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CONVEX FUNCTIONS ON σ-ALGEBRAS OF NONATOMIC MEASURE SPACES*

Nobusumi Sagara† and Milan Vlach

Abstract: The purpose of this paper is to present a convex-like structure of set functions on σ-algebras of nonatomic finite measure spaces. Using the nonatomicity of measures, we introduce a convex subset of a σ-algebra, a *μ-convex set*, and a convex set function, a *μ-convex function*, in a reasonably standard way analogous to convex analysis. We prove Jensen inequality for μ-convex functions and show that the set of minimizers of μ-convex functions is μ-convex. We then metrize σ-algebras and study the continuity of set functions on σ-algebras as continuous functions on metric spaces. Specifically, we prove a minimax theorem for set functions and investigate how the μ-convexity and the absolute continuity of set functions, and the continuity and the countable additivity of finitely additive set functions, are mutually related.

Key words: *nonatomic finite measure space, μ-convex set; μ-convex functions on σ-algebras, supermodularity, lower semicontinuity, minimax theorem, countable additivity*

Mathematics Subject Classification: *28A10, 52A01; 91A12, 91B32*

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1 Introduction

A recent development of the study of nonadditive measures and discrete convex analysis exemplifies the affinity of the submodularity of set functions with standard convex analysis. A prominent result in this respect is that a set function on an algebra is submodular if and only if its *Choquet integral* is a convex function on the space of bounded measurable (with respect to the algebra) functions (see Marinacci and Montrucchio [9]). Since submodular functions have convex extensions (*Lovász extensions* in the terminology of discrete convex analysis) to vector spaces, it is possible to apply concepts and results of duality theory, such as the Fenchel–Legendre transform, subgradients and separation theorems (see Marinacci and Montrucchio [8] and Murota [10]).

The main purpose of this paper is to present another convex-like structure of set functions on σ-algebras of nonatomic finite measure spaces and to illustrate its usefulness in optimization. A key notion in our approach to the convexity of families of measurable sets and set functions is that of convex combinations of measurable sets along the lines of Halmos [6]. This notion, which uses the nonatomicity of measures in an essential way, makes it possible to introduce a convex subset of a σ-algebra, a *μ-convex set*, and a convex set function, a μ-

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†Corresponding author.

convex function, in a reasonably standard way and to derive a number of results analogous to those of standard convex analysis. In particular, in Section 2, we prove Jensen inequality for μ-convex functions, show that the set of minimizers of μ-convex functions is μ-convex and demonstrate that the μ-convexity of set functions introduced in this paper inherently differs from the notion of the supermodularity of set functions on σ-algebras.

In Section 3, we metrize σ-algebras and study the continuity of set functions on σ-algebras as continuous functions on metric spaces. Specifically, we prove a minimax theorem for set functions and investigate how the μ-convexity and the absolute continuity of set functions, and the continuity and the countable additivity of finitely additive set functions, are mutually related.

Unlike Choquet integrals of submodular functions on a σ-algebra, our notions of the convexity and the continuity of set functions are insufficient to obtain convex extensions to the space of bounded measurable functions. Nevertheless, they are useful for investigating the properties of cores of nonatomic cooperative transferable utility games with an infinite set of players, as demonstrated in Sagara and Vlach [13] and characterizing solutions to fair division problems in a measurable space (see Sagara [11]). Another application is presented in Sagara and Vlach [12], who consider the representability of preference relations on σ-algebras in terms of the continuous μ-convex functions.

2 Convex Functions on σ-Algebras

2.1 μ-Convex Sets

Let $\mathscr{F}$ be a σ-algebra of subsets of Ω. An extended real-valued set function $\mu : \mathscr{F} \to \mathbb{R} \cup \{+\infty\}$ is a *signed charge* if μ is finitely additive and $\mu(\emptyset) = 0$. A nonnegative signed charge is called a *charge*. A countably additive signed charge is called a *signed measure*. A nonnegative signed measure is called a *measure*. A signed charge μ is *finite* if $\sup_{A\in\mathscr{F}} |\mu(A)| < +\infty$. A measure μ is said to be *nonatomic* if every set $A \in \mathscr{F}$ with $\mu(A) > 0$ includes a set $E \in \mathscr{F}$ such that $0 < \mu(E) < \mu(A)$.

Let $(\Omega, \mathscr{F}, \mu)$ be a nonatomic finite measure space, where $\mathscr{F}$ is a σ-algebra of subsets of a nonempty set Ω and μ is a nonatomic finite measure on $\mathscr{F}$. It follows from the convexity of the range of a nonatomic finite measure that, for every $A \in \mathscr{F}$ and $\alpha \in [0, \mu(A)]$, there exists some $E \in \mathscr{F}$ with $E \subset A$ satisfying $\mu(E) = \alpha$ and, for every $B \in \mathscr{F}$ and $\beta \in [\mu(B), \mu(\Omega)]$, there exists some $F \in \mathscr{F}$ with $B \subset F$ such that $\mu(F) = \beta$ (see Halmos [7, Section 41(2) and (3)]).

Let $A \in \mathscr{F}$ and $t \in [0, 1]$ be given arbitrarily. We define the family $\mathscr{K}_t^{\mu}(A)$ of measurable subsets of A by:

$$\mathscr{K}_t^{\mu}(A) = \{E \in \mathscr{F} \mid \mu(E) = t\mu(A),\ E \subset A\}.$$

The nonatomicity of μ implies that $\mathscr{K}_t^{\mu}(A)$ is nonempty for every $A \in \mathscr{F}$ and $t \in [0, 1]$. Note that $E \in \mathscr{K}_t^{\mu}(A)$ if and only if $A \setminus E \in \mathscr{K}_{1-t}^{\mu}(A)$, and that $\mu(A) = 0$ if and only if $\mathscr{K}_t^{\mu}(A)$ contains the empty set for every $t \in [0, 1]$.

Let Δ^{n-1} denote the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$; that is:

$$\Delta^{n-1} = \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \alpha_i \geq 0,\ i = 1, \dots, n \right\}.$$

For arbitrarily given $A_1, \dots, A_n \in \mathscr{F}$ and $(t_1, \dots, t_n) \in \Delta^{n-1}$, we denote by

$\mathscr{K}^{\mu}_{t_1,\dots,t_n}(A_1,\dots,A_n)$ the family of sets $E \in \mathscr{F}$ such that E is the union of some pairwise disjoint sets $E_1 \in \mathscr{K}^{\mu}_{t_1}(A_1),\dots,E_n \in \mathscr{K}^{\mu}_{t_n}(A_n)$. When $n=2$, we shall simply write $\mathscr{K}^{\mu}_{t}(A,B)$ instead of $\mathscr{K}^{\mu}_{t,1-t}(A,B)$.

