii) *If $x \notin H_1$, $y \notin H_2$ and $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$ then $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$.*

Proof. Let $x, y \in X$. Since $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ then the interval $[x,y]_{\bar{\mathcal{G}}}$ is connected in topology $\mathcal{T}'_X$.

(i) Let $H_1, H_2 \in \mathcal{H}$ be such that $x \in H_1$, $y \in H_2$ and $[x,y]_{\bar{\mathcal{G}}} \subset H_1 \cup H_2$. Since $[x,y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}'_X$ and both H_1 and H_2 are closed in $\mathcal{T}'_X$ then the intersection $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2$ is not empty.

(ii) Let $H_1, H_2 \in \mathcal{H}$ be such that $x \notin H_1$, $y \notin H_2$ and $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$. If either $x \notin H_2$ or $y \notin H_1$ then $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$, because either $x \notin H_1 \cup H_2$ or $y \notin H_1 \cup H_2$. So, assume that $x \in H_2$ and $y \in H_1$. Equality $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$ implies inclusion $[x,y]_{\bar{\mathcal{G}}} \subset (X \backslash H_1) \cup (X \backslash H_2)$. Note that $(X \backslash H_1)$ and $(X \backslash H_2)$ are open in topology $\mathcal{T}'_X$. Since $[x,y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}'_X$ then the intersection $[x,y]_{\bar{\mathcal{G}}} \cap (X \backslash H_1) \cap (X \backslash H_2)$ is not empty. It means that $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$. □

For any $G \subset X$ let $\mathrm{co}_{(X,\mathcal{H})}G$ denote the set defined by:

$$\mathrm{co}_{(X,\mathcal{H})}G = \bigcap\{H \in \mathcal{H} : \ G \subset H\}. \tag{2.1}$$

If $G \not\subset H$ for all $H \in \mathcal{H}$ then we set $\mathrm{co}_{(X,\mathcal{H})}G = X$.

Due to Proposition 1.3, we have $\mathrm{co}_{(X,\mathcal{H})}G = \mathrm{conv}_{\mathcal{G}}G$ whenever $G \subset X$ is finite. Notice that, in general, this equality is not valid for arbitrary subsets $G \subset X$. We only have the inclusion $\mathrm{conv}_{\mathcal{G}}G \subset \mathrm{co}_{(X,\mathcal{H})}G$.

The following separation theorem gives a description of sets $G \subset X$, which can be represented as the intersection of a subfamily of $\mathcal{H}$. In other words, we describe elements of the protopology on X generated by $\mathcal{H}$.

Theorem 2.5. *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Let $G \subset X$.*

1. *The following conditions are equivalent:*

   (i) *For every $g \in X \backslash G$ a set $H \in \mathcal{H}$ exists such that $G \subset H$ and $g \notin H$.*

   (ii) *G is closed in topology $\mathcal{T}'_X$ and convex in convexity $\mathcal{G}$.*

   (iii) *G is closed in topology $\mathcal{T}'_X$ and $[x,y]_{\mathcal{G}} \subset G$ for all $x, y \in G$.*

   (iv) *G is closed in topology $\mathcal{T}'_X$ and $[x,y]_{\bar{\mathcal{G}}} \subset G$ for all $x, y \in G$.*

2. *If $[x,y]_{\bar{\mathcal{G}}} \subset G$ for all $x, y \in G$ then*

$$\mathrm{co}_{(X,\mathcal{H})}G = \mathrm{cl}_{\mathcal{T}'_X} G, \tag{2.2}$$

   where $\mathrm{cl}_{\mathcal{T}'_X} G$ is the closure of G in topology $\mathcal{T}'_X$.

Proof. We first prove (2.2). Since each set $H \in \mathcal{H}$ is closed in topology $\mathcal{T}'_X$ then $\mathrm{cl}_{\mathcal{T}'_X} G \subset \mathrm{co}_{(X,\mathcal{H})}G$. In order to prove the inclusion $\mathrm{co}_{(X,\mathcal{H})}G \subset \mathrm{cl}_{\mathcal{T}'_X} G$ we will check that $g \notin \mathrm{co}_{(X,\mathcal{H})}G$ whenever $g \notin \mathrm{cl}_{\mathcal{T}'_X} G$. Note that $g \notin \mathrm{co}_{(X,\mathcal{H})}G$ if and only if a set $H \in \mathcal{H}$ exists with $g \notin H$ and $G \subset H$. So let $g \notin \mathrm{cl}_{\mathcal{T}'_X} G$.

Since the topology $\mathcal{T}'_X$ is generated by $\mathcal{H}' = \{X \backslash H : \ H \in \mathcal{H}\}$ then a finite collection $\{H_1, \ldots, H_n\} \subset \mathcal{H}$ exists such that $g \in \bigcap_i (X \backslash H_i) \subset X \backslash G$. In other words, $g \notin \bigcup_i H_i$ and $G \subset \bigcup_i H_i$. If $n = 1$ then the set H_1 possesses required properties: $G \subset H_1$ and $g \notin H_1$. Let $n > 1$.

We will prove that a set $H_0 \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$ exists such that $G \subset \bigcup_{i \geq 3} H_i \cup H_0$. Then, by induction, there is a set $H \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, \ldots, H_n\}$ with $G \subset H$. Moreover, $g \notin H$ because $g \notin \bigcup_i H_i$ and $H \subset \bigcup_i H_i$ (see Proposition 2.3), hence $g \notin \mathrm{co}_{(X,\mathcal{H})}G$.

