Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 6, No. 1, pp. 75–87, January 2010

CHARACTERIZING LOCALLY EFFICIENT SOLUTIONS OF FUZZY MULTICRITERIA LOCATION PROBLEMS WITH RECTILINEAR NORM

Masamichi Kon

Abstract: A fuzzy multicriteria location problem with rectilinear norm on the plane is considered. We give some properties of efficient and locally efficient solutions of the problem, and give characterizations of locally efficient solutions of the problem. Such characterizations are then used to readily find all locally efficient solutions of the problem.

Key words: *fuzzy location problem, multicriteria problem, rectilinear norm, efficiency*

Mathematics Subject Classification: *90B85*

1 Preliminaries

In a general continuous location model, finitely many points called demand points in $\mathbb{R}^2$, modeling existing facilities or customers, are given. Let $\boldsymbol{d}_i \equiv (a_i, b_i) \in \mathbb{R}^2$, $i = 1, 2, \cdots, n (\geq 2)$ be distinct demand points. We put $I \equiv \{1, 2, \cdots, n\}$ and $D \equiv \{\boldsymbol{d}_i \colon i \in I\}$. Then a problem to locate a new facility in $\mathbb{R}^2$ is called a single facility location problem. If one prefers the location of the facility near demand points, then the problem is formulated as follows:

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} g(\gamma_1(\boldsymbol{x} - \boldsymbol{d}_1), \gamma_2(\boldsymbol{x} - \boldsymbol{d}_2), \cdots, \gamma_n(\boldsymbol{x} - \boldsymbol{d}_n))$$

where $\boldsymbol{x} \in \mathbb{R}^2$ is the variable location of the facility. It is often assumed that $g \colon \mathbb{R}^n \to \mathbb{R}$ is non-decreasing and convex or that $g \colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $g(\boldsymbol{z}) = \boldsymbol{z}$ for all $\boldsymbol{z} \in \mathbb{R}^n$. It is also often assumed that $\gamma_i \colon \mathbb{R}^2 \to \mathbb{R}$, $i \in I$ are norms or gauges, and each $\gamma_i(\boldsymbol{x} - \boldsymbol{d}_i)$, $i \in I$ represents the distance from $\boldsymbol{d}_i$ to $\boldsymbol{x}$. In this paper, it is assumed that all γ_i, $i \in I$ are the same rectilinear norm $\|\cdot\|_1$. See [2, 5] for gauges. Then *a multicriteria location problem* (MCP) is formulated as follows:

$$\min_{\boldsymbol{x} \in \mathbb{R}^2} \boldsymbol{f}(\boldsymbol{x}) \equiv (\|\boldsymbol{x} - \boldsymbol{d}_1\|_1, \|\boldsymbol{x} - \boldsymbol{d}_2\|_1, \cdots, \|\boldsymbol{x} - \boldsymbol{d}_n\|_1).$$

For example, MCP is considered in [1, 4, 6].

Definition 1.1. (i) A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *an efficient solution* of MCP if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$ and $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{f}(\boldsymbol{x}_0)$, and is called *a locally efficient solution* of MCP if for some $\varepsilon > 0$, there is no $\boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}_0)$ such that $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$ and $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{f}(\boldsymbol{x}_0)$, where $N_\varepsilon(\boldsymbol{x}_0) \equiv \{\boldsymbol{y} \in \mathbb{R}^2\colon \|\boldsymbol{y} - \boldsymbol{x}_0\|_2 < \varepsilon\}$ and $\|\cdot\|_2$ is Euclidean norm. Let E(D) and LE(D) be sets of all efficient and locally efficient solutions of MCP, respectively.

(ii) A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *a strictly efficient solution* of MCP if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{x} \neq \boldsymbol{x}_0$ and $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$, and is called *a locally strictly efficient solution* of MCP if for some $\varepsilon > 0$, there is no $\boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}_0)$ such that $\boldsymbol{x} \neq \boldsymbol{x}_0$ and $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x}_0)$. Let SE($D$) and LSE($D$) be sets of all strictly and locally strictly efficient solutions of MCP, respectively.

(iii) A point $\boldsymbol{x}_0 \in \mathrm{E}(D) \setminus \mathrm{SE}(D)$ is called *an alternately efficient solution* of MCP, and $\boldsymbol{x}_0 \in \mathrm{LE}(D) \setminus \mathrm{LSE}(D)$ is called *a locally alternately efficient solution* of MCP. We put $\mathrm{AE}(D) \equiv \mathrm{E}(D) \setminus \mathrm{SE}(D)$ and $\mathrm{LAE}(D) \equiv \mathrm{LE}(D) \setminus \mathrm{LSE}(D)$.

(iv) A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *a quasiefficient solution* of MCP if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{f}(\boldsymbol{x}_0)$, and is called *a locally quasiefficient solution* of MCP if for some $\varepsilon > 0$, there is no $\boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}_0)$ such that $\boldsymbol{f}(\boldsymbol{x}) < \boldsymbol{f}(\boldsymbol{x}_0)$. Let QE($D$) and LQE($D$) be sets of all quasiefficient and locally quasiefficient solutions of MCP, respectively.

From Definition 1.1 and the definition of $\boldsymbol{f}$, $D \subseteq \mathrm{SE}(D) \subseteq \mathrm{E}(D) \subseteq \mathrm{QE}(D)$. Since each $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1$, $i \in I$ is convex in $\boldsymbol{x} \in \mathbb{R}^2$, $\mathrm{SE}(D) = \mathrm{LSE}(D)$, $\mathrm{AE}(D) = \mathrm{LAE}(D)$, $\mathrm{E}(D) = \mathrm{LE}(D)$ and $\mathrm{QE}(D) = \mathrm{LQE}(D)$.

Formulation of MCP is natural if one prefers the location of the facility near demand points. However, for the location of the facility, degrees of satisfaction with respect to demand points may be different even if distances from demand points to the facility are the same. Furthermore, for example, if the facility is an airport, then one may not prefer the location of the facility near demand points because of the noise. In order to deal with such situations, we consider membership functions, which represent degrees of satisfaction for the location of the facility with respect to demand points, and a maximization problem with an objective function involving membership functions. Membership functions come from fuzzy set theory which was first proposed in [7]. It is assumed that membership functions $\mu_i\colon \mathbb{R} \to [0,1] \equiv \{x \in \mathbb{R}\colon 0 \leq x \leq 1\}$, $i \in I$ are given. For each $\boldsymbol{x} \in \mathbb{R}^2$ and $i \in I$, $\mu_i(\|\boldsymbol{x} - \boldsymbol{d}_i\|_1)$ represents the degree of satisfaction for the location $\boldsymbol{x}$ with respect to the demand point $\boldsymbol{d}_i$. Throughout this paper, it is assumed that for each $\mu_i, i \in I$, (i) $\mu_i(x) = 0$ for $x < 0$, (ii) $\mu_i(m_i) = 1$ for some $m_i \geq 0$ and (iii) μ_i is strictly increasing on $[0, m_i]$ and strictly decreasing on $[m_i, \infty) \equiv \{x \in \mathbb{R} : x \geq m_i\}$. Then *a fuzzy multicriteria location problem* (FMCP) is formulated as follows:

$$\max_{\boldsymbol{x} \in \mathbb{R}^2} \boldsymbol{\mu}(\boldsymbol{x}) \equiv (\mu_1(\|\boldsymbol{x} - \boldsymbol{d}_1\|_1), \mu_2(\|\boldsymbol{x} - \boldsymbol{d}_2\|_1), \cdots, \mu_n(\|\boldsymbol{x} - \boldsymbol{d}_n\|_1)).$$

For example, FMCP is considered in [3].