Theorem 2.1. *For each $n \geq 2$, $\mathscr{K}^{\mu}_{t_1,\dots,t_n}(A_1,\dots,A_n)$ is nonempty for every finite sequence of sets $A_1,\dots,A_n$ in $\mathscr{F}$ and every element $(t_1,\dots,t_n)$ in Δ^{n-1}.*

Proof. Let $A_1,\dots,A_n \in \mathscr{F}$ and $(t_1,\dots,t_n) \in \Delta^n$ be given arbitrarily. Choose any $E_i \in \mathscr{K}^{\mu}_{t_i}(A_i \setminus \bigcup_{j\neq i}^n A_j)$ and $F_i \in \mathscr{K}^{\mu}_{t_i}(A_i \cap \bigcup_{j\neq i}^n A_j)$ for each $i=1,\dots,n$. By construction, the sets $E_1,\dots,E_n,F_1,\dots,F_n$ are pairwise disjoint and

$$\mu(E_i \cup F_i) = t_i\mu\left(A_i \setminus \bigcup_{j\neq i}^{n} A_j\right) + t_i\mu\left(A_i \cap \bigcup_{j\neq i}^{n} A_j\right) = t_i\mu(A_i),$$

and hence $E_i \cup F_i \in \mathscr{K}^{\mu}_{t_i}(A_i)$ for each i. Therefore, we obtain $\bigcup_{i=1}^n (E_i \cup F_i) \in \mathscr{K}^{\mu}_{t_1,\dots,t_n}(A_1,\dots,A_n)$. □

Definition 2.2. A subset $\mathscr{X}$ of $\mathscr{F}$ is *μ-convex* if $\mathscr{K}^{\mu}_{t}(A,B) \subset \mathscr{X}$ for every $A,B \in \mathscr{X}$ and $t \in [0,1]$.

It is easy to verify that the intersection of an arbitrary family of μ-convex sets is μ-convex.

Lemma 2.3. *Let $A_1,\dots,A_n$ be a finite sequence of sets in $\mathscr{F}$ and $t_1,\dots,t_n$ be nonnegative real numbers satisfying $\sum_{i=1}^n t_i \leq 1$ with $n \geq 2$. If $E_1 \in \mathscr{K}^{\mu}_{t_1}(A_1),\dots,E_n \in \mathscr{K}^{\mu}_{t_n}(A_n)$ are pairwise disjoint, then for every real number $s_1,\dots,s_n$ satisfying $\sum_{i=1}^n s_i \leq 1$ and $t_i \leq s_i$ for each $i=1,\dots,n$, there exist pairwise disjoint sets $F_1 \in \mathscr{K}^{\mu}_{s_1}(A_1),\dots,F_n \in \mathscr{K}^{\mu}_{s_n}(A_n)$ such that $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^n F_i$.*

Proof. The argument is based on induction. Let A_1 and A_2 be sets in $\mathscr{F}$, let t_1 and t_2 be nonnegative real numbers with $t_1+t_2 \leq 1$, let $E_1 \in \mathscr{K}^{\mu}_{t_1}(A_1)$ and $E_2 \in \mathscr{K}^{\mu}_{t_2}(A_2)$ be disjoint sets and let s_1 and s_2 be real numbers with $s_1+s_2 \leq 1$, $t_1 \leq s_1$ and $t_2 \leq s_2$. Without loss of generality, we assume that $\mu(A_1) \leq \mu(A_2)$. By the nonatomicity of μ, there exists some $F_1 \in \mathscr{K}^{\mu}_{s_1}(A_1)$ such that $E_1 \subset F_1$. We then have:

$$\begin{aligned}\mu(A_2 \setminus F_1) \geq \mu(A_2) - \mu(F_1) &= \mu(A_2) - s_1\mu(A_1)\\ &\geq \mu(A_2) - s_1\mu(A_2) \geq s_2\mu(A_2).\end{aligned}$$

By the nonatomicity of μ, there exists some $F_2 \in \mathscr{K}^{\mu}_{s_2}(A_2)$ such that $E_2 \setminus F_1 \subset F_2 \subset A_2 \setminus F_1$. By construction, we have $F_1 \cap F_2 = \emptyset$ and $E_1 \cup E_2 \subset F_1 \cup F_2$. Thus, the result is true for $n=2$.

Suppose that the result is true for $n \geq 2$. Let $A_1,\dots,A_{n+1}$ be sets in $\mathscr{F}$, let $t_1,\dots,t_{n+1}$ be nonnegative real numbers satisfying $\sum_{i=1}^{n+1} t_i \leq 1$, let $E_1 \in \mathscr{K}^{\mu}_{t_1}(A_1),\dots,E_{n+1} \in \mathscr{K}^{\mu}_{t_{n+1}}(A_{n+1})$ be disjoint sets and let $s_1,\dots,s_{n+1}$ be real numbers with $\sum_{i=1}^{n+1} s_i \leq 1$ and $t_i \leq s_i$ for each $i=1,\dots,n+1$. Without loss of generality, we assume that $\mu(A_i) \leq \mu(A_{n+1})$ for each $i=1,\dots,n$. By the induction hypothesis, there exist pairwise disjoint sets $F_1,\dots,F_n$ in $\mathscr{F}$ such that $F_i \in \mathscr{K}^{\mu}_{s_i}(A_i)$ for each $i=1,\dots,n$ and $\bigcup_{i=1}^n E_i \subset \bigcup_{i=1}^n F_i$. It follows that

$\mu(E_{n+1}) \le s_{n+1}\mu(A_{n+1})$ and

$$\begin{aligned}\mu\left(A_{n+1} \setminus \bigcup_{i=1}^{n} F_i\right) &\ge \mu(A_{n+1}) - \mu\left(\bigcup_{i=1}^{n} F_i\right) = \mu(A_{n+1}) - \sum_{i=1}^{n}\mu(F_i)\\ &= \mu(A_{n+1}) - \sum_{i=1}^{n} s_i\mu(A_i) \ge \mu(A_{n+1}) - \sum_{i=1}^{n} s_i\mu(A_{n+1})\\ &\ge s_{n+1}\mu(A_{n+1}).\end{aligned}$$

Therefore, the nonatomicity of μ guarantees the existence of a set $F_{n+1} \in \mathscr{K}^{\mu}_{s_{n+1}}(A_{n+1})$ such that $E_{n+1} \setminus \bigcup_{i=1}^{n} F_i \subset F_{n+1} \subset A_{n+1} \setminus \bigcup_{i=1}^{n} F_i$. Then, the sets $F_1 \in \mathscr{K}^{\mu}_{s_1}(A_1), \dots, F_{n+1} \in \mathscr{K}^{\mu}_{s_{n+1}}(A_{n+1})$ are pairwise disjoint and satisfy $\bigcup_{i=1}^{n+1} E_i \subset \bigcup_{i=1}^{n+1} F_i$ by construction. Therefore, the result is true for $n+1$ and the proof is complete. □

Theorem 2.4. *A subset $\mathscr{X}$ of $\mathscr{F}$ is μ-convex if and only if, for each $n \ge 2$, $\mathscr{K}^{\mu}_{t_1,\dots,t_n}(A_1,\dots,A_n) \subset \mathscr{X}$ for every finite sequence of sets $A_1,\dots,A_n$ in $\mathscr{X}$ and every element $(t_1,\dots,t_n)$ in Δ^{n-1}.*

Proof. The sufficient condition is clearly satisfied. Thus, we need only dem-onstrate the necessary condition, for which we use induction on n. Suppose that $\mathscr{X}$ is μ-convex. According to the definition of μ-convexity, the result is true for $n = 2$. Now, suppose that the result is true for $n \ge 2$. Let $A_1,\dots,A_{n+1} \in \mathscr{X}$ and let $(t_1,\dots,t_{n+1}) \in \Delta^n$. Without loss of generality, we assume that $t_{n+1} < 1$. Arbitrarily choose $E_i \in \mathscr{K}^{\mu}_{t_i}(A_i)$ for each $i = 1,\dots,n+1$ satisfying $E_i \cap E_j = \emptyset$ for $i \ne j$. This is allowed by Theorem 2.1. Define $s_i = (1-t_{n+1})^{-1}t_i$ for $i = 1,\dots,n$. We then have $\sum_{i=1}^{n} s_i = 1$ and $t_i \le s_i$ for $i = 1,\dots,n$. By Lemma 2.3, there exist pairwise disjoint sets $F_1 \in \mathscr{K}^{\mu}_{s_1}(A_1),\dots,F_n \in \mathscr{K}^{\mu}_{s_n}(A_n)$ such that $\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{n} F_i$. It follows from the induction hypothesis that $\bigcup_{i=1}^{n} F_i \in \mathscr{K}^{\mu}_{s_1,\dots,s_n}(A_1,\dots,A_n) \subset \mathscr{X}$. We then have