Consider two sets:

$$\begin{aligned} Z_1 &= \left\{ H \in \mathcal{H} : \ G \subset \bigcup_{i \geq 3} H_i \cup H \cup H_2 \right\}, \\ Z_2 &= \left\{ H \in \mathcal{H} : \ G \subset \bigcup_{i \geq 3} H_i \cup H \cup H_1 \right\}. \end{aligned}$$

Prove that Z_1 and Z_2 cover the interval $[H_1, H_2]_{\bar{\mathcal{G}}^*}$. Assume it is not true. Then a set $H \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ exists such that $H \notin Z_1 \cup Z_2$. In other words, there are two points $x, y \in G$

with $x \notin H_1$, $y \notin H_2$ and $x, y \notin \bigcup_{i\geq 3} H_i \cup H$. Since $x, y \notin H$ then, by Proposition 2.3, $[x,y]_{\bar{\mathcal{G}}} \cap H = \emptyset$. Proposition 2.3 implies also that $H_1 \cap H_2 \subset H$, because $H \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$. Hence $[x,y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 = \emptyset$. Due to Lemma 2.4, $[x,y]_{\bar{\mathcal{G}}} \not\subset H_1 \cup H_2$. Then a point $z \in [x,y]_{\bar{\mathcal{G}}}$ exists with $z \notin H_1 \cup H_2$. Since $x, y \notin \bigcup_{i\geq 3} H_i$ then, by Proposition 2.3, $z \notin \bigcup_{i\geq 3} H_i$. Thus, the point z does not belong to the union $\bigcup_{i\geq 1} H_i$, which contradicts the assumption that $[x,y]_{\bar{\mathcal{G}}} \subset G$, because $z \in [x,y]_{\bar{\mathcal{G}}}$ and $G \subset \bigcup_{i\geq 1} H_i$. Therefore we conclude that $[H_1, H_2]_{\bar{\mathcal{G}}^*} \subset Z_1 \cup Z_2$.

It is easy to see that both Z_1 and Z_2 are closed in topology $\mathcal{T}'_{\mathcal{H}}$. For example, the set Z_1 can be represented as the intersection of sets $\{H \in \mathcal{H} : \ x \in \bigcup_{i\geq 2} H_i \cup H\}$ with $x \in G$. Each of them is closed in $\mathcal{T}'_{\mathcal{H}}$, because it is either $\mathcal{H}_x$ (if $x \notin \bigcup_{i\geq 2} H_i$) or $\mathcal{H}$ (if $x \in \bigcup_{i\geq 2} H_i$).

So, we have: $H_1 \in Z_1$, $H_2 \in Z_2$, $[H_1, H_2]_{\bar{\mathcal{G}}^*} \subset Z_1 \cup Z_2$. Moreover, Z_1 and Z_2 are closed in $\mathcal{T}'_{\mathcal{H}}$. Since $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ then a set $H_0 \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ exists such that $H_0 \in Z_1 \cap Z_2$.

At last, we need to check the inclusion $G \subset \bigcup_{i\geq 3} H_i \cup H_0$. Since $H_0 \in Z_1 \cap Z_2$ and $H_1 \cap H_2 \subset H_0$ then

$$G \subset \left(\bigcup_{i\geq 3} H_i \cup H_0 \cup H_2 \right) \bigcap \left(\bigcup_{i\geq 3} H_i \cup H_0 \cup H_1 \right) \subset \bigcup_{i\geq 3} H_i \cup H_0.$$

Thus, (2.2) is valid. Now prove the equivalence of (i)–(iv). Clearly condition (i) means that $G = \bigcap\{H \in \mathcal{H} : \ G \subset H\} = \mathrm{co}_{(X,\mathcal{H})} G$.

(i) $\Longrightarrow$ (ii) Since all sets $H \in \mathcal{H}$ are closed in topology $\mathcal{T}'_X$ and convex in convexity $\mathcal{G}$ then condition (ii) holds true.

(ii) $\Longrightarrow$ (iii) It is obvious because $[x,y]_{\mathcal{G}} \subset G$ for all $x, y \in G$ whenever $G \in \mathcal{G}$.

(iii) $\Longrightarrow$ (iv) It is sufficient to note that $[x,y]_{\bar{\mathcal{G}}} \subset [x,y]_{\mathcal{G}}$ for all $x, y \in X$.

(iv) $\Longrightarrow$ (i) Since G is closed in topology $\mathcal{T}'_X$ then $\mathrm{cl}_{\mathcal{T}'_X} G = G$. Moreover, by (2.2), $\mathrm{co}_{(X,\mathcal{H})} G = \mathrm{cl}_{\mathcal{T}'_X} G$ because $[x,y]_{\bar{\mathcal{G}}} \subset G$ for all $x, y \in G$. Hence $\mathrm{co}_{(X,\mathcal{H})} G = G$. □

Remark 2.6. If $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ then for any $G \subset X$

$$\mathrm{co}_{(X,\mathcal{H})} G = \mathrm{cl}_{\mathcal{T}'_X} \mathrm{conv}_{\mathcal{G}} G. \tag{2.3}$$

Indeed, equality $\mathrm{co}_{(X,\mathcal{H})} \mathrm{conv}_{\mathcal{G}} G = \mathrm{cl}_{\mathcal{T}'_X} \mathrm{conv}_{\mathcal{G}} G$ follows from (2.2) because $[x,y]_{\bar{\mathcal{G}}} \subset \mathrm{conv}_{\mathcal{G}} G$ for all $x, y \in \mathrm{conv}_{\mathcal{G}} G$. At the same time, since for every $H \in \mathcal{H}$ inclusions $G \subset H$ and $\mathrm{conv}_{\mathcal{G}} G \subset H$ are equivalent, then $\mathrm{co}_{(X,\mathcal{H})} \mathrm{conv}_{\mathcal{G}} G = \mathrm{co}_{(X,\mathcal{H})} G$.

The next theorem states that, under some conditions, two convex sets, one of which is closed in $\mathcal{T}'_X$ and the other one is compact in $\mathcal{T}'_X$, can be separated by a set $H \in \mathcal{H}$.