Definition 1.2. (i) A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *an efficient solution* of FMCP if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{x}_0)$ and $\boldsymbol{\mu}(\boldsymbol{x}) \neq \boldsymbol{\mu}(\boldsymbol{x}_0)$, and is called *a locally efficient solution* of FMCP if for some $\varepsilon > 0$, there is no $\boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}_0)$ such that $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{x}_0)$ and $\boldsymbol{\mu}(\boldsymbol{x}) \neq \boldsymbol{\mu}(\boldsymbol{x}_0)$. Let FE($D$) and FLE($D$) be sets of all efficient and locally efficient solutions of FMCP, respectively.

(ii) A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *a strictly efficient solution* of FMCP if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{x} \neq \boldsymbol{x}_0$ and $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{x}_0)$, and is called *a locally strictly efficient solution* of FMCP if for some $\varepsilon > 0$, there is no $\boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}_0)$ such that $\boldsymbol{x} \neq \boldsymbol{x}_0$ and $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{x}_0)$. Let FSE($D$) and FLSE($D$) be sets of all strictly and locally strictly efficient solutions of FMCP, respectively.

(iii) A point $\boldsymbol{x}_0 \in \mathrm{FE}(D) \setminus \mathrm{FSE}(D)$ is called *an alternately efficient solution* of FMCP, and

$\boldsymbol{x}_0 \in \mathrm{FLE}(D) \setminus \mathrm{FLSE}(D)$ is called *a locally alternately efficient solution* of FMCP. We put $\mathrm{FAE}(D) \equiv \mathrm{FE}(D) \setminus \mathrm{FSE}(D)$ and $\mathrm{FLAE}(D) \equiv \mathrm{FLE}(D) \setminus \mathrm{FLSE}(D)$.

(iv) A point $\boldsymbol{x}_0 \in \mathbb{R}^2$ is called *a quasiefficient solution* of FMCP if there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{\mu}(\boldsymbol{x}) > \boldsymbol{\mu}(\boldsymbol{x}_0)$, and is called *a locally quasiefficient solution* of FMCP if for some $\varepsilon > 0$, there is no $\boldsymbol{x} \in N_\varepsilon(\boldsymbol{x}_0)$ such that $\boldsymbol{\mu}(\boldsymbol{x}) > \boldsymbol{\mu}(\boldsymbol{x}_0)$. Let $\mathrm{FQE}(D)$ and $\mathrm{FLQE}(D)$ be sets of all quasiefficient and locally quasiefficient solutions of FMCP, respectively.

From Definition 1.2, $\mathrm{FSE}(D) \subseteq \mathrm{FE}(D) \subseteq \mathrm{FQE}(D)$ and $\mathrm{FLSE}(D) \subseteq \mathrm{FLE}(D) \subseteq \mathrm{FLQE}(D)$.

In this paper, the fuzzy multicriteria location problem with rectilinear norm on the plane is considered. We give some properties of efficient and locally efficient solutions of the problem, and give characterizations of locally efficient solutions of the problem. Such characterizations are then used to readily find all locally efficient solutions of the problem.

2 Efficiency and Local Efficiency for FMCP

In this section, we give some properties of efficient and locally efficient solutions of FMCP.

We put

$$B_i(r) \equiv \{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 \leq r\}$$

for $r \geq 0$ and $i \in I$, and put

$$B \equiv \bigcup_{i \in I} B_i(m_i)$$

where $m_i, i \in I$ are the same as the ones used in membership functions $\mu_i, i \in I$. Let $B_i^0(r)$ and $\partial B_i(r)$ be the interior and the boundary of $B_i(r)$, respectively. For $i \in I$, we define $\overline{\mu}_i : \mathbb{R}^2 \to [0,1]$ as follows:

$$\overline{\mu}_i(\boldsymbol{x}) \equiv \mu_i(\|\boldsymbol{x} - \boldsymbol{d}_i\|_1), \quad \boldsymbol{x} \in \mathbb{R}^2.$$

For $\alpha \in [0,1]$ and $i \in I$, we put

$$\begin{array}{ll} [\mu_i]_{\geq}(\alpha) \equiv \{x \in \mathbb{R} : \mu_i(x) \geq \alpha\}, & [\mu_i]_{>}(\alpha) \equiv \{x \in \mathbb{R} : \mu_i(x) > \alpha\}, \\ [\overline{\mu}_i]_{\geq}(\alpha) \equiv \{\boldsymbol{x} \in \mathbb{R}^2 : \overline{\mu}_i(\boldsymbol{x}) \geq \alpha\}, & [\overline{\mu}_i]_{>}(\alpha) \equiv \{\boldsymbol{x} \in \mathbb{R}^2 : \overline{\mu}_i(\boldsymbol{x}) > \alpha\}. \end{array}$$

Proposition 2.1. *Assume that all $\mu_i, i \in I$ are upper semicontinuous on $[0,\infty)$. Then there exists an efficient solution of FMCP, namely, $FE(D) \neq \emptyset$.*

Proof. For each $i \in I$, since μ_i is upper semicontinuous on $[0,\infty)$ and $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1$ is continuous in $\boldsymbol{x} \in \mathbb{R}^2$, $\overline{\mu}_i$ is upper semicontinuous on $\mathbb{R}^2$. Put $S_1 \equiv \partial B_1(m_1)$. Since S_1 is a nonempty compact set, we can define nonempty compact sets $S_2, \cdots, S_n$ such that $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n$ as

$$S_i \equiv \left\{\boldsymbol{x} \in S_{i-1} : \overline{\mu}_i(\boldsymbol{x}) = \max_{\boldsymbol{y} \in S_{i-1}} \overline{\mu}_i(\boldsymbol{y})\right\}, \quad i = 2, \cdots, n$$

from Weierstrass's theorem.