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{n} E_i\right) &= \sum_{i=1}^{n}\mu(E_i) = \sum_{i=1}^{n} t_i\mu(A_i) = \sum_{i=1}^{n}(1-t_{n+1})s_i\mu(A_i)\\ &= (1-t_{n+1})\sum_{i=1}^{n}\mu(F_i) = (1-t_{n+1})\mu\left(\bigcup_{i=1}^{n} F_i\right).\end{aligned}$$

Hence, $\bigcup_{i=1}^{n} E_i \in \mathscr{K}^{\mu}_{1-t_{n+1}}(\bigcup_{i=1}^{n} F_i)$. Thus, by the μ-convexity of $\mathscr{X}$, we obtain

$$\bigcup_{i=1}^{n+1} E_i \in \mathscr{K}^{\mu}_{t_{n+1}}\left(A_{n+1}, \bigcup_{i=1}^{n} F_i\right) \subset \mathscr{X}.$$

Therefore, we have $\mathscr{K}^{\mu}_{t_1,\dots,t_{n+1}}(A_1,\dots,A_{n+1}) \subset \mathscr{X}$ for every $(t_1,\dots,t_{n+1}) \in \Delta^n$. □

2.2 μ-Convex Functions

Let $\mathscr{X}$ be a subset of $\mathscr{F}$. The *effective domain* of a set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\operatorname{dom} f = \{A \in \mathscr{X} \mid f(A) < +\infty\}.$$

The set function f is *proper* if $\operatorname{dom} f$ is nonempty. The *epigraph* of f is the set defined by

$$\operatorname{epi} f = \{(A,\alpha) \in \mathscr{X} \times \mathbb{R} \mid f(A) \le \alpha\}.$$

Then, $\operatorname{dom} f$ is the projection of $\operatorname{epi} f$ into $\mathscr{X}$. The *level set* of f at $\alpha \in \mathbb{R}$ is the α-section of $\operatorname{epi} f$; that is,

$$\operatorname{lev}_\alpha f = \{A \in \mathscr{X} \mid f(A) \leq \alpha\}.$$

Recall that the symmetric difference, $A \triangle B$, of sets A and B is given by $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Definition 2.5. Let $\mathscr{X}$ be a μ-convex subset of $\mathscr{F}$. A set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is:

(i) *μ-convex* if $A, B \in \mathscr{X}$ and $t \in [0,1]$ imply

$$f(C) \leq tf(A) + (1-t)f(B) \quad \text{for every } C \in \mathscr{K}_t^\mu(A,B);$$

(ii) *strictly μ-convex* if $A, B \in \mathscr{X}$ with $\mu(A \triangle B) > 0$ and $t \in (0,1)$ imply

$$f(C) < tf(A) + (1-t)f(B) \quad \text{for every } C \in \mathscr{K}_t^\mu(A,B).$$

A set function $f : \mathscr{X} \to \mathbb{R} \cup \{-\infty\}$ is (*strictly*) *μ-concave* if $-f$ is (strictly) μ-convex. A real-valued function on $\mathscr{F}$ is *μ-additive* if it is both μ-convex and μ-concave on $\mathscr{F}$.

The following properties of μ-convex functions are similar to those of convex functions on vector spaces (see Attouch et al. [1, Section 3.3]; Ekeland and Témam [5, Section 1.2]).

- The *indicator function* $\delta_{\mathscr{X}} : \mathscr{F} \to \mathbb{R} \cup \{+\infty\}$ of a subset $\mathscr{X}$ of $\mathscr{F}$ defined by

$$\delta_{\mathscr{X}}(A) = \begin{cases} 0 & \text{if } A \in \mathscr{X} \\ +\infty & \text{otherwise} \end{cases}$$

  is μ-convex if and only if $\mathscr{X}$ is μ-convex.
- If f and g are μ-convex functions on $\mathscr{X}$ into $\mathbb{R} \cup \{+\infty\}$ and $\alpha \geq 0$, then αf and $f + g$ defined by

$$(\alpha f)(A) = \alpha f(A), \quad (f+g)(A) = f(A) + g(A)$$

  are μ-convex functions.
- If $\{f_i\}_{i \in I}$ is a family of μ-convex functions on $\mathscr{X}$ into $\mathbb{R} \cup \{+\infty\}$, then their pointwise supremum $f = \sup_{i \in I} f_i$ is a μ-convex function.

  If f is a μ-convex function on $\mathscr{X}$ into $\mathbb{R} \cup \{+\infty\}$, then $\operatorname{lev}_\alpha f$ is μ-convex for every $\alpha \in \mathbb{R}$.
- The effective domain of a μ-convex function is μ-convex.

A subset X of $\mathscr{F} \times \mathbb{R}$ is *μ-convex* if (A, α) and (B, β) in X and $t \in [0,1]$ imply $(C, t\alpha + (1-t)\beta) \in X$ for every $C \in \mathscr{K}_t^\mu(A,B)$.

Theorem 2.6. *A set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is μ-convex if and only if its epigraph is μ-convex.*

Proof. Let f be μ-convex, and then take (A, α) and (B, β) in $\operatorname{epi} f$ arbitrarily. Then, it necessarily follows that $f(A) \leq \alpha < +\infty$ and $f(B) \leq \beta < +\infty$, and for every $t \in [0,1]$, from the μ-convexity of f, we have

$$f(C) \leq tf(A) + (1-t)f(B) \leq t\alpha + (1-t)\beta$$

for every $C \in \mathscr{K}_t^{\mu}(A,B)$. This means that $(C, t\alpha + (1-t)\beta) \in \operatorname{epi} f$.

Conversely, let $\operatorname{epi} f$ be μ-convex. Its projection $\operatorname{dom} f$ is therefore μ-convex and it is sufficient to verify the inequality in Definition 2.5 over $\operatorname{dom} f$. Take A and B in $\operatorname{dom} f$, and α and β in $\mathbb{R}$ satisfying $f(A) \le \alpha$ and $f(B) \le \beta$. By hypothesis, $(C, t\alpha + (1-t)\beta) \in \operatorname{epi} f$ for every $C \in \mathscr{K}_t^{\mu}(A,B)$ and $t \in [0,1]$ so that $f(C) \le t\alpha + (1-t)\beta$. Given that $f(A)$ and $f(B)$ are finite, it is sufficient to take $\alpha = f(A)$ and $\beta = f(B)$. □

Theorem 2.7 (Jensen inequality). *A set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is μ-convex if and only if, for each $n \ge 2$, for every finite sequence of sets $A_1, \dots, A_n$ in $\mathscr{X}$ and for every element $(t_1, \dots, t_n)$ in Δ^{n-1},*

$$f(C) \le \sum_{i=1}^{n} t_i f(A_i) \quad \textit{for every } C \in \mathscr{K}^{\mu}_{t_1,\dots,t_n}(A_1, \dots, A_n).$$