Theorem 2.7. *Let $\mathcal{T}$ be a topology on $\mathcal{H}$ such that $\{H \in \mathcal{H} : \ K \cap H = \emptyset\}$ is open in $\mathcal{T}$ whenever K is compact in the topology $\mathcal{T}'_{\mathcal{H}}$. Assume that $(\mathcal{H}, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Let $G, K \subset X$ be such that $G \cap K = \emptyset$. Assume that $[x,y]_{\bar{\mathcal{G}}} \subset G \ \forall x, y \in G$ and $[x,y]_{\bar{\mathcal{G}}} \subset K \ \forall x, y \in K$. If G is closed in topology $\mathcal{T}'_X$ and K is compact in $\mathcal{T}'_X$ then a set $H \in \mathcal{H}$ exists with $G \subset H$ and $K \subset X \backslash H$.*

Proof. It is easy to see that $\mathcal{T}'_{\mathcal{H}} \subset \mathcal{T}$. Indeed, the collection $\mathcal{H}^{*\prime}$ consists of sets $\mathcal{H} \backslash \mathcal{H}_x = \{H \in \mathcal{H} : \ \{x\} \cap H = \emptyset\}$, which are open in $\mathcal{T}$. Since $\mathcal{T}'_{\mathcal{H}}$ is generated by $\mathcal{H}^{*\prime}$ then $\mathcal{T}'_{\mathcal{H}} \subset \mathcal{T}$.

So, the space $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ (see Remark 2.2). Since G and K are disjoint then, by Theorem 2.5, for every $g \in K$ a set $H \in \mathcal{H}$ exists such that $G \subset H$ and

$g \notin H$. Hence $K \subset \bigcup\{X \backslash H \in \mathcal{H}' : \ G \subset H\}$. Since K is compact in topology $\mathcal{T}'_X$ and all sets $X \backslash H \in \mathcal{H}'$ are open in $\mathcal{T}'_X$ then there exists a finite collection $\{H_1, \dots, H_n\} \subset \mathcal{H}$ such that $G \subset \bigcap_i H_i$ and

$$K \subset \bigcup_{i \geq 1} (X \backslash H_i). \tag{2.4}$$

Let $n > 1$.

We need to find a set $H_0 \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$, which satisfies inclusion

$$K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash H_0). \tag{2.5}$$

Due to Proposition 2.3, $H_1 \cap H_2 \subset H_0$ whenever $H_0 \in \mathrm{conv}_{\bar{\mathcal{G}}^*}\{H_1, H_2\}$, therefore $G \subset \bigcap_{i \geq 3} H_i \cap H_0$. Then, by induction, there is a set $H \in \mathcal{H}$ with $G \subset H$ and $K \subset X \backslash H$.

Consider the following sets:

$$Z_1 = \left\{ H \in \mathcal{H} : \ K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash H) \cup (X \backslash H_2) \right\},$$

$$Z_2 = \left\{ H \in \mathcal{H} : \ K \subset \bigcup_{i \geq 3} (X \backslash H_i) \cup (X \backslash H) \cup (X \backslash H_1) \right\}.$$

First, prove that $[H_1, H_2]_{\bar{\mathcal{G}}^*} \subset Z_1 \cup Z_2$. Assume it is not true. Let $H \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ be such that $H \notin Z_1$ and $H \notin Z_2$. In view of (2.4), there exist $x, y \in K$ such that

$$x, y \in H, \quad x, y \in H_i \ \ \forall\, i \geq 3, \quad x \in H_1, \quad y \in H_2.$$

Since $x, y \in H$ and $H \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ then $[x, y]_{\bar{\mathcal{G}}} \subset H \subset H_1 \cup H_2$. Lemma 2.4 implies that $[x, y]_{\bar{\mathcal{G}}} \cap H_1 \cap H_2 \neq \emptyset$. Let $z \in [x, y]_{\bar{\mathcal{G}}}$ be a point with $z \in H_1 \cap H_2$. Since $x, y \in \bigcap_{i \geq 3} H_i$ then $z \in [x, y]_{\bar{\mathcal{G}}} \subset \bigcap_{i \geq 3} H_i$. Hence $z \in \bigcap_{i \geq 1} H_i \implies z \notin K$, which contradicts the inclusion $z \in [x, y]_{\bar{\mathcal{G}}}$, because, by conditions of theorem, $[x, y]_{\bar{\mathcal{G}}} \subset K$. Consequently, the sets Z_1 and Z_2 cover the interval $[H_1, H_2]_{\bar{\mathcal{G}}^*}$.

Now prove that Z_1 and Z_2 are open in the topology $\mathcal{T}$. We have

$$Z_1 = \{H \in \mathcal{H} : \ K_1 \cap H = \emptyset\}, \qquad Z_2 = \{H \in \mathcal{H} : \ K_2 \cap H = \emptyset\},$$

where

$$K_1 = \bigcap_{i \geq 3} K \cap H_i \cap H_2 \quad \text{and} \quad K_2 = \bigcap_{i \geq 3} K \cap H_i \cap H_1.$$

Since K is compact in the topology $\mathcal{T}'_X$ and all sets H_i are closed in $\mathcal{T}'_X$ then the sets K_1 and K_2 are compact in $\mathcal{T}'_X$. Therefore, by conditions of theorem, both Z_1 and Z_2 are open in $\mathcal{T}$.

So, $[H_1, H_2]_{\bar{\mathcal{G}}^*} \subset Z_1 \cup Z_2$ and $Z_1, Z_2 \in \mathcal{T}$. Moreover, $H_1 \in Z_1$ and $H_2 \in Z_2$. Since $[H_1, H_2]_{\bar{\mathcal{G}}^*}$ is connected in $\mathcal{T}$ then a set $H_0 \in [H_1, H_2]_{\bar{\mathcal{G}}^*}$ exists such that $H_0 \in Z_1$ and $H_0 \in Z_2$. Then the inclusion (2.5) is valid for H_0, because $H_0 \subset H_1 \cup H_2$. The proof is completed. □

3 Convex Hull of a Finite Union of Convex Sets

Here we give a description of the convex hull $\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i$ and the set $\mathrm{co}_{(X,\mathcal{H})} \bigcup_{i=1}^{n} G_i$, where $\{G_1, \dots, G_n\}$ is a finite collection of convex sets. Note that the set $\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i$ can be described via convex hulls of unions of two convex sets, because $\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i = G^n$, where $G^1 = G_1$ and $G^i = \mathrm{conv}_{\mathcal{G}}(G^{i-1} \cup G_i)$ for $i = 2, \dots, n$.