For $\boldsymbol{x}_0 \in S_n$, suppose that $\boldsymbol{x}_0 \notin \mathrm{FE}(D)$ in order to show that $\boldsymbol{x}_0 \in \mathrm{FE}(D)$. Then there exists $\overline{\boldsymbol{x}} \in \mathbb{R}^2$ such that $\boldsymbol{\mu}(\overline{\boldsymbol{x}}) \geq \boldsymbol{\mu}(\boldsymbol{x}_0)$ and $\boldsymbol{\mu}(\overline{\boldsymbol{x}}) \neq \boldsymbol{\mu}(\boldsymbol{x}_0)$. Put

$$i_0 \equiv \min\{i \in I : \overline{\mu}_i(\overline{\boldsymbol{x}}) > \overline{\mu}_i(\boldsymbol{x}_0)\}.$$

Since

$$1 \geq \overline{\mu}_1(\overline{\boldsymbol{x}}) \geq \overline{\mu}_1(\boldsymbol{x}_0) = 1,$$

$i_0 \geq 2$. Since

$$\overline{\mu}_i(\overline{\boldsymbol{x}}) = \overline{\mu}_i(\boldsymbol{x}_0), \quad i = 1, 2, \cdots, i_0 - 1,$$

$\overline{\boldsymbol{x}} \in S_{i_0-1}$ from the definition of S_{i_0-1}. In this case, $\boldsymbol{x}_0 \in S_{i_0-1}$. Since $\overline{\mu}_{i_0}(\overline{\boldsymbol{x}}) > \overline{\mu}_{i_0}(\boldsymbol{x}_0)$, $\boldsymbol{x}_0 \notin S_{i_0}$ from the definition of S_{i_0}. This contradicts that $\boldsymbol{x}_0 \in S_n \subseteq S_{i_0}$. Therefore, $\boldsymbol{x}_0 \in \mathrm{FE}(D)$. □

The following proposition shows that FMCP is reduced to MCP when $m_i = 0, i \in I$.

Proposition 2.2. *If $m_i = 0, i \in I$, then the following statements hold.*

(i) *FLE(D) = FE(D) = E(D).*

(ii) *FLSE(D) = FSE(D) = SE(D).*

(iii) *FLAE(D) = FAE(D) = AE(D).*

(iv) *FLQE(D) = FQE(D) = QE(D).*

Proof. Since each μ_i, $i \in I$ is strictly decreasing on $[0, \infty)$, for each $i \in I$ and $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^2$, $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1 \leq \|\boldsymbol{y} - \boldsymbol{d}_i\|_1$ if and only if $\overline{\mu}_i(\boldsymbol{x}) \geq \overline{\mu}_i(\boldsymbol{y})$, and $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1 < \|\boldsymbol{y} - \boldsymbol{d}_i\|_1$ if and only if $\overline{\mu}_i(\boldsymbol{x}) > \overline{\mu}_i(\boldsymbol{y})$. Thus from Definition 1.1 and 1.2, we have

$$\mathrm{FE}(D) = \mathrm{E}(D), \quad \mathrm{FSE}(D) = \mathrm{SE}(D), \quad \mathrm{FAE}(D) = \mathrm{AE}(D), \quad \mathrm{FQE}(D) = \mathrm{QE}(D)$$

and

$$\mathrm{FLE}(D) = \mathrm{LE}(D),\ \mathrm{FLSE}(D) = \mathrm{LSE}(D),\ \mathrm{FLAE}(D) = \mathrm{LAE}(D),\ \mathrm{FLQE}(D) = \mathrm{LQE}(D).$$

Since

$$\mathrm{E}(D) = \mathrm{LE}(D), \quad \mathrm{SE}(D) = \mathrm{LSE}(D), \quad \mathrm{AE}(D) = \mathrm{LAE}(D), \quad \mathrm{QE}(D) = \mathrm{LQE}(D),$$

we have the conclusion. □

Proposition 2.3. *For $\boldsymbol{x} \in \mathbb{R}^2$, if $\boldsymbol{x} \notin B$, then the following statements hold.*

(i) $\boldsymbol{x} \in \mathit{FLE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{FE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{E}(D)$.

(ii) $\boldsymbol{x} \in \mathit{FLSE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{FSE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{SE}(D)$.

(iii) $\boldsymbol{x} \in \mathit{FLAE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{FAE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{AE}(D)$.

(iv) $\boldsymbol{x} \in \mathit{FLQE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{FQE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathit{QE}(D)$.

Proof. Suppose that $\boldsymbol{x} \notin B$. Since $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1 > m_i$, $i \in I$, for sufficiently small $\delta > 0$, each μ_i, $i \in I$ is strictly decreasing on $\{x \in \mathbb{R}\colon \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 - \delta < x < \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 + \delta\}$. For each $i \in I$ and sufficiently small $\varepsilon > 0$, since $\|\boldsymbol{y} - \boldsymbol{d}_i\|_1$ is continuous at $\boldsymbol{y} = \boldsymbol{x}$, $\|\boldsymbol{x} - \boldsymbol{d}_i\|_1 - \delta < \|\boldsymbol{y} - \boldsymbol{d}_i\|_1 < \|\boldsymbol{x} - \boldsymbol{d}_i\|_1 + \delta$ for any $\boldsymbol{y} \in N_\varepsilon(\boldsymbol{x})$. Thus for each $i \in I$ and any $\boldsymbol{y} \in N_\varepsilon(\boldsymbol{x})$, $\|\boldsymbol{y} - \boldsymbol{d}_i\|_1 \leq \|\boldsymbol{x} - \boldsymbol{d}_i\|_1$ if and only if $\overline{\mu}_i(\boldsymbol{y}) \geq \overline{\mu}_i(\boldsymbol{x})$, and $\|\boldsymbol{y} - \boldsymbol{d}_i\|_1 < \|\boldsymbol{x} - \boldsymbol{d}_i\|_1$ if and only if $\overline{\mu}_i(\boldsymbol{y}) > \overline{\mu}_i(\boldsymbol{x})$. Therefore, from Definition 1.1 and 1.2, we have

$$\begin{gathered}
\boldsymbol{x} \in \mathrm{FLE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathrm{LE}(D) = \mathrm{E}(D), \\
\boldsymbol{x} \in \mathrm{FLSE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathrm{LSE}(D) = \mathrm{SE}(D), \\
\boldsymbol{x} \in \mathrm{FLAE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathrm{LAE}(D) = \mathrm{AE}(D), \\
\boldsymbol{x} \in \mathrm{FLQE}(D) \overset{\text{iff}}{\Leftrightarrow} \boldsymbol{x} \in \mathrm{LQE}(D) = \mathrm{QE}(D).
\end{gathered}$$

It is trivial that

$$\begin{gathered}\boldsymbol{x} \in \mathrm{FE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FLE}(D),\\ \boldsymbol{x} \in \mathrm{FSE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FLSE}(D),\\ \boldsymbol{x} \in \mathrm{FAE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FLAE}(D),\\ \boldsymbol{x} \in \mathrm{FQE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FLQE}(D).\end{gathered}$$

We shall show that

$$\begin{array}{ll}\text{(a)} & \boldsymbol{x} \in \mathrm{FLE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FE}(D),\\ \text{(b)} & \boldsymbol{x} \in \mathrm{FLSE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FSE}(D),\\ \text{(c)} & \boldsymbol{x} \in \mathrm{FLAE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FAE}(D),\\ \text{(d)} & \boldsymbol{x} \in \mathrm{FLQE}(D) \Rightarrow \boldsymbol{x} \in \mathrm{FQE}(D).\end{array}$$