Proof. Because the sufficient condition is clearly satisfied, we need only prove the necessary condition, for which we use induction on n. Let f be μ-convex. For $n = 2$, the result clearly follows from Definition 2.5. Suppose that the result is true for $n \ge 2$. Let $A_1, \dots, A_{n+1} \in \mathscr{F}$ and let $(t_1, \dots, t_{n+1}) \in \Delta^n$. Choose $C \in \mathscr{K}^{\mu}_{t_1,\dots,t_{n+1}}(A_1, \dots, A_{n+1})$ arbitrarily. Then, $C = \bigcup_{i=1}^{n+1} E_i$ is the union of pairwise disjoint sets $E_1 \in \mathscr{K}_{t_1}^{\mu}(A_1), \dots, E_{n+1} \in \mathscr{K}^{\mu}_{t_{n+1}}(A_{n+1})$. Without loss of generality, one may assume that $t_{n+1} < 1$. Define $s_i = (1 - t_{n+1})^{-1} t_i$ for each $i = 1, \dots, n$. By Lemma 2.3, there exists a set $F \in \mathscr{K}^{\mu}_{s_1,\dots,s_n}(A_1, \dots, A_n)$ containing $\bigcup_{i=1}^{n} E_i$ such that $\bigcup_{i=1}^{n+1} E_i \in \mathscr{K}^{\mu}_{t_{n+1}}(A_{n+1}, F)$. Because of the μ-convexity of f and given the induction hypothesis, we have:

$$\begin{aligned} f(C) = f\left(\bigcup_{i=1}^{n+1} E_i\right) &\le t_{n+1} f(A_{n+1}) + (1 - t_{n+1}) f(F) \\ &\le t_{n+1} f(A_{n+1}) + (1 - t_{n+1}) \sum_{i=1}^{n} s_i f(A_i) = \sum_{i=1}^{n+1} t_i f(A_i). \end{aligned}$$

Hence, the result is true for $n + 1$. □

Example 2.8. Let φ be a real-valued function on the closed interval $[0, \mu(\Omega)]$. Define the set function f_φ on $\mathscr{F}$ by $f_\varphi = \varphi \circ \mu$. Because $C \in \mathscr{K}_t^{\mu}(A,B)$ implies $\mu(C) = t\mu(A) + (1-t)\mu(B)$, if φ is convex, then, for every $C \in \mathscr{K}_t^{\mu}(A,B)$ and $t \in [0,1]$, we have:

$$\begin{aligned} f_\varphi(C) = \varphi(t\mu(A) + (1-t)\mu(B)) &\le t\varphi(\mu(A)) + (1-t)\varphi(\mu(B)) \\ &= t f_\varphi(A) + (1-t) f_\varphi(B). \end{aligned}$$

Hence, f_φ is μ-convex on $\mathscr{F}$.

Conversely, suppose that φ is such that f_φ is μ-convex on $\mathscr{F}$. Choose $x, y \in [0, \mu(\Omega)]$ and $t \in [0,1]$ arbitrarily. By the nonatomicity of μ, there exist A and B in $\mathscr{F}$ such that $\mu(A) = x$ and $\mu(B) = y$. Then, by Theorem 2.1, there exist $E \in \mathscr{K}_t^{\mu}(A)$ and $F \in \mathscr{K}^{\mu}_{1-t}(B)$ such that $E \cap F = \emptyset$. We then have:

$$\begin{aligned} \varphi(tx + (1-t)y) = \varphi(t\mu(A) + (1-t)\mu(B)) &= \varphi(\mu(E) + \mu(F)) = f_\varphi(E \cup F) \\ &\le t f_\varphi(A) + (1-t) f_\varphi(B) = t\varphi(x) + (1-t)\varphi(y). \end{aligned}$$

Hence, φ is convex on $[0, \mu(\Omega)]$.

Consequently, f_φ is (strictly) μ-convex on $\mathscr{F}$ if and only if φ is (strictly) convex on $[0, \mu(\Omega)]$.

$\boxed{2.3}$ Minimization of μ-Convex Functions

Sets A and B in $\mathscr{F}$ are *μ-equivalent* if $\mu(A \triangle B) = 0$. The μ-equivalence defines an equivalence relation (reflexive, symmetric, transitive binary relation) on $\mathscr{F}$.

Theorem 2.9. (i) *For every μ-convex function on a μ-convex set, the set of its minimizers is μ-convex.*

(ii) *For every strictly μ-convex function on a μ-convex set, its minimizer is unique up to μ-equivalence.*

Proof. (i) Let A and B be minimizers of a μ-convex function f defined on a μ-convex set. Denote its minimum value by α. Then, for every $t \in [0,1]$ and $C \in \mathscr{K}_t^\mu(A,B)$, we have $f(C) \le tf(A) + (1-t)f(B) = \alpha$, and hence $f(C) = \alpha$. Therefore, C is a minimizer of f.

(ii) Assume in the above that f is strictly μ-convex. If $\mu(A \triangle B) > 0$, then for every $t \in (0,1)$ and $C \in \mathscr{K}_t^\mu(A,B)$, we have $f(C) < tf(A) + (1-t)f(B) = \alpha$, which is a contradiction. Therefore, we obtain $\mu(A \triangle B) = 0$. $\square$

Definition 2.10. Let $(\Omega, \mathscr{F}, \mu)$ and $(\Omega', \mathscr{F}', \mu')$ be nonatomic finite measure spaces, let $\mathscr{X}$ be a μ-convex subset of $\mathscr{F}$ and let $\mathscr{X}'$ be a μ'-convex subset of $\mathscr{F}'$. A set function $L : \mathscr{X} \times \mathscr{X}' \to \mathbb{R} \cup \{+\infty\}$ is *(μ,μ')-convex* if for every (A,A') and (B,B') in $\mathscr{X} \times \mathscr{X}'$ and $t \in [0,1]$,

$$L(C,C') \le tL(A,A') + (1-t)L(B,B')$$

for every $(C,C') \in \mathscr{K}_t^\mu(A,B) \times \mathscr{K}_t^{\mu'}(A',B')$.

Definition 2.11. A subset $\mathscr{G}$ of $\mathscr{F} \times \mathscr{F}'$ is *(μ,μ')-convex* if $\mathscr{K}_t^\mu(A,B) \times \mathscr{K}_t^{\mu'}(A',B') \subset \mathscr{G}$ for every $(A,A'), (B,B') \in \mathscr{G}$ and $t \in [0,1]$.

Definition 2.12. Let $\mathscr{P}(\mathscr{X}')$ be the family of all subsets of $\mathscr{X}'$. A set-valued mapping $\Gamma : \mathscr{X} \to \mathscr{P}(\mathscr{X}')$ is *(μ,μ')-convex* if the graph of Γ is a (μ,μ')-convex subset of $\mathscr{F} \times \mathscr{F}'$.

Theorem 2.13. *Let $\mathscr{X}$ be a μ-convex subset of $\mathscr{F}$ and let $\mathscr{X}'$ be a μ'-convex subset of $\mathscr{F}'$. If $L : \mathscr{X} \times \mathscr{X}' \to \mathbb{R} \cup \{+\infty\}$ is a (μ,μ')-convex function and $\Gamma : \mathscr{X} \to \mathscr{P}(\mathscr{X}')$ is a (μ,μ')-convex set-valued mapping, then the marginal function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(A) = \inf_{A' \in \Gamma(A)} L(A,A')$$

is μ-convex and the set of minimizers defined by

$$\arg\min L(A) = \{A' \in \Gamma(A) \mid L(A,A') = f(A)\}$$

is μ'-convex in $\mathscr{F}'$ for every $A \in \mathscr{X}$.