We need the following result (see [10]).

Theorem 3.1. *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let $\mathcal{T}$ be a topology on X such that for any $F \in [X]^{<\omega}$ and $Z \subset X$*

$$\bigcap_{z \in Z} \mathrm{conv}_{\mathcal{G}}(F \cup \{z\}) = \mathrm{conv}_{\mathcal{G}} F \quad \text{whenever } Z \text{ has a limit point in } F. \tag{3.1}$$

Let F be a finite subset of X and $x, y \in F$. Assume that $[x, y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}$. Then

$$\mathrm{conv}_{\mathcal{G}} F = \bigcup_{z \in [x,y]_{\bar{\mathcal{G}}}} \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \setminus \{x, y\})). \tag{3.2}$$

It is easy to check that the condition (3.1) is valid for the topology $\mathcal{T} = \mathcal{T}_X$. However, in this paper we are mainly interested in topologies $\mathcal{T}$ on X such that $\mathcal{T}'_X \subset \mathcal{T}$.

Proposition 3.2. *Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Let $N \geq 1$. Assume that $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$, where $\mathcal{T}$ is a topology on X, which enjoys (3.1). Then for any $G_1, \dots, G_n \in \mathcal{G}$*

$$\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i = \bigcup_{F_i \in [G_i]^{\leq N}} \mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} F_i. \tag{3.3}$$

Proof. If $F_i \in [G_i]^{\leq N}$ for all i then $\bigcup_i F_i \subset \bigcup_i G_i$, hence $\mathrm{conv}_{\mathcal{G}} \bigcup_i F_i \subset \mathrm{conv}_{\mathcal{G}} \bigcup_i G_i$.

Now we need to check the inclusion

$$\mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} G_i \subset \bigcup_{F_i \in [G_i]^{\leq N}} \mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} F_i.$$

Let $a \in \mathrm{conv}_{\mathcal{G}} \bigcup_i G_i$. Then, by Proposition 1.2, there exists a finite subset $F \subset \bigcup_i G_i$ with $a \in \mathrm{conv}_{\mathcal{G}} F$.

If $F \cap G_i \in [G_i]^{\leq N}$ for all $i \leq n$ then $a \in \mathrm{conv}_{\mathcal{G}} \bigcup_i F_i$, where $F_i = F \cap G_i \in [G_i]^{\leq N}$.

Let $F \cap G_i \notin [G_i]^{\leq N}$ for some i. In other words, F contains m different points of G_i and $m > N$. Since $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$ then two points $x, y \in F \cap G_i$ exist such that the interval $[x, y]_{\bar{\mathcal{G}}}$ is connected in $\mathcal{T}$. By Theorem 3.1, the equality (3.2) is valid. Therefore $a \in \mathrm{conv}_{\mathcal{G}}(\{z\} \cup (F \setminus \{x, y\}))$ for some $z \in [x, y]_{\bar{\mathcal{G}}}$. Since G_i is convex and $x, y \in G_i$ then $z \in [x, y]_{\bar{\mathcal{G}}} \subset [x, y]_{\mathcal{G}} \subset G_i$. Hence the set $\{z\} \cup (F \setminus \{x, y\})$ contains $(m-1)$ points of G_i.

By induction, there is a set $F_i \in [G_i]^{\leq N}$ such that $a \in \mathrm{conv}_{\mathcal{G}}(F_i \cup (F \setminus G_i))$. By repeating this process for each $i = 1, \dots, n$, we will find n sets $F_i \in [G_i]^{\leq N}$ with $a \in \mathrm{conv}_{\mathcal{G}} \bigcup_{i=1}^{n} F_i$. ☐

Remark 3.3. Recall that a convexity space $(X, \mathcal{G})$ is called join-hull commutative (see [3]) provided for each subset $F \subset X$ and $x \in X$ the following holds

$$\mathrm{conv}_{\mathcal{G}}(F \cup \{x\}) = \bigcup_{y \in \mathrm{conv}_{\mathcal{G}} F} [x, y]_{\mathcal{G}}.$$

Assume that one of the spaces $(\mathcal{H}, \mathcal{T}_{\mathcal{H}})$ or $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to the convexity $\bar{\mathcal{G}}^*$. Assume also that $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$, where $\mathcal{T}$ is a topology on X, which enjoys (3.1). Then the convexity space $(X, \mathcal{G})$ is join-hull commutative. Indeed, due to Proposition 3.2, we have for any $F \subset X$ and $x \in X$

$$\begin{aligned}\operatorname{conv}_{\mathcal{G}}(F \cup \{x\}) &= \operatorname{conv}_{\mathcal{G}}(\operatorname{conv}_{\mathcal{G}} F \cup \operatorname{conv}_{\mathcal{G}}\{x\}) \\ &= \bigcup \{[z, y]_{\mathcal{G}} : \ y \in \operatorname{conv}_{\mathcal{G}} F, \ \ z \in \operatorname{conv}_{\mathcal{G}}\{x\}\} = \bigcup_{y \in \operatorname{conv}_{\mathcal{G}} F} [x, y]_{\mathcal{G}}.\end{aligned}$$

Now consider a description of the set $\operatorname{co}_{(X,\mathcal{H})} \bigcup_{i=1}^n G_i$, where $G_i \in \mathcal{G}$.