(a) Suppose that $\boldsymbol{x} \in \mathrm{FLE}(D)$. In order to show that $\boldsymbol{x} \in \mathrm{FE}(D)$, suppose that $\boldsymbol{x} \notin \mathrm{FE}(D)$. Then there exists $\boldsymbol{y} \in \mathbb{R}^2$ such that $\boldsymbol{\mu}(\boldsymbol{y}) \geq \boldsymbol{\mu}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{y}) \neq \boldsymbol{\mu}(\boldsymbol{x})$. Since

$$\boldsymbol{y} \in \bigcap_{i\in I} [\overline{\mu}_i]_{\geq} (\overline{\mu}_i(\boldsymbol{x})) \subseteq \bigcap_{i\in I} B_i(\|\boldsymbol{x} - \boldsymbol{d}_i\|_1)$$

and

$$\boldsymbol{y} \in \left[\overline{\mu}_j\right]_{>} (\overline{\mu}_j(\boldsymbol{x})) \subseteq B_j^0(\|\boldsymbol{x} - \boldsymbol{d}_j\|_1)$$

for some $j \in I$, $\boldsymbol{x} \notin \mathrm{E}(D)$. This contradicts that $\boldsymbol{x} \in \mathrm{E}(D)$ from the first part of this proof. Therefore, $\boldsymbol{x} \in \mathrm{FE}(D)$.

(b) Suppose that $\boldsymbol{x} \in \mathrm{FLSE}(D)$. In order to show that $\boldsymbol{x} \in \mathrm{FSE}(D)$, suppose that $\boldsymbol{x} \notin \mathrm{FSE}(D)$. Then there exists $\boldsymbol{y} \in \mathbb{R}^2$ such that $\boldsymbol{y} \neq \boldsymbol{x}$ and $\boldsymbol{\mu}(\boldsymbol{y}) \geq \boldsymbol{\mu}(\boldsymbol{x})$. Since

$$\boldsymbol{y} \in \bigcap_{i\in I} [\overline{\mu}_i]_{\geq} (\overline{\mu}_i(\boldsymbol{x})) \subseteq \bigcap_{i\in I} B_i(\|\boldsymbol{x} - \boldsymbol{d}_i\|_1),$$

$\boldsymbol{x} \notin \mathrm{SE}(D)$. This contradicts that $\boldsymbol{x} \in \mathrm{SE}(D)$ from the first part of this proof. Therefore, $\boldsymbol{x} \in \mathrm{FSE}(D)$.

(c) Suppose that $\boldsymbol{x} \in \mathrm{FLAE}(D)$. In order to show that $\boldsymbol{x} \in \mathrm{FAE}(D)$, suppose that $\boldsymbol{x} \notin \mathrm{FAE}(D)$. Since $\boldsymbol{x} \in \mathrm{FLAE}(D) \subseteq \mathrm{FLE}(D)$, $\boldsymbol{x} \in \mathrm{FE}(D)$ from (a). Since $\boldsymbol{x} \notin \mathrm{FAE}(D)$, $\boldsymbol{x} \in \mathrm{FSE}(D) \subseteq \mathrm{FLSE}(D)$ from Definition 1.2 (iii). Again from Definition 1.2 (iii), $\boldsymbol{x} \notin \mathrm{FLAE}(D)$. This contradicts that $\boldsymbol{x} \in \mathrm{FLAE}(D)$. Therefore, $\boldsymbol{x} \in \mathrm{FAE}(D)$.

(d) Suppose that $\boldsymbol{x} \in \mathrm{FLQE}(D)$. In order to show that $\boldsymbol{x} \in \mathrm{FQE}(D)$, suppose that $\boldsymbol{x} \notin \mathrm{FQE}(D)$. Then there exists $\boldsymbol{y} \in \mathbb{R}^2$ such that $\boldsymbol{\mu}(\boldsymbol{y}) > \boldsymbol{\mu}(\boldsymbol{x})$. Since

$$\boldsymbol{y} \in \bigcap_{i\in I} [\overline{\mu}_i]_{>} (\overline{\mu}_i(\boldsymbol{x})) \subseteq \bigcap_{i\in I} B_i^0(\|\boldsymbol{x} - \boldsymbol{d}_i\|_1),$$

$\boldsymbol{x} \notin \mathrm{QE}(D)$. This contradicts that $\boldsymbol{x} \in \mathrm{QE}(D)$ from the first part of this proof. Therefore, $\boldsymbol{x} \in \mathrm{FQE}(D)$. □

$\mathrm{E}(D)$, $\mathrm{SE}(D)$ and $\mathrm{AE}(D)$ can be determined by using algorithms in [1, 4]. From Proposition 2 in [4], $\mathrm{QE}(D) = \{(x, y) \in \mathbb{R}^2\colon \min\{a_i\colon i \in I\} \leq x \leq \max\{a_i\colon i \in I\}, \min\{b_i\colon i \in I\} \leq y \leq \max\{b_i\colon i \in I\}\}$. In the case $\boldsymbol{x} \in B$, even if $\boldsymbol{x} \in \mathrm{FLE}(D) \subseteq \mathrm{FLQE}(D)$, $\boldsymbol{x} \notin \mathrm{FE}(D)$ and $\boldsymbol{x} \notin \mathrm{FQE}(D)$ generally.

3 Local Efficiency for FMCP and Summary Diagrams

In this section, we introduce the concept of the summary diagram, and give characterizations of locally efficient solutions of FMCP. Such characterizations are then used to readily find all locally efficient solutions of FMCP.

Following [6], we introduce the concept of the summary diagram in order to check that a given point in $\mathbb{R}^2$ is locally efficient or strictly efficient or quasiefficient in FMCP or not. In [6], the summary diagram is introduced for multicriteria location problems with one-infinity norm in $\mathbb{R}^2$. Roughly speaking, the summary diagram represents locations of demand points from a given point $\boldsymbol{x} \in \mathbb{R}^2$ (or equivalently the location of $\boldsymbol{x}$ from demand points), and the summary diagram is very useful because the local efficiency of $\boldsymbol{x}$ can be determined easily by using the summary diagram as shown later in this section. We put

$$\begin{aligned} O_1 &\equiv \{(x,0) \in \mathbb{R}^2 : x \geq 0\}, \\ O_2 &\equiv \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}, \\ O_3 &\equiv \{(0,y) \in \mathbb{R}^2 : y \geq 0\}, \\ O_4 &\equiv \{(x,y) \in \mathbb{R}^2 : x < 0, y > 0\} \end{aligned}$$

and

$$O_{-j} \equiv -O_j, \quad j = 1,2,3,4$$

and

$$O_j(\boldsymbol{x}) \equiv \boldsymbol{x} + O_j, \quad j = \pm 1, \pm 2, \pm 3, \pm 4$$

for $\boldsymbol{x} \in \mathbb{R}^2$.