Proof. Let $A, B \in \mathscr{X}$, $t \in [0,1]$ and $C \in \mathscr{K}_t^\mu(A,B)$ be arbitrary. Choose any $A' \in \Gamma(A)$ and $B' \in \Gamma(B)$. Because Γ is (μ,μ')-convex, every $C' \in \mathscr{K}_t^{\mu'}(A',B')$ belongs to $\Gamma(C)$. Given the (μ,μ')-convexity of L, we have

$$f(C) \le L(C,C') \le tL(A,A') + (1-t)L(B,B').$$

Because $A' \in \Gamma(A)$ and $B' \in \Gamma(B)$ are arbitrary, we obtain $f(C) \le tf(A) + (1-t)f(B)$. Therefore, f is μ-convex.

Because $A' \mapsto L(A,A')$ is μ-convex on the μ-convex set $\Gamma(A)$, the μ-convexity of $\arg\min L(A)$ follows from Theorem 2.9(i). $\square$

3 Continuous Functions on σ-Algebras

We denote the μ-equivalence class of $A \in \mathscr{F}$ by $[A]$ and denote the set of μ-equivalence classes in $\mathscr{F}$ by $\mathscr{F}_\mu$. For any two μ-equivalence classes $[A]$ and $[B]$, we define the metric d_μ on $\mathscr{F}_\mu$ by $d_\mu([A],[B]) = \mu(A \triangle B)$. If $\mathscr{F}$ is countably generated, then the metric space $(\mathscr{F}_\mu, d_\mu)$ is complete and separable (see Dunford and Schwartz [4, Lemma III.7.1]; Halmos [7, Theorem 40.B]).

A subset $\mathscr{X}$ of $\mathscr{F}$ is *μ-open* in $\mathscr{F}$ if $\mathscr{X}_\mu = \{[A] \in \mathscr{F}_\mu \mid A \in \mathscr{X}\}$ is open in $\mathscr{F}_\mu$ and is *μ-closed* in $\mathscr{F}$ if its complement $\mathscr{F} \setminus \mathscr{X}$ is μ-open in $\mathscr{F}$. A subset $\mathscr{X}$ of $\mathscr{F}$ is *μ-compact* in $\mathscr{F}$ if $\mathscr{X}_\mu$ is compact in $\mathscr{F}_\mu$. A subset X of $\mathscr{F} \times \mathbb{R}$ is μ-open in $\mathscr{F} \times \mathbb{R}$ if $X_\mu = \{([A],x) \in \mathscr{F}_\mu \times \mathbb{R} \mid (A,x) \in X\}$ is open in the product topology of $\mathscr{F}_\mu \times \mathbb{R}$. Similarly, a subset X of $\mathscr{F} \times \mathbb{R}$ is μ-closed in $\mathscr{F} \times \mathbb{R}$ if its complement $(\mathscr{F} \times \mathbb{R}) \setminus X$ is μ-open in $\mathscr{F} \times \mathbb{R}$.

3.1 μ-Lower Semicontinuous Functions

Definition 3.1. A set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is *μ-lower semicontinuous* at $A \in \mathscr{X}$ if for every real number $z < f(A)$, there exists some $\delta > 0$ such that $\mu(A \triangle B) < \delta$ implies $z < f(B)$. If f is μ-lower semicontinuous at every set in $\mathscr{X}$, then it is said to be μ-lower semicontinuous. A set function $f : \mathscr{X} \to \mathbb{R} \cup \{-\infty\}$ is *μ-upper semicontinuous* if $-f$ is μ-lower semicontinuous.

The following properties are similar to those of lower semicontinuous functions on topological spaces (see Attouch et al. [1, Section 3.2]; Ekeland and Témam [5, Section 1.2]).

- If f and g are μ-lower semicontinuous functions on $\mathscr{X}$ into $\mathbb{R} \cup \{+\infty\}$ and $\alpha \geq 0$, then αf and $f + g$ defined by
$$(\alpha f)(A) = \alpha f(A), \quad (f+g)(A) = f(A) + g(A)$$
are μ-lower semicontinuous functions.
- If $\{f_i\}_{i \in I}$ is a family of μ-lower semicontinuous functions on $\mathscr{X}$ into $\mathbb{R} \cup \{+\infty\}$, then their pointwise supremum $f = \sup_{i \in I} f_i$ is a μ-lower semicontinuous function.
- The effective domain of a μ-lower semicontinuous function is μ-closed.
- A subset $\mathscr{X}$ of $\mathscr{F}$ is μ-closed if and only if $\delta_{\mathscr{X}}$ is μ-lower semicontinuous.

Definition 3.2. A set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is *μ-invariant* if $A, B \in \mathscr{X}$ with $\mu(A \triangle B) = 0$ implies $f(A) = f(B)$.

Theorem 3.3. *Every μ-lower semicontinuous function is μ-invariant.*

Proof. Let $f : \mathscr{X} \to \mathbb{R} \cup \{-\infty\}$ be a μ-lower semicontinuous function. Suppose that f is not μ-invariant. Then there exist $A, B \in \mathscr{X}$ such that $\mu(A \triangle B) = 0$ and $f(A) \neq f(B)$. Without loss of generality, assume that $f(A) > f(B)$. Because f is μ-lower semicontinuous at A, there exists some $\delta > 0$ such that, for every $A' \in \mathscr{X}$, $\mu(A \triangle A') < \delta$ implies $f(A') > f(B)$. By choosing $A' = B$, we have $f(B) > f(B)$, which is a contradiction. Therefore, f is μ-invariant. □

Proposition 3.4. *If $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is a μ-lower semicontinuous set function, then the following equivalent conditions hold:*

(i) $\operatorname{epi} f$ *is* μ*-closed;*

(ii) $\operatorname{lev}_\alpha f$ *is* μ*-closed for every* $\alpha \in \mathbb{R}$*;*

(iii) $\{A \in \mathscr{X} \mid f(A) > \alpha\}$ *is* μ*-open for every* $x \in \mathbb{R}$*;*

(iv) $f(A) \leq \liminf_{\mu(A \triangle B) \to 0} f(B)$ *for every* $A \in \mathscr{X}$*.*

Proof. It follows from Theorem 3.3 that f 'induces' a function f_μ on $\mathscr{X}_\mu$ defined by $f_\mu([A]) = f(A)$. Thus, the above conditions correspond exactly to those relating to the lower semicontinuous function f_μ on the metric space $(\mathscr{X}_\mu, d_\mu)$. The proof is standard; see, for example, Attouch et al. [1, Proposition 3.2.2]. □

$\boxed{3.2}$ The Minimax Theorem for Set Functions

Definition 3.5. Let $\mathscr{X}$ and $\mathscr{X}'$ be subsets of σ-algebras of Ω and Ω', respectively. A pair $(A_0, A_0') \in \mathscr{X} \times \mathscr{X}'$ is a *saddle point* of the set function $L : \mathscr{X} \times \mathscr{X}' \to \mathbb{R}$ if

$$L(A_0, A') \leq L(A_0, A_0') \leq L(A, A_0') \quad \text{for every } (A, A') \in \mathscr{X} \times \mathscr{X}'.$$

It is well known that the minimax value of L is attained at its saddle points (see Ekeland and Témam [5, Proposition VI.1.2]).