Proposition 3.4. *Let $\mathcal{T}$ be a topology on X such that $\mathcal{T}'_X \subset \mathcal{T}$ and (3.1) is valid for $\mathcal{T}$. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$. Then for any $G_1, \ldots, G_n \in \mathcal{G}$*

$$\operatorname{co}_{(X,\mathcal{H})} \bigcup_{i=1}^n G_i = \operatorname{cl}_{\mathcal{T}'_X} \left(\operatorname{conv}_{\mathcal{G}} \bigcup_{i=1}^n G_i \right) = \operatorname{cl}_{\mathcal{T}'_X} \left(\bigcup_{g_i \in G_i} \operatorname{conv}_{\mathcal{G}}\{g_1, \ldots, g_n\} \right). \tag{3.4}$$

Proof. Since $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$ and $\mathcal{T}'_X \subset \mathcal{T}$ then $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$ as well (see Remark 2.2). It follows from (2.3) that

$$\operatorname{co}_{(X,\mathcal{H})} \bigcup_{i=1}^n G_i = \operatorname{cl}_{\mathcal{T}'_X} \left(\operatorname{conv}_{\mathcal{G}} \bigcup_{i=1}^n G_i \right),$$

and, by Proposition 3.2 (with $N = 1$),

$$\operatorname{conv}_{\mathcal{G}} \bigcup_{i=1}^n G_i = \bigcup_{g_i \in G_i} \operatorname{conv}_{\mathcal{G}}\{g_1, \ldots, g_n\}.$$

□

4 Description of Abstract Convex Functions

Let L be a set of functions $l : Y \to \mathbb{R}$ defined on a set Y. Let $X = Y \times \mathbb{R}$ and $\mathcal{H}$ be the collection of all epigraphs $\operatorname{epi} l = \{(y, c) \in Y \times \mathbb{R} : \ l(y) \le c\}$ with $l \in L$.

First consider segments $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ and $[\operatorname{epi} l_1, \operatorname{epi} l_2]_{\bar{\mathcal{G}}^*}$ for $(y_1, c_1), (y_2, c_2) \in Y \times \mathbb{R}$ and $l_1, l_2 \in L$.

It is easy to see that the set $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ consists of all points $(y, c) \in Y \times \mathbb{R}$ such that for any $l \in L$ the following implications hold:

$$\begin{array}{rcl} \max\{l(y_1) - c_1, l(y_2) - c_2\} \le 0 & \Longrightarrow & l(y) \le c, \\ l(y) \le c & \Longrightarrow & \min\{l(y_1) - c_1, l(y_2) - c_2\} \le 0. \end{array} \tag{4.1}$$

In particular, $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ contains all (y, c) such that

$$\min\{l(y_1) - c_1, l(y_2) - c_2\} \le l(y) - c \le \max\{l(y_1) - c_1, l(y_2) - c_2\} \quad \forall\, l \in L.$$

For every pair $l_1, l_2 \in L$ we have:

$$[\operatorname{epi} l_1, \operatorname{epi} l_2]_{\bar{\mathcal{G}}^*} = \{\operatorname{epi} l : \ l \in L, \ \min\{l_1(y), l_2(y)\} \le l(y) \le \max\{l_1(y), l_2(y)\} \ \forall\, y \in Y\}. \tag{4.2}$$

We begin with the description of L-convex functions on finite subsets of Y. Let Z be a subset of Y. Recall (see [8]) that a function $f : Y \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ is called L-convex on Z if a subfamily $T \subset L$ exists such that $f(z) = \sup_{l \in T} l(z)$ for all $z \in Z$. The following proposition holds (see [10]).

Proposition 4.1. *Let $N \geq 2$ and $\mathcal{T}$ be a topology on X, which enjoys (3.1). Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is N-connected with respect to $\bar{\mathcal{G}}$. Then for any function $f : Y \to \mathbb{R}_{+\infty}$ the following conditions are equivalent:*

(i) *For all $y, y_1, \dots, y_N \in Y$*

$$f(y) \leq \sup\{l(y) : \ l \in L, \ l(y_i) \leq f(y_i) \ \forall\, i = 1, \dots, N\}. \tag{4.3}$$

(ii) *f is L-convex on every finite subset of Y.*

Now consider the case, when $(X, \mathcal{T}'_X)$ is connected (one-connected) with respect to $\bar{\mathcal{G}}$. This allows us to give a description of L-convex functions on the whole set Y.

Proposition 4.2. *Let $\mathcal{L}$ be the collection of all functions $\ell(x) = \min_{l \in T} l(x)$ with $T \in [L]^{<\omega}$, where $[L]^{<\omega}$ is the collection of all finite subsets of L. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Then a function $f : Y \to \mathbb{R}_{+\infty}$ is L-convex if and only if it is $\mathcal{L}$-convex and*

$$f(y) \leq \sup\{l(y) : \ l \in L, \ l(y_1) \leq f(y_1), \ l(y_2) \leq f(y_2)\} \qquad \forall\, y, y_1, y_2 \in Y. \tag{4.4}$$

Proof. If f is L-convex then inequalities (4.4) obviously hold. Moreover, since $L \subset \mathcal{L}$ then f is $\mathcal{L}$-convex as well.

Conversely, assume that f is $\mathcal{L}$-convex and enjoys (4.4). It is clear that for every $\ell \in \mathcal{L}$ its epigraph $\operatorname{epi} \ell$ is closed in topology $\mathcal{T}'_X$, because it is the union of a finite number of epigraphs of functions $l \in L$. Since f is $\mathcal{L}$-convex then the epigraph $\operatorname{epi} f$ is also closed in topology $\mathcal{T}'_X$. Moreover, inequalities (4.4) imply that $[(y_1, c_1), (y_2, c_2)]_{\mathcal{G}} \subset \operatorname{epi} f$ for any $(y_1, c_1), (y_2, c_2) \in \operatorname{epi} f$. Indeed, if $(y_1, c_1), (y_2, c_2) \in \operatorname{epi} f$ and $(y, c) \in [(y_1, c_1), (y_2, c_2)]_{\mathcal{G}}$ then

$$\begin{aligned} f(y) &\leq \sup\{l(y) : \ l \in L, \ l(y_1) \leq f(y_1), \ l(y_2) \leq f(y_2)\} \\ &\leq \sup\{l(y) : \ l \in L, \ l(y_1) \leq c_1, \ l(y_2) \leq c_2\} \\ &\leq \sup\{l(y) : \ l \in L, \ l(y) \leq c\} \leq c. \end{aligned}$$