For $\boldsymbol{x} \in \mathbb{R}^2$ with $\boldsymbol{x} \notin D$, we put

$$I_1 \equiv \{i \in I : \boldsymbol{x} \in B_i(m_i)\}, \quad I_2 \equiv \{i \in I : \boldsymbol{x} \in (B_i^0(m_i))^c\}$$

and

$$\begin{aligned} J_1 &\equiv \{j \in \{\pm 2, \pm 4\} : \boldsymbol{x} \in O_j(\boldsymbol{d}_i) \text{ for some } i \in I_1\}, \\ J_2 &\equiv -\{j \in \{\pm 2, \pm 4\} : \boldsymbol{x} \in \overline{O}_j(\boldsymbol{d}_i) \text{ for some } i \in I_2\}, \\ J_3 &\equiv -\{j \in \{\pm 1, \pm 3\} : \boldsymbol{x} \in O_j(\boldsymbol{d}_i) \text{ for some } i \in I_1\} \end{aligned}$$

where $(B_i^0(m_i))^c$ is the complement of $B_i^0(m_i)$ and $\overline{O}_j(\boldsymbol{d}_i)$ is the closure of $O_j(\boldsymbol{d}_i)$. Then $\mathrm{SD}(\boldsymbol{x}) \equiv J_1 \cup J_2 \cup J_3$ is called *the summary diagram* of $\boldsymbol{x}$. Conveniently, $\mathrm{SD}(\boldsymbol{x})$ is represented in diagram form as follows: First, we put $\boldsymbol{v}_1 \equiv (1,0)$, $\boldsymbol{v}_2 \equiv (1,1)$, $\boldsymbol{v}_3 \equiv (0,1)$, $\boldsymbol{v}_4 \equiv (-1,1)$ and $\boldsymbol{v}_{-j} \equiv -\boldsymbol{v}_j$, $j = 1, 2, 3, 4$; Next, for each $j \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$, dot $\boldsymbol{v}_j$ if $j \in \mathrm{SD}(\boldsymbol{x})$.

Example. We set $\boldsymbol{d}_1 = (1,5)$, $\boldsymbol{d}_2 = (3,3)$, $\boldsymbol{d}_3 = (4,0)$, $\boldsymbol{d}_4 = (0,1)$ $\boldsymbol{x} = (0,3)$ and $m_1 = 3$, $m_2 = 4$, $m_3 = m_4 = 2$, and consider the summary diagram of $\boldsymbol{x}$, $\mathrm{SD}(\boldsymbol{x})$. Since $\|\boldsymbol{x} - \boldsymbol{d}_1\|_1 = 3 = m_1 = 3$, $\|\boldsymbol{x} - \boldsymbol{d}_2\|_1 = 3 \leq m_2 = 4$, $\|\boldsymbol{x} - \boldsymbol{d}_3\|_1 = 7 \geq m_3 = 2$, $\|\boldsymbol{x} - \boldsymbol{d}_4\|_1 = 2 = m_4 = 2$, we have $\boldsymbol{x} \in B_1(m_1) \cap (B_1^0(m_1))^c$, $\boldsymbol{x} \in B_2(m_2)$, $\boldsymbol{x} \in (B_3^0(m_3))^c$, $\boldsymbol{x} \in B_4(m_4) \cap (B_4^0(m_4))^c$. Thus we have $I_1 = \{1,2,4\}$, $I_2 = \{1,3,4\}$. Since $1 \in I_1$, $\boldsymbol{x} \in O_{-2}(\boldsymbol{d}_1)$, we have $J_1 = \{-2\}$. Since $1 \in I_2$, $\boldsymbol{x} \in \overline{O}_{-2}(\boldsymbol{d}_1)$, $3 \in I_2$, $\boldsymbol{x} \in \overline{O}_4(\boldsymbol{d}_3)$, $4 \in I_2$, $\boldsymbol{x} \in \overline{O}_2(\boldsymbol{d}_4) \cap \overline{O}_4(\boldsymbol{d}_4)$, we have $J_2 = \{2,-4,-2\}$. Since $2 \in I_1$, $\boldsymbol{x} \in O_{-1}(\boldsymbol{d}_2)$, $4 \in I_1$, $\boldsymbol{x} \in O_3(\boldsymbol{d}_4)$, we have $J_3 = \{1,-3\}$. Therefore, we have

$$\mathrm{SD}(\boldsymbol{x}) = \{-4,-3,-2,1,2\}.$$

Fig.1 shows its summary diagram in diagram form. We put

$$U_\varepsilon(\boldsymbol{x}) \equiv \left(\bigcap_{i \in I} [\overline{\mu}_i]_{\geq} (\overline{\mu}_i(\boldsymbol{x})) \right) \cap N_\varepsilon(\boldsymbol{x}), \quad W_\varepsilon(\boldsymbol{x}) \equiv \left(\bigcap_{i \in I} [\overline{\mu}_i]_{>} (\overline{\mu}_i(\boldsymbol{x})) \right) \cap N_\varepsilon(\boldsymbol{x})$$

for $\varepsilon > 0$. The summary diagram of $\boldsymbol{x}$, $\mathrm{SD}(\boldsymbol{x})$, or the summary diagram of $\boldsymbol{x}$ in diagram form is closely related with sets $U_\varepsilon(\boldsymbol{x})$ and $W_\varepsilon(\boldsymbol{x})$ for sufficiently small $\varepsilon > 0$, and the local efficiency of $\boldsymbol{x}$ can be determined by using these sets. We put

$$V_j \equiv \{\boldsymbol{y} \in \mathbb{R}^2 : \langle \boldsymbol{v}_j, \boldsymbol{y} \rangle \geq 0\}, \quad V_j^0 \equiv \{\boldsymbol{y} \in \mathbb{R}^2 : \langle \boldsymbol{v}_j, \boldsymbol{y} \rangle > 0\}$$

for $j \in \mathrm{SD}(\boldsymbol{x}) \cap \{\pm 2, \pm 4\} = \{-4, -2, 2\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product, and

$$V_j \equiv \{\boldsymbol{y} \in \mathbb{R}^2 : \|\boldsymbol{y} - \boldsymbol{v}_j\|_1 \geq 1\}, \quad V_j^0 \equiv \{\boldsymbol{y} \in \mathbb{R}^2 : \|\boldsymbol{y} - \boldsymbol{v}_j\|_1 > 1\}$$

for $j \in \mathrm{SD}(\boldsymbol{x}) \cap \{\pm 1, \pm 3\} = \{-3, 1\}$. From $\mathrm{SD}(\boldsymbol{x})$ or $\mathrm{SD}(\boldsymbol{x})$ in diagram form, we have

$$U_\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \left(\bigcap_{j \in \mathrm{SD}(\boldsymbol{x})} V_j \right) \cap N_\varepsilon(\boldsymbol{0}) = \boldsymbol{x} + \left\{ \boldsymbol{y} \in \mathbb{R}^2 : \boldsymbol{y} = \frac{\lambda}{\sqrt{2}} \boldsymbol{v}_{-4}, \lambda \in [0, \varepsilon] \right\}$$

and

$$W_\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \left(\bigcap_{j \in \mathrm{SD}(\boldsymbol{x})} V_j^0 \right) \cap N_\varepsilon(\boldsymbol{0}) = \emptyset$$

for sufficiently small $\varepsilon > 0$. Since $W_\varepsilon(\boldsymbol{x}) = \emptyset$ for sufficiently small $\varepsilon > 0$, we have $\boldsymbol{x} \in \mathrm{FLQE}(D)$. On the other hand, since $U_\varepsilon(\boldsymbol{x}) \cap ([\overline{\mu}_3]_> (\overline{\mu}_3(\boldsymbol{x})) \cap N_\varepsilon(\boldsymbol{x})) \neq \emptyset$ for any $\varepsilon > 0$, we have $\boldsymbol{x} \notin \mathrm{FLE}(D)$.