Proposition 3.6. *A set function* $L : \mathscr{X} \times \mathscr{X}' \to \mathbb{R}$ *has a saddle point* $(A_0, A_0') \in \mathscr{X} \times \mathscr{X}'$ *if and only if*

$$\min_{A \in \mathscr{X}} \sup_{A' \in \mathscr{X}'} L(A, A') = \max_{A' \in \mathscr{X}'} \inf_{A \in \mathscr{X}} L(A, A').$$

This value coincides with $L(A_0, A_0')$*.*

Theorem 3.7 (Minimax theorem). *Let* $\mathscr{X}$ *be a* μ*-convex,* μ*-compact subset of* $\mathscr{F}$*, let* $\mathscr{X}'$ *be a* μ'*-convex,* μ'*-compact subset of* $\mathscr{F}'$ *and let* $L : \mathscr{X} \times \mathscr{X}' \to \mathbb{R}$ *be a set function with the following properties:*

(i) *for every* $A \in \mathscr{X}$*, the function* $A' \mapsto L(A, A')$ *is* μ'*-concave and* μ'*-upper semicontinuous;*

(ii) *for every* $A' \in \mathscr{X}'$*, the function* $A \mapsto L(A, A')$ *is* μ*-convex and* μ*-lower semicontinuous.*

Then, L *has a saddle point* $(A_0, A_0') \in \mathscr{X} \times \mathscr{X}'$ *and*

$$\min_{A \in \mathscr{X}} \max_{A' \in \mathscr{X}'} L(A, A') = \max_{A' \in \mathscr{X}'} \min_{A \in \mathscr{X}} L(A, A') = L(A_0, A_0').$$

Proof. We first consider the case in which, for every $A' \in \mathscr{X}'$, the function $A \mapsto L(A, A')$ is strictly μ-convex. From condition (i), the function f, defined by $f(A') = \min_{A \in \mathscr{X}} L(A, A')$, is a μ'-concave, μ'-upper semicontinuous function as the pointwise infimum of a family of μ'-concave, μ'-upper semicontinuous functions $A' \mapsto L(A, A')$ with $A \in \mathscr{X}$. Given that, for every $A' \in \mathscr{X}'$, the minimizer of a μ-lower semicontinuous function $A \mapsto L(A, A')$ is unique up to μ-equivalence according to Theorem 2.9(ii), there exists a mapping $\varphi : \mathscr{X}' \to \mathscr{X}$ such that $f(A') = L(\varphi(A'), A')$ for every $A' \in \mathscr{X}'$. Because f is μ'-upper semicontinuous on the μ'-compact set $\mathscr{X}'$, it attains its maximum value at the set $A_0' \in \mathscr{X}'$:

$$f(A_0') = \max_{A' \in \mathscr{X}'} f(A') = \max_{A' \in \mathscr{X}'} \min_{A \in \mathscr{X}} L(A, A'). \tag{3.1}$$

For $A' \in \mathscr{X}'$, $t \in (0,1)$ and $B'_t \in \mathscr{K}^{\mu}_t(A', A'_0)$, and given the μ'-concavity described in condition (i), we have

$$f(A'_0) \geq f(B'_t) = L(\varphi(B'_t), B'_t) \geq tL(\varphi(B'_t), A') + (1-t)L(\varphi(B'_t), A'_0)$$
$$\geq tL(\varphi(B'_t), A') + (1-t)f(A'_0).$$

Hence,

$$L(\varphi(B'_t), A') \leq f(A'_0) \quad \text{for every } A' \in \mathscr{X} \text{ and } t \in (0,1). \tag{3.2}$$

We show that $\mu(\varphi(B'_t) \triangle \varphi(A'_0)) \to 0$ as $t \to 0$. From the definition of $\varphi(B'_t)$, it follows that $L(\varphi(B'_t), B'_t) \leq L(A, B'_t)$ for every $A \in \mathscr{X}$ and $t \in (0,1)$. Thus, given the μ'-concavity described in condition (i), we have

$$tL(\varphi(B'_t), A') + (1-t)L(\varphi(B'_t), A'_0) \leq L(A, B'_t) \tag{3.3}$$

for every $A \in \mathscr{X}$ and $t \in (0,1)$. By Theorem 2.1, there exist $E \in \mathscr{K}^{\mu}_t(A')$ and $F \in \mathscr{K}^{\mu}_{1-t}(A'_0)$ such that $B'_t = E \cup F$ and $E \cap F = \emptyset$. Because $B'_t \triangle A'_0 \subset E \cup (A'_0 \setminus F)$, we have $\mu(B'_t \triangle A'_0) \leq t(\mu(A') + \mu(A'_0)) \to 0$ as $t \to 0$. Let $\{t^\nu\}$ be any sequence in $(0,1)$ with $t^\nu \to 0$. Given that $\mathscr{X}$ is μ-compact and $B^\nu = \varphi(B'_{t^\nu}) \in \mathscr{X}$ for each ν, there exists a subsequence $\{B^{\nu_k}\}$ of $\{B^\nu\}$ such that $\mu(B^{\nu_k} \triangle B) \to 0$ for some $B \in \mathscr{X}$. The passage to the limit in the inequality (3.3) with $t^{\nu_k} \to 0$ yields

$$L(B, A'_0) \leq \liminf_{k\to+\infty} L(B^{\nu_k}, A'_0) \leq \limsup_{k\to+\infty} L(A, B'_{t^{\nu_k}}) \leq L(A, A'_0)$$

for every $A \in \mathscr{X}$ given the μ'-upper semicontinuity and the μ-lower semicontinuity described in conditions (i) and (ii). Thus, $\min_{A\in\mathscr{X}} L(A, A'_0) = L(B, A'_0)$. Because the minimizer $\varphi(A'_0)$ is unique up to μ-equivalence by Theorem 2.9(ii), we have $\mu(\varphi(A'_0) \triangle B) = 0$. Therefore, $\mu(B^{\nu_k} \triangle \varphi(A'_0)) \to 0$. Given that $\mu(\varphi(B'_{t^\nu}) \triangle \varphi(A'_0)) \to 0$ for every subsequence $\{\varphi(B'_{t^\nu})\}$ of $\{\varphi(B'_t)\}$, it follows that $\mu(\varphi(B'_t) \triangle \varphi(A'_0)) \to 0$.

Given the μ-lower semicontinuity described in condition (ii), letting $t \to 0$ in inequality (3.2) yields

$$L(\varphi(A'_0), A') \leq \liminf_{t\to 0} L(\varphi(B'_t), A') \leq f(A'_0) \quad \text{for every } A' \in \mathscr{X}.$$

Because the minimax inequality $\max\min L \leq \min\max L$ is satisfied, we have $f(A'_0) \leq L(A, A'_0)$ for every $A \in \mathscr{X}$ by (3.1). By setting $A_0 = \varphi(A'_0)$, we obtain

$$L(A_0, A') \leq f(A'_0) = L(A_0, A'_0) \leq L(A, A'_0) \quad \text{for every } (A, A') \in \mathscr{X} \times \mathscr{X}'.$$

Therefore, $(A_0, A'_0) \in \mathscr{X} \times \mathscr{X}'$ is a saddle point of L. The result follows from Proposition 3.6.