Due to Theorem 2.5, for each $(y, c) \notin \operatorname{epi} f$ a set $\operatorname{epi} l \in \mathcal{H}$ exists such that $\operatorname{epi} f \subset \operatorname{epi} l$ and $(y, c) \notin \operatorname{epi} l$. This means that f is L-convex. □

Next proposition shows that, in some cases, $\mathcal{L}$-convexity of f can be interchanged with the lower semicontinuity. Recall that L is said to be closed under vertical shifts if $(l - c) \in L$ for all $l \in L$ and $c \in \mathbb{R}$.

Proposition 4.3. *Assume that L is closed under vertical shifts. Let Y be equipped with a topology such that Y is compact and all functions $l \in L$ are continuous. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Then a function $f : Y \to \mathbb{R}_{+\infty}$ is L-convex if and only if it is lower semicontinuous and possesses (4.4).*

Proof. Since L consists of continuous functions then every L-convex function is lower semicontinuous. Inequalities (4.4) for L-convex functions f are trivial.

Now assume that f is lower semicontinuous and possesses (4.4). Let us prove that f is $\mathcal{L}$-convex, where $\mathcal{L}$ is the collection of all minimums of finite subfamilies of L. Then, by Proposition 4.2, f is L-convex.

Take an arbitrary $y \in Y$. It follows from (4.4) that

$$f(y) \leq \sup\{l(y): \ l \in L, \ l(y) \leq f(y), \ l(z) \leq f(z)\} \quad \forall\, z \in Y,$$

hence

$$f(y) = \sup\{l(y): \ l \in L, \ l(y) \leq f(y), \ l(z) \leq f(z)\} \quad \forall\, z \in Y. \tag{4.5}$$

Let $\varepsilon > 0$. If $f(y) < +\infty$ then, by (4.5), for each $z \in Y$ a function $l_z \in L$ exists such that $l_z(z) \leq f(z)$ and $f(y) - \varepsilon/2 \leq l_z(y) \leq f(y)$. If $f(y) = +\infty$ then for each $z \in Y$ a function $l_z \in L$ exists such that $l_z(z) \leq f(z)$ and $1/\varepsilon \leq l_z(y) \leq f(y)$. Since L is closed under vertical shifts then every function $h_z(x) = l_z(x) - \varepsilon/2$ belongs to L. We have:

$$h_z(y) \leq f(y) - \varepsilon/2, \qquad h_z(z) \leq f(z) - \varepsilon/2$$

and

$$f(y) - \varepsilon \leq h_z(y) \quad \text{if } f(y) < +\infty, \qquad 1/\varepsilon - \varepsilon/2 \leq h_z(y) \quad \text{if } f(y) = +\infty.$$

Since f is lower semicontinuous, h_z is continuous and $h_z(z) < f(z)$ then for each $z \in Y$ a neighbourhood U_z of z exists such that $h_z(x) < f(x)$ for all $x \in U_z$. Due to compactness of Y, there is a finite collection $\{z_1, \ldots, z_m\} \subset Y$ with $U_{z_1} \cup \ldots \cup U_{z_m} = Y$. Consider the function $\ell(x) = \min_i h_{z_i}(x)$. Then $\ell \in \mathcal{L}$ and $\ell(x) < f(x)$ for all $x \in Y$. Moreover, $f(y) - \varepsilon \leq \ell(y)$ if $f(y) < +\infty$ and $1/\varepsilon - \varepsilon/2 \leq \ell(y)$ if $f(y) = +\infty$.

Thus, we have proved that, for any $y \in Y$ and $\varepsilon > 0$, a function $\ell \in \operatorname{supp}(f, \mathcal{L})$ exists such that $f(y) - \varepsilon \leq \ell(y)$ for $f(y) < +\infty$ and $1/\varepsilon - \varepsilon/2 \leq \ell(y)$ for $f(y) = +\infty$. This means that f is $\mathcal{L}$-convex. □

5 Description of Abstract Convex Sets

Let L be a set of functions defined on a set Y. Let $X = L$ and $\mathcal{H}$ be the collection of all subsets $\{l \in L: \ l(y) \leq c\} \subset X$, where $(y, c) \in Y \times \mathbb{R}$.

Then for any $l_1, l_2 \in L$

$$\begin{aligned} [l_1, l_2]_{\mathcal{G}} &= \bigcap \{H \in \mathcal{H}: \ l_1, l_2 \in H\} \\ &= \{l \in L: \ l(y) \leq c \text{ whenever } \max\{l_1(y), l_2(y)\} \leq c\} \\ &= \{l \in L: \ l(y) \leq \max\{l_1(y), l_2(y)\} \ \forall\, y \in Y\}. \end{aligned}$$

Similarly,

$$[l_1, l_2]_{\bar{\mathcal{G}}} = \{l \in L: \ \min\{l_1(y), l_2(y)\} \leq l(y) \leq \max\{l_1(y), l_2(y)\} \ \forall\, y \in Y\}.$$

Let $(y_1, c_1), (y_2, c_2) \in Y \times \mathbb{R}$ and $H_i = \{l \in L: \ l(y_i) \leq c_i\}$ $(i = 1, 2)$. Then, by Proposition 2.3, $[H_1, H_2]_{\bar{\mathcal{G}}^*} = \{H \in \mathcal{H}: \ H_1 \cap H_2 \subset H \subset H_1 \cup H_2\}$. In other words, a set $H = \{l \in L: \ l(y) \leq c\}$ belongs to $[H_1, H_2]_{\bar{\mathcal{G}}^*}$ if and only if for each $l \in L$ the following implications hold:

$$\begin{aligned} \max\{l(y_1) - c_1, l(y_2) - c_2\} \leq 0 \quad &\Longrightarrow \quad l(y) \leq c, \\ l(y) \leq c \quad &\Longrightarrow \quad \min\{l(y_1) - c_1, l(y_2) - c_2\} \leq 0. \end{aligned}$$