For $\boldsymbol{x} \in \mathbb{R}^2$ with $\boldsymbol{x} \notin D$, $\mathrm{SD}(\boldsymbol{x})$ in diagram form as in Fig.1 is called the pattern of $\mathrm{SD}(\boldsymbol{x})$, where we identify patterns if they are the same pattern by rotation. Fig.2 shows important patterns in order to check that $\boldsymbol{x}$ is locally efficient or strictly efficient or quasiefficient in FMCP or not.

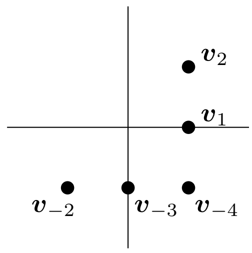
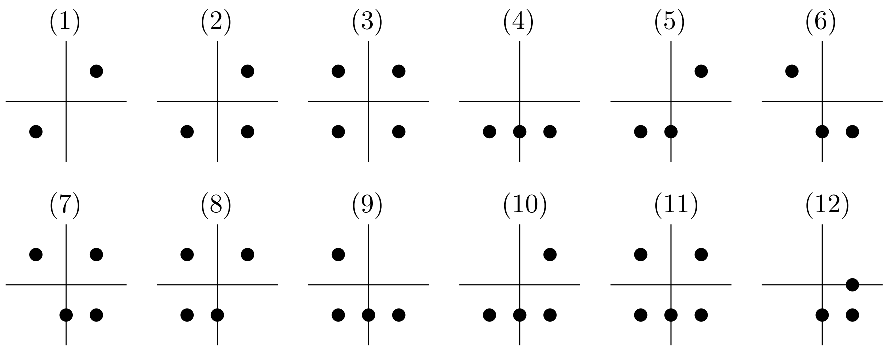


Figure 1: $\mathrm{SD}(\boldsymbol{x}) = \{-4, -3, -2, 1, 2\}$.



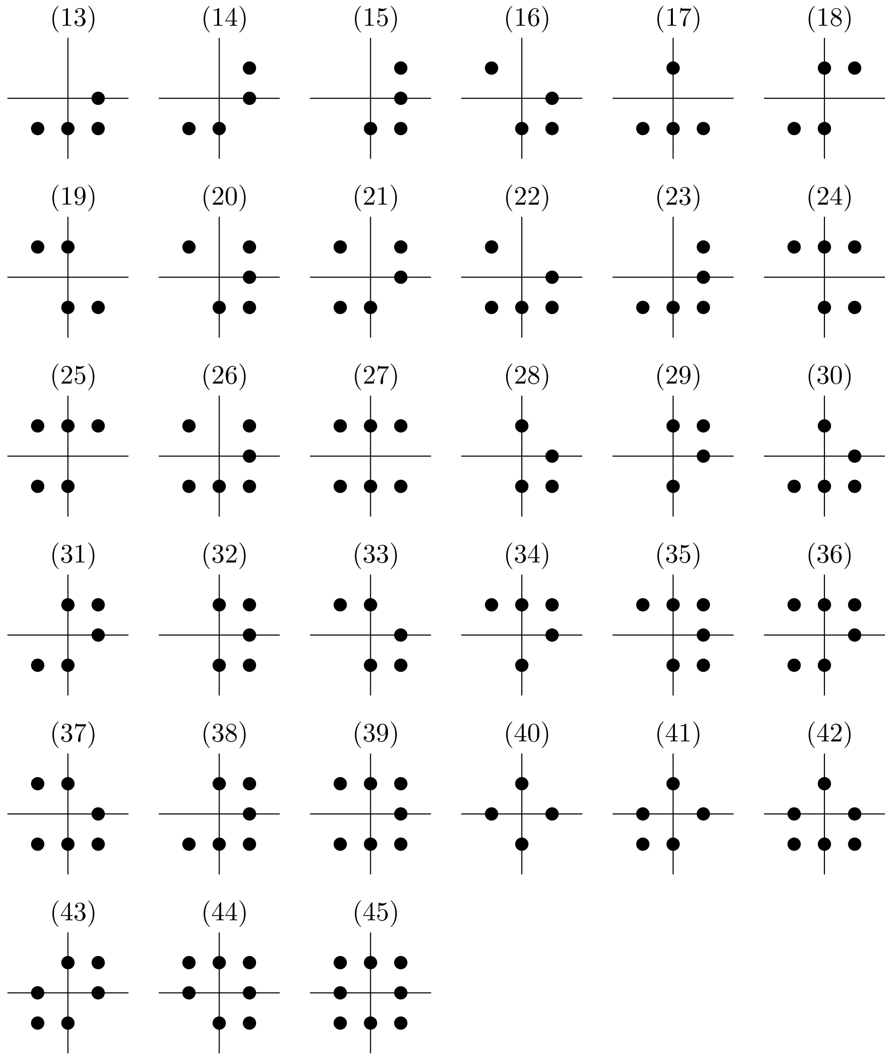


Figure 2: Patterns of summary diagrams.

Proposition 3.1. *Assume that* $\boldsymbol{x} \in \mathbb{R}^2$ *and* $\boldsymbol{x} \notin D$. *Then* $\boldsymbol{x} \in FLQE(D)$ *if and only if the pattern of* $SD(\boldsymbol{x})$ *coincides with one of patterns in Fig.2. If the pattern of* $SD(\boldsymbol{x})$ *coincides with one of patterns in Fig.2, then the following statements hold, where it is assumed that* $\boldsymbol{d}_i$, $i \in I$ *are rotated around* $\boldsymbol{x}$ *to fit the pattern of* $SD(\boldsymbol{x})$.

(i) $\boldsymbol{x} \in FLSE(D)$ *if and only if the pattern of* $SD(\boldsymbol{x})$ *coincides with one of patterns (3), (11), (26), (27), (39), (45) in Fig.2.*

(ii) *If the pattern of* $SD(\boldsymbol{x})$ *coincides with the pattern (1) in Fig.2, then* $\boldsymbol{x} \in FLAE(D)$.

(iii) *If the pattern of* $SD(\boldsymbol{x})$ *coincides with one of patterns (5)-(8), (12)-(22), (24), (25), (28)-(38), (40)-(44) in Fig.2, then* $\boldsymbol{x} \in FLQE(D) \setminus FLE(D)$.