Next, we consider the general case. We introduce a perturbation $L_\varepsilon : \mathscr{X} \times \mathscr{X}' \to \mathbb{R}$ of L defined by

$$L_\varepsilon(A, A') = L(A, A') + \varepsilon\mu(A)^2, \quad \varepsilon > 0.$$

Because μ^2 is strictly μ-convex (see Example 2.8), for every $A' \in \mathscr{X}$, the function $A \mapsto L_\varepsilon(A, A')$ is strictly μ-convex given the μ-convexity described in condition (ii). By applying the above reasoning, we demonstrate the existence for L_ε of a saddle point $(A_\varepsilon, A'_\varepsilon) \in \mathscr{X} \times \mathscr{X}'$: for every $(A, A') \in \mathscr{X} \times \mathscr{X}'$,

$$L(A_\varepsilon, A') + \varepsilon\mu(A_\varepsilon)^2 \leq L(A_\varepsilon, A'_\varepsilon) + \varepsilon\mu(A_\varepsilon)^2 \leq L(A, A'_\varepsilon) + \varepsilon\mu(A)^2. \tag{3.4}$$

Because $\mathscr{X}$ and $\mathscr{X}'$ are μ-compact and μ'-compact, respectively, there exists a sequence $\{\varepsilon^\nu\}$ with $\varepsilon^\nu \to 0$ such that $\mu(A_{\varepsilon^\nu} \triangle A_0) \to 0$ for some $A_0 \in \mathscr{X}$ and $\mu'(A'_{\varepsilon^\nu} \triangle A'_0) \to 0$ for some $A'_0 \in \mathscr{X}'$. Because μ^2 is μ-continuous (see Example 3.10), passing to the limit in (3.4) by using the μ'-upper semicontinuity and the μ-lower semicontinuity described in conditions (i) and (ii) yields $L(A_0, A') \leq L(A, A'_0)$ for every $(A, A') \in \mathscr{X} \times \mathscr{X}$. This proves that (A_0, A'_0) is a saddle point of L and thus proves the theorem. □

3.3 μ-Absolutely Continuous Functions

Definition 3.8. (i) A set function $f : \mathscr{X} \to \mathbb{R}$ is *μ-continuous* if it is both μ-lower semicontinuous and μ-upper semicontinuous.

(ii) A set function $f : \mathscr{X} \to \mathbb{R} \cup \{+\infty\}$ is *μ-absolutely continuous* if $\mu(N) = 0$ implies $f(A \cup N) = f(A)$ for every $A \in \mathscr{X}$ with $A \cup N \in \mathscr{X}$.

Theorem 3.9. *Let $f : \mathscr{F} \to \mathbb{R}$ be a set function with $f(\emptyset) = 0$. If f is μ-invariant, then it is μ-absolutely continuous. If f is a finite measure, then the following conditions are equivalent:*

(i) *f is μ-invariant;*

(ii) *f is μ-continuous;*

(iii) *f is μ-absolutely continuous.*

Proof. Let f be a μ-invariant function with $f(\emptyset) = 0$. Because $\mu(N) = 0$ yields $\mu((A \cup N) \triangle A) = 0$, we have $f(A \cup N) = f(A)$ for every $A \in \mathscr{F}$.

Let f be a finite measure. Implication (ii) $\Rightarrow$ (i) follows from Theorem 3.3 and implication (i) $\Rightarrow$ (iii) is immediate from the above argument.

We demonstrate implication (iii) $\Rightarrow$ (ii). If f is not μ-continuous, then for some $\varepsilon > 0$, there exist sets A^ν and A such that $\mu(A^\nu \triangle A) < \frac{1}{2^\nu}$ and $|f(A^\nu) - f(A)| \geq \varepsilon$ for each ν. Given that $|f(A^\nu) - f(A)| \leq f(A^\nu \setminus A) + f(A \setminus A^\nu) = f(A^\nu \triangle A)$, we have $f(A^\nu \triangle A) \geq \varepsilon$. If $B = \limsup_\nu (A^\nu \triangle A)$, then $\mu(B) = 0$ by the first Borel–Cantelli lemma (see Billingsley [2, Theorem 4.3]). However, $f(B) \geq \varepsilon$ because $\limsup_\nu f(A^\nu \triangle A) \leq f(B)$ (see Billingsley [2, Theorem 4.1]). Therefore, f is not μ-absolutely continuous. □

Example 3.10. Let $\mu_1, \dots, \mu_n$ be nonatomic finite measures of a measurable space $(\Omega, \mathscr{F})$ and define the nonatomic finite measure by $\mu = \frac{1}{n} \sum_{i=1}^n \mu_i$. Denote by S the range of the vector measure $(\mu_1, \dots, \mu_n)$, that is, $S = \{(\mu_1(A), \dots, \mu_n(A)) \in \mathbb{R}^n \mid A \in \mathscr{F}\}$. Let φ be a real-valued function on S. The set function f_φ defined by

$$f_\varphi(A) = \varphi(\mu_1(A), \dots, \mu_n(A)) \quad \text{for } A \in \mathscr{F}$$

is μ-continuous if φ is continuous. To demonstrate this, we let $\{A^\nu\}$ be a sequence in $\mathscr{F}$ such that $\mu(A^\nu \triangle A) \to 0$. Because $\mu(A^\nu \triangle A) \to 0$ implies $\mu_i(A^\nu \triangle A) \to 0$ for each i, we have $\mu_i(A^\nu) \to \mu_i(A)$ given that $|\mu_i(A^\nu) - \mu_i(A)| \leq \mu_i(A^\nu \triangle A)$ for each ν. Therefore, $\lim_\nu f_\varphi(A^\nu) = f_\varphi(A)$, and hence f_φ is μ-continuous.

Suppose that φ is discontinuous at some point x in S. Because S is convex by the Lyapunov convexity theorem (see Dubins and Spanier [3] and Halmos [6]) and contains the origin of $\mathbb{R}^n$, $x^\nu = (1 - \frac{1}{\nu})x \in S$ for each ν. Since $x^\nu \to x$, there exists some $\varepsilon > 0$ such that $|\varphi(x^\nu) - \varphi(x)| \geq \varepsilon$ for each ν. Let $A \in \mathscr{F}$ be such that $x = (\mu_1(A), \dots, \mu_n(A))$. Given the nonatomicity of μ_i, there exist sets $A^\nu \subset A$ such that $(\mu_1(A^\nu), \dots, \mu_n(A^\nu)) =$

$(1-\frac{1}{\nu})(\mu_1(A),\dots,\mu_n(A))$ (see Dubins and Spanier [3, Lemma 5.3]). Since $\mu_i(A^\nu \triangle A) = \mu_i(A \setminus A^\nu) = \frac{1}{\nu}\mu_i(A)$ for each $i = 1,\dots,n$, we have $\mu(A^\nu \triangle A) \to 0$. It follows from $|f_\varphi(A^\nu) - f_\varphi(A)| = |\varphi(x^\nu) - \varphi(x)| \geq \varepsilon$ for each ν that f_φ is discontinuous at A.

Therefore, f_φ is μ-continuous if and only if φ is continuous on S.

Convex functions on normed spaces are locally Lipschitz continuous if they are locally bounded above (see Ekeland and Témam [5, Corollary I.2.4]). As the following example demonstrates, this is not the case for μ-convex functions.

Example 3.11. Define the set function f by

$$f(A) = \begin{cases} 1 & \text{if } \mu(A) = \mu(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

Note that f is μ-absolutely continuous. Let $A, B \in \mathscr{F}$, $t \in [0,1]$ and $C \in \mathscr{K}_t^\mu(A,B)$ be arbitrary. If $\mu(C) = \mu(\Omega)$, then $\mu(A) = \mu(B) = \mu(\Omega)$ whenever $t \in (0,1)$; $\mu(A) = \mu(\Omega)$ whenever $t = 1$; $\mu(B) = \mu(\Omega)$ whenever $t = 0$. Thus, we have $f(C) = 1 = tf(A)+(1-t)f(B)$. If $\mu(C) < \mu(\Omega)$, then $0 = f(C) \leq tf(A)+(1-t)f(B)$ given the nonnegative of f. Therefore, f is μ-convex. Although f is μ-upper semicontinuous on $\mathscr{F}$, it is not μ-lower semicontinuous at Ω.

$\boxed{3.4}$ Countable Additivity of μ-Convex Charges

Theorem 3.12. *Let $\mathscr{X}$ be a μ-convex set containing the empty set and let $f : \mathscr{X} \to \mathbb{R}$ be a set function with $f(\emptyset) = 0$. If f is μ-convex, then $\mu(A)f(B) \leq \mu(B)f(A)$ for every A and B in $\mathscr{X}$ with $B \subset A$.*

Proof. Let f be μ-convex and let $f(\emptyset) = 0$. Choose A and B in $\mathscr{X}$ arbitrarily such that $B \subset A$. If $\mu(A) = 0$, then $\mu(B) = 0$ from the monotonicity of μ. Thus, the above inequality holds trivially. If $\mu(A) > 0$, we can define $t = \frac{\mu(B)}{\mu(A)}$. Because $f(\emptyset) = 0$ and $B \in \mathscr{K}_t^\mu(A,\emptyset)$, we have $f(B) \leq tf(A) + (1-t)f(\emptyset) = tf(A)$ by the μ-convexity of f, which yields $\mu(A)f(B) \leq \mu(B)f(A)$. $\square$

Theorem 3.12 implies that every μ-convex function on $\mathscr{F}$ with $f(\emptyset) = 0$ is dominated by the nonatomic finite measure $\mu_f = \frac{f(\Omega)}{\mu(\Omega)}\mu$; that is, $f(A) \leq \mu_f(A)$ for every $A \in \mathscr{F}$. This observation enables us to state the following corollaries.

Corollary 3.13. *If $f : \mathscr{F} \to \mathbb{R}$ is a monotone μ-convex set function with $f(\emptyset) = 0$, then $f(N) = 0$ for every μ-null set N.*

Proof. Let N be a μ-null set. We then have $0 \leq f(N) \leq \mu_f(N) = 0$. $\square$

Corollary 3.14. *A set function $f : \mathscr{F} \to \mathbb{R}$ with $f(\emptyset) = 0$ is μ-additive if and only if $f = \mu_f$.*

Proof. If f is μ-additive, then $f \leq \mu_f$ by its μ-convexity and $f \geq \mu_f$ by its μ-concavity. Hence, $f = \mu_f$. The converse implication is obvious. $\square$

For a finite charge, the converse of Theorem 3.12 is true.

Theorem 3.15. *A finite charge f is μ-convex if and only if $\mu(A)f(B) \leq \mu(B)f(A)$ for every A and B in $\mathscr{F}$ with $B \subset A$.*

Proof. Suppose that the inequality given in the theorem is satisfied. Let $A, B \in \mathscr{F}$, $t \in [0,1]$ and $C \in \mathscr{K}_t^\mu(A,B)$ be arbitrary. We must show that $f(C) \leq tf(A)+(1-t)f(B)$. If $f(A) = 0$, then $f(B) = 0$ from the monotonicity of f, and hence $f(C) = tf(A) + (1-t)f(B) = 0$ given that $f(C) \leq f(A \cup B) = 0$. If $\mu(A) = 0$, then $\mu(B) = 0$ because of the monotonicity of μ, and hence $\mu(C) = 0$. Given the μ-absolute continuity of f by Corollary 3.13, it follows that $f(A) = f(B) = f(C) = 0$. Hence, $f(C) = tf(A) + (1-t)f(B) = 0$. Suppose that $\mu(A) > 0$ and $f(A) > 0$. By Theorem 2.1, there exist $E \in \mathscr{K}_t^\mu(A)$ and $F \in \mathscr{K}_{1-t}^\mu(B)$ such that $C = E \cup F$ and $E \cap F = \emptyset$. If $f(B) = 0$, then $f(F) = 0$ given that $F \subset B$. We then have

$$f(C) = f(E) = \frac{f(E)}{f(A)}f(A) \leq \frac{\mu(E)}{\mu(A)}f(A) = tf(A) + (1-t)f(B),$$

for which we have used the condition $\mu(A)f(E) \leq \mu(E)f(A)$. If $f(B) > 0$, then we have

$$\begin{aligned} f(C) = f(E) + f(F) = \frac{f(E)}{f(A)}f(A) + \frac{f(F)}{f(B)}f(B) &\leq \frac{\mu(E)}{\mu(A)}f(A) + \frac{\mu(F)}{\mu(B)}f(B) \\ &= tf(A) + (1-t)f(B), \end{aligned}$$

for the second line of which we have used

$$\mu(A)f(E) \leq \mu(E)f(A) \text{ and } \mu(B)f(F) \leq \mu(F)f(B).$$

Hence, f is μ-convex.

The converse implication follows from Theorem 3.12. □

Recall that if a function on a vector space vanishing at the origin is both convex and concave, then it is an additive function. A similar property holds for μ-additive set functions.

Theorem 3.16. (i) *A finite signed charge f coincides with μ_f if and only if f is μ-additive.*

(ii) *A μ-continuous charge is a measure.*

(iii) *A μ-convex finite charge is a μ-continuous measure.*

Proof. (i) If $f = \mu_f$, it is clear that f is a μ-additive nonatomic finite signed measure. Suppose, conversely, that f is a μ-additive finite signed charge. The μ-additivity of f implies $\mu(A)f(B) = \mu(B)f(A)$ for every A and B in $\mathscr{F}$ with $B \subset A$ by Theorem 3.15. Letting $B = \Omega$ in this equality yields $f = \mu_f$.

(ii) Let f be a μ-continuous charge. It is sufficient to demonstrate its countable additivity. Let A^ν, $\nu = 1, 2, \dots$, be pairwise disjoint sets of $\mathscr{F}$ and $A = \bigcup_{\nu=1}^\infty A^\nu$. If $f(A) < +\infty$, let $E^k = A \setminus \bigcup_{\nu=1}^k A^\nu$ for each k. Then, $E^\nu \downarrow \emptyset$ and $\mu(E^\nu \triangle \emptyset) = \mu(E^\nu) \to 0$. Because f is μ-continuous, we obtain $\lim_\nu f(E^\nu) = f(\emptyset) = 0$. By letting $k \to \infty$ in both sides of the equality $f(E^k) = f(A) - \sum_{\nu=1}^k f(A^\nu)$, we obtain $f(A) = \sum_{\nu=1}^\infty f(A^\nu)$. Given $f(A) = +\infty$, we let $E^k = \bigcup_{\nu=1}^k A^\nu$ for each k. Then, $E^\nu \uparrow A$ and $\mu(E^\nu \triangle A) = \mu(A \setminus E^\nu) \to 0$. Because f is μ-continuous, we obtain $\lim_\nu f(E^\nu) = f(A)$. By letting $k \to \infty$ in both sides of the equality $f(E^k) = \sum_{\nu=1}^k f(A^\nu)$, we obtain $f(A) = \sum_{\nu=1}^\infty f(A^\nu) = +\infty$.

(iii) μ-continuity follows from Theorems 3.9 and 3.15 and countable additivity follows from (ii) above. □

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Nobusumi Sagara
Faculty of Economics, Hosei University
4342, Aihara, Machida, Tokyo 194–0298, Japan
E-mail address: `nsagara@hosei.ac.jp`

Milan Vlach
School of Mathematics and Physics, Charles University
Malostranské náměstí 25, 118 00 Praha 1, Czech Republic
E-mail address: `milan.vlach@mfff.cuni.cz`