Thus, our formulas for $[l_1, l_2]_{\bar{\mathcal{G}}}$ and $[H_1, H_2]_{\bar{\mathcal{G}}^*}$ coincide with the corresponding formulas for $[\operatorname{epi} l_1, \operatorname{epi} l_2]_{\bar{\mathcal{G}}^*}$ and $[(y_1, c_1), (y_2, c_2)]_{\bar{\mathcal{G}}}$ in the case, when $X = Y \times \mathbb{R}$ and $\mathcal{H} = \{\operatorname{epi} l : l \in L\}$ (see (4.1) and (4.2)).

Recall that a set $U \subset L$ is called (L, Y)-convex if $U = \mathrm{co}_L U$, where $\mathrm{co}_L U = \{l \in L : l(y) \le \sup_{u \in U} u(y) \ \forall y \in Y\}$. Then we have $\mathrm{co}_L U = \mathrm{co}_{(X,\mathcal{H})} U$, where $\mathrm{co}_{(X,\mathcal{H})} U$ is defined by (2.1). Indeed,

$$\begin{aligned} \mathrm{co}_{(X,\mathcal{H})} U &= \bigcap \{H \in \mathcal{H} : \ U \subset H\} \\ &= \{l \in L : \ l(y) \le c \text{ whenever } u(y) \le c \ \forall u \in U\} \\ &= \left\{ l \in L : \ l(y) \le \sup_{u \in U} u(y) \ \forall y \in Y \right\} = \mathrm{co}_L U. \end{aligned}$$

Proposition 5.1. *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. Then a set $U \subset L$ is (L, Y)-convex if and only if it is closed in the topology $\mathcal{T}'_X$ and*

$$\{l \in L : \ \min\{l_1(y), l_2(y)\} \le l(y) \le \max\{l_1(y), l_2(y)\} \ \forall y \in Y\} \subset U \quad \forall l_1, l_2 \in U. \tag{5.1}$$

Proof. Let $U \subset L = X$. Theorem 2.5 states that $U = \mathrm{co}_{(X,\mathcal{H})} U$ if and only if U is closed in topology $\mathcal{T}'_X$ and $[l_1, l_2]_{\bar{\mathcal{G}}} \subset U$ for all $l_1, l_2 \in U$. Since $\mathrm{co}_L U = \mathrm{co}_{(X,\mathcal{H})} U$ then U is (L, Y)-convex if and only if it is closed in topology $\mathcal{T}'_X$ and possesses (5.1). □

A set $U \subset L$ is closed if and only if it contains each $l \in L$ such that every neighbourhood of l contains an element of U. Since the topology $\mathcal{T}'_X$ is generated by the collection of all sets $\{l \in L : \ l(y) > c\}$ with $(y, c) \in Y \times \mathbb{R}$, then U is closed in topology $\mathcal{T}'_X$ if and only if it contains all $l \in L$ such that for every finite subset $F \subset Y$ and for every $\varepsilon > 0$ a function $u \in U$ exists with $u(y) > l(y) - \varepsilon \ \forall y \in F$.

Let $\mathcal{T}$ be the topology of pointwise convergence on L. It is clear that condition (3.1) is valid for $\mathcal{T}$. Indeed, let U be a finite subset of L and a set $Z \subset L$ be such that $u' \in U$ is a limit point of Z. Then $\inf_{z \in Z} z(y) \le u'(y) \le \max_{u \in U} u(y)$ for all $y \in Y$, and we have

$$\begin{aligned} \bigcap_{z \in Z} \mathrm{conv}_{\mathcal{G}}(U \cup \{z\}) &= \left\{ l \in L : \ l(y) \le \max\left\{ z(y), \max_{u \in U} u(y) \right\} \ \forall y \in Y, \ z \in Z \right\} \\ &= \left\{ l \in L : \ l(y) \le \max\left\{ \inf_{z \in Z} z(y), \max_{u \in U} u(y) \right\} \ \forall y \in Y \right\} \\ &= \left\{ l \in L : \ l(y) \le \max_{u \in U} u(y) \ \forall y \in Y \right\} = \mathrm{conv}_{\mathcal{G}} U. \end{aligned}$$

Moreover, since every set $H = \{l \in L : \ l(y) \le c\} \in \mathcal{H}$ is closed in topology $\mathcal{T}$ then $\mathcal{T}'_X \subset \mathcal{T}$.

Proposition 5.2. *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T}'_X)$ is connected with respect to $\bar{\mathcal{G}}$. If L is compact in the topology of pointwise convergence then a set $U \subset L$ is (L, Y)-convex if and only if it is closed in $\mathcal{T}$ and*

$$\{l \in L : \ l(y) \le \max\{l_1(y), l_2(y)\} \ \forall y \in Y\} \subset U \quad \forall l_1, l_2 \in U. \tag{5.2}$$

Proof. By Theorem 2.5, U is (L, Y)-convex if and only if it is closed in topology $\mathcal{T}'_X$ and $[l_1, l_2]_{\mathcal{G}} \subset U$ for all $l_1, l_2 \in U$.

Inclusions $[l_1, l_2]_{\mathcal{G}} \subset U$ for $l_1, l_2 \in U$ are equivalent to (5.2). If U is closed in topology $\mathcal{T}'_X$ then it is closed in $\mathcal{T}$ as well, because $\mathcal{T}'_X \subset \mathcal{T}$.