(iv) *Assume that the pattern of* $SD(\boldsymbol{x})$ *coincides with one of patterns (2), (10), (23) in Fig.2. If there exists* $\boldsymbol{d}_i$, $i \in I$ *such that* $\boldsymbol{x} \in (O_4(\boldsymbol{d}_i) \setminus B_i(m_i)) \cup (O_{-4}(\boldsymbol{d}_i) \cap B_i^0(m_i))$, *then* $\boldsymbol{x} \in FLQE(D) \setminus FLE(D)$, *otherwise* $\boldsymbol{x} \in FLAE(D)$.

(v) *Assume that the pattern of SD*$(\boldsymbol{x})$ *coincides with the pattern (4) in Fig.2. If there exists* $\boldsymbol{d}_i$, $i \in I$ *such that* $\boldsymbol{x} \in (O_4(\boldsymbol{d}_i) \setminus B_i(m_i)) \cup (O_{-4}(\boldsymbol{d}_i) \cap B_i^0(m_i)) \cup (O_2(\boldsymbol{d}_i) \setminus B_i(m_i)) \cup (O_{-2}(\boldsymbol{d}_i) \cap B_i^0(m_i))$*, then* $\boldsymbol{x} \in$ *FLQE*$(D) \setminus$ *FLE*(D)*, otherwise* $\boldsymbol{x} \in$ *FLAE*(D).

(vi) *Assume that the pattern of SD*$(\boldsymbol{x})$ *coincides with the pattern (9) in Fig.2. If there exists* $\boldsymbol{d}_i$, $i \in I$ *such that* $\boldsymbol{x} \in (O_2(\boldsymbol{d}_i) \setminus B_i(m_i)) \cup (O_{-2}(\boldsymbol{d}_i) \cap B_i^0(m_i))$*, then* $\boldsymbol{x} \in$ *FLQE*$(D) \setminus$ *FLE*(D)*, otherwise* $\boldsymbol{x} \in$ *FLAE*(D).

Proof. Suppose that the pattern of SD$(\boldsymbol{x})$ coincides with one of patterns (3), (11), (26), (27), (39), (45) in Fig.2. Then for sufficiently small $\varepsilon > 0$, it can be seen easily that

$$\left(\bigcap_{i \in I} [\overline{\mu}_i]_{\geq} (\overline{\mu}_i(\boldsymbol{x})) \right) \cap N_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{x}\}.$$

Similarly, for the other patterns, investigating

$$\left(\bigcap_{i \in I} [\overline{\mu}_i]_{\geq} (\overline{\mu}_i(\boldsymbol{x})) \right) \cap N_\varepsilon(\boldsymbol{x}), \quad \left(\bigcap_{i \in I} [\overline{\mu}_i]_{>} (\overline{\mu}_i(\boldsymbol{x})) \right) \cap N_\varepsilon(\boldsymbol{x})$$

and $[\overline{\mu}_i]_{>} (\overline{\mu}_i(\boldsymbol{x})) \cap N_\varepsilon(\boldsymbol{x})$, $i \in I$ for sufficiently small $\varepsilon > 0$, we have the conclusion. □

From Proposition 3.1, we have the following corollary.

Corollary 3.2. *For* $\boldsymbol{x} \in \mathbb{R}^2$ *with* $\boldsymbol{x} \notin D$, $\boldsymbol{x} \in$ *FLSE*(D) *if and only if* $\{\pm 2, \pm 4\} \subseteq$ *SD*$(\boldsymbol{x})$.

Proposition 3.3. *For each* $\boldsymbol{d}_i$, $i \in I$, $\boldsymbol{d}_i \in$ *FSE*(D) *if* $m_i = 0$*, and the following statements hold if* $m_i > 0$.

(i) $\boldsymbol{d}_i \in$ *FLSE*$(D \setminus \{\boldsymbol{d}_i\}) \Rightarrow \boldsymbol{d}_i \in$ *FLSE*(D).

(ii)$\boldsymbol{d}_i \in$ *FLAE*$(D \setminus \{\boldsymbol{d}_i\}) \Rightarrow \boldsymbol{d}_i \in$ *FLQE*$(D) \setminus$ *FLE*(D).

(iii)$\boldsymbol{d}_i \in$ *FLQE*$(D \setminus \{\boldsymbol{d}_i\}) \setminus$ *FLE*$(D \setminus \{\boldsymbol{d}_i\}) \Rightarrow \boldsymbol{d}_i \in$ *FLQE*$(D) \setminus$ *FLE*(D).

(iv)$\boldsymbol{d}_i \notin$ *FLQE*$(D \setminus \{\boldsymbol{d}_i\}) \Rightarrow \boldsymbol{d}_i \notin$ *FLQE*(D).

Proof. Suppose that $m_i = 0$ for $\boldsymbol{d}_i$, $i \in I$, and we shall show that $\boldsymbol{d}_i \in$ FSE(D). Since

$$[\overline{\mu}_i]_{\geq} (\overline{\mu}_i(\boldsymbol{d}_i)) = [\overline{\mu}_i]_{\geq} (\mu_i(0)) = [\overline{\mu}_i]_{\geq} (1) = \{\boldsymbol{d}_i\},$$

we have

$$\bigcap_{j \in I} [\overline{\mu}_j]_{\geq} (\overline{\mu}_j(\boldsymbol{d}_i)) = \{\boldsymbol{d}_i\}.$$

This means that there is no $\boldsymbol{x} \in \mathbb{R}^2$ such that $\boldsymbol{x} \neq \boldsymbol{d}_i$ and $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{d}_i)$. Therefore, $\boldsymbol{d}_i \in$ FSE(D).

(i) Since $\boldsymbol{d}_i \in$ FLSE$(D \setminus \{\boldsymbol{d}_i\})$, there exists $\varepsilon > 0$ such that

$$\{\boldsymbol{d}_i\} = \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_{\geq} (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_\varepsilon(\boldsymbol{d}_i) \supseteq \left(\bigcap_{j \in I} [\overline{\mu}_j]_{\geq} (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_\varepsilon(\boldsymbol{d}_i) \supseteq \{\boldsymbol{d}_i\}.$$

Therefore, $\boldsymbol{d}_i \in$ FLSE(D).

(ii) Fix any $\varepsilon > 0$. Since $m_i > 0$ and μ_i is strictly increasing on $[0, m_i]$,

$$N_{\varepsilon_0}(\boldsymbol{d}_i) \setminus \{\boldsymbol{d}_i\} \subseteq [\overline{\mu}_i]_>(\overline{\mu}_i(\boldsymbol{d}_i)) \subseteq [\overline{\mu}_i]_\geq(\overline{\mu}_i(\boldsymbol{d}_i))$$

for sufficiently small $\varepsilon_0 > 0$. Since $\boldsymbol{d}_i \notin \text{FLSE}(D \setminus \{\boldsymbol{d}_i\})$,

$$\{\boldsymbol{d}_i\} \neq \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_\geq (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) \supseteq \{\boldsymbol{d}_i\}.$$

Thus there exists $\boldsymbol{x} \neq \boldsymbol{d}_i$ such that

$$\boldsymbol{x} \in \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_\geq (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i)$$

and

$$\boldsymbol{x} \in [\overline{\mu}_i]_>(\overline{\mu}_i(\boldsymbol{d}_i)) \subseteq [\overline{\mu}_i]_\geq(\overline{\mu}_i(\boldsymbol{d}_i)).$$

Therefore, $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{d}_i)$, $\boldsymbol{\mu}(\boldsymbol{x}) \neq \boldsymbol{\mu}(\boldsymbol{d}_i)$ and $\boldsymbol{x} \in N_{\varepsilon_0}(\boldsymbol{d}_i) \subseteq N_\varepsilon(\boldsymbol{d}_i)$. By arbitrariness of ε, $\boldsymbol{d}_i \notin \text{FLE}(D)$.

On the other hand, since $\boldsymbol{d}_i \in \text{FLQE}(D \setminus \{\boldsymbol{d}_i\})$, there exists $\varepsilon_1 > 0$ such that

$$\emptyset = \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_> (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_1}(\boldsymbol{d}_i) \supseteq \left(\bigcap_{j \in I} [\overline{\mu}_j]_> (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_1}(\boldsymbol{d}_i).$$

Therefore, $\boldsymbol{d}_i \in \text{FLQE}(D)$.

(iii) Fix any $\varepsilon > 0$. Since $m_i > 0$ and μ_i is strictly increasing on $[0, m_i]$,

$$N_{\varepsilon_0}(\boldsymbol{d}_i) \subseteq [\overline{\mu}_i]_\geq(\overline{\mu}_i(\boldsymbol{d}_i))$$

for sufficiently small $\varepsilon_0 > 0$. Since $\boldsymbol{d}_i \notin \text{FLE}(D \setminus \{\boldsymbol{d}_i\})$, there exists $k \in I \setminus \{i\}$ such that

$$\left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_\geq (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) \cap [\overline{\mu}_k]_>(\overline{\mu}_k(\boldsymbol{d}_i)) \neq \emptyset.$$

Thus we have

$$\begin{aligned} &\left(\bigcap_{j \in I} [\overline{\mu}_j]_\geq (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) \cap [\overline{\mu}_k]_>(\overline{\mu}_k(\boldsymbol{d}_i)) \\ &= \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_\geq (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) \cap [\overline{\mu}_k]_>(\overline{\mu}_k(\boldsymbol{d}_i)) \\ &\neq \emptyset. \end{aligned}$$

Therefore, there exists $\boldsymbol{x} \in N_{\varepsilon_0}(\boldsymbol{d}_i) \subseteq N_\varepsilon(\boldsymbol{d}_i)$ such that $\boldsymbol{\mu}(\boldsymbol{x}) \geq \boldsymbol{\mu}(\boldsymbol{d}_i)$ and $\boldsymbol{\mu}(\boldsymbol{x}) \neq \boldsymbol{\mu}(\boldsymbol{d}_i)$. By arbitrariness of ε, $\boldsymbol{d}_i \notin \text{FLE}(D)$.

On the other hand, since $\boldsymbol{d}_i \in \text{FLQE}(D \setminus \{\boldsymbol{d}_i\})$, it can be seen that $\boldsymbol{d}_i \in \text{FLQE}(D)$ similarly as the last part of (ii) of this proof.

(iv) Fix any $\varepsilon > 0$. Since $m_i > 0$ and μ_i is strictly increasing on $[0, m_i]$,

$$N_{\varepsilon_0}(\boldsymbol{d}_i) \setminus \{\boldsymbol{d}_i\} \subseteq [\overline{\mu}_i]_> (\overline{\mu}_i(\boldsymbol{d}_i))$$

for sufficiently small $\varepsilon_0 > 0$. Since $\boldsymbol{d}_i \notin \mathrm{FLQE}(D \setminus \{\boldsymbol{d}_i\})$,

$$\left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_> (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) \neq \emptyset.$$

Thus we have

$$\begin{aligned} \left(\bigcap_{j \in I} [\overline{\mu}_j]_> (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) &\supseteq \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_> (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap (N_{\varepsilon_0}(\boldsymbol{d}_i) \setminus \{\boldsymbol{d}_i\}) \\ &= \left(\bigcap_{j \in I \setminus \{i\}} [\overline{\mu}_j]_> (\overline{\mu}_j(\boldsymbol{d}_i)) \right) \cap N_{\varepsilon_0}(\boldsymbol{d}_i) \\ &\neq \emptyset. \end{aligned}$$

Therefore, there exists $\boldsymbol{x} \in N_{\varepsilon_0}(\boldsymbol{d}_i) \subseteq N_{\varepsilon}(\boldsymbol{d}_i)$ such that $\boldsymbol{\mu}(\boldsymbol{x}) > \boldsymbol{\mu}(\boldsymbol{d}_i)$. By arbitrariness of ε, $\boldsymbol{d}_i \notin \mathrm{FLQE}(D)$. □

For each demand point $\boldsymbol{d}_i$, $i \in I$, we draw 0 and $\frac{\pi}{2}$-oriented lines passing through the demand point $\boldsymbol{d}_i$ and $\partial B_i(m_i)$ which consists of four line segments (see Fig.3). Then the plane is divided into *subregions* and *edges* and *corners*, where each subregion does not contain its boundary and each edge does not contain its end point(s) and each corner is an intersection point of some of drawn lines and $\partial B_i(m_i)$, $i \in I$.

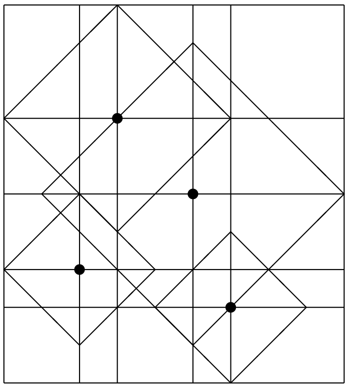


Figure 3: Subregions, edges and corners. (•: demand points)

Proposition 3.4. *Let $S \subseteq \mathbb{R}^2$ be a subregion or an edge. For $\boldsymbol{x} \in S$, the following statements hold.*

(i) $\boldsymbol{x} \in \mathit{FLE}(D) \Rightarrow S \subseteq \mathit{FLE}(D)$.

(ii) $\boldsymbol{x} \in \mathit{FLSE}(D) \Rightarrow S \subseteq \mathit{FLSE}(D)$.

(iii) $\boldsymbol{x} \in \mathit{FLAE}(D) \Rightarrow S \subseteq \mathit{FLAE}(D)$.

(iv) $\boldsymbol{x} \in \mathit{FLQE}(D) \Rightarrow S \subseteq \mathit{FLQE}(D)$.

Proof. Let $S \subseteq \mathbb{R}^2$ be a subregion or an edge. Suppose that $\boldsymbol{x} \in S$. From the definition of the summary diagram, $\mathrm{SD}(\boldsymbol{x}) = \mathrm{SD}(\boldsymbol{y})$ for any $\boldsymbol{y} \in S$. Therefore, we have the conclusion from Proposition 3.1. $\square$

Numerical example. We set $\boldsymbol