Conversely, let $U \subset L$ be closed in the topology of pointwise convergence and enjoy (5.2). Assume that L is compact in $\mathcal{T}$. Then U is also compact in $\mathcal{T}$. We need to check that U is closed in the topology $\mathcal{T}'_X$. Let $l \in L \backslash U$. It follows from (5.2) that for every $u \in U$ a point $y_u \in Y$ exists with $l(y_u) > u(y_u)$. Let $c_u = (u(y_u) + l(y_u))/2$. Then for each $u \in U$ the set $\{l' \in L : \ l'(y_u) < c_u\}$ is a neighbourhood of u (i.e. it is open in topology $\mathcal{T}$ and contains u), and $l(y_u) > c_u$. Since U is compact then there is a finite collection $\{(y_1, c_1), \ldots, (y_n, c_n)\} \subset Y \times \mathbb{R}$ such that $\min_i(u(y_i) - c_i) < 0 < \min_i(l(y_i) - c_i)$ for all $u \in U$. Hence $l \notin \bigcup_i H_i$ and $U \subset \bigcup_i H_i$, where $H_i = \{l' \in L : \ l'(y_i) \le c_i\} \in \mathcal{H}$. This means that l does not belong to the closure $\mathrm{cl}_{\mathcal{T}'_X} U$, because $\bigcup_i H_i$ is closed in topology $\mathcal{T}'_X$. Thus, U is closed in $\mathcal{T}'_X$. □

Proposition 5.3. *Assume that L is closed under vertical shifts. Let Y be equipped with a topology such that Y is compact and all functions $l \in L$ are continuous on Y. Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$. Then a set $U \subset L$ is (L, Y)-convex if and only if (5.2) holds and U contains every $l \in L$ such that $(l - \varepsilon) \in U$ for any $\varepsilon > 0$.*

Proof. If U is (L, Y)-convex then (5.2) is valid. Moreover, since $l(y) = \sup_{\varepsilon > 0}(l(y) - \varepsilon)$ then U contains every $l \in L$ such that $(l - \varepsilon) \in U$ for all $\varepsilon > 0$.

Conversely, assume that $U \subset L$ possesses (5.2) and $l \in U$ whenever $(l - \varepsilon) \in U$ for all $\varepsilon > 0$. Let $l \in L$ be such that $l(y) \le \sup_{u \in U} u(y)$ for all $y \in Y$. We show that $(l - \varepsilon) \in U$ for any positive ε. Then l belongs to U as well, and therefore U is (L, Y)-convex.

So let $\varepsilon > 0$. Since $l(y) - \varepsilon < \sup_{u \in U} u(y) \ \ \forall y \in Y$ then for each $y \in Y$ a function $u_y \in U$ exists with $l(y) - \varepsilon < u_y(y)$. Due to continuity of l and u_y, the inequality $l(z) - \varepsilon < u_y(z)$ holds for all z from a neighbourhood of y. Then, by compactness of Y, a finite collection $\{u_1, \ldots, u_n\} \subset U$ exists such that $l(y) - \varepsilon < \max_i u_i(y)$ for all $y \in Y$.

Since the topology $\mathcal{T}$ enjoys condition (3.1) then (see [10]) the convexity $\mathcal{G}$ is of arity 2. It follows from (5.2) that $[l_1, l_2]_{\mathcal{G}} \subset U$ for any $l_1, l_2 \in U$. Hence U is convex. This implies that $\mathrm{conv}_{\mathcal{G}}\{u_1, \ldots, u_n\} \subset U$. In other words,

$$\Big\{u \in L : \ u(y) \le \max_i u_i(y) \ \ \forall y \in Y\Big\} \subset U.$$

In particular, U contains the function $h(y) = l(y) - \varepsilon$. □

At last, we derive a formula for the (L, Y)-convex hull of a finite union of (L, Y)-convex sets. This is important for the description of the support set and the subdifferential of the maximum of a finite collection of abstract convex functions. Indeed, for every L-convex functions $f_1, \ldots, f_n$ we have

$$\mathrm{supp}\left(\max_{i=1,\ldots,n} f_i, L\right) = \mathrm{co}_L \bigcup_{i=1}^{n} \mathrm{supp}\,(f_i, L).$$

The subdifferential of the maximum of a finite collection of abstract convex functions have been considered in ([9], Corollary 4.1).

Proposition 5.4. *Assume that $(\mathcal{H}, \mathcal{T}'_{\mathcal{H}})$ is connected with respect to $\bar{\mathcal{G}}^*$ and $(X, \mathcal{T})$ is connected with respect to $\bar{\mathcal{G}}$. Then for any (L, Y)-convex sets $U_1, \ldots, U_n$*

$$\mathrm{co}_L \bigcup_{i=1}^{n} U_i = \mathrm{cl}_{\mathcal{T}'_X}\left(\bigcup_{u_i \in U_i}\left\{l \in L : \ l(y) \le \max_{i=1,\ldots,n} u_i(y) \ \ \forall y \in Y\right\}\right). \tag{5.3}$$

Proof. Since $\mathcal{T}'_X \subset \mathcal{T}$ and condition (3.1) is valid for $\mathcal{T}$ then we can apply Proposition 3.4. Let $U_1, \ldots, U_n \subset L$ be (L, Y)-convex. Then $U_1, \ldots, U_n \in \mathcal{G}$ and, by (3.4),

$$\begin{aligned}\mathrm{co}_L \bigcup_{i=1}^{n} U_i &= \mathrm{co}_{(X,\mathcal{H})} \bigcup_{i=1}^{n} U_i = \mathrm{cl}_{\mathcal{T}'_X} \left(\bigcup_{u_i \in U_i} \mathrm{conv}_{\mathcal{G}}\{u_1, \ldots, u_n\} \right) \\ &= \mathrm{cl}_{\mathcal{T}'_X} \left(\bigcup_{u_i \in U_i} \{l \in L : \ l(y) \leq \max_{i=1,\ldots,n} u_i(y) \ \forall\, y \in Y\} \right).\end{aligned}$$

□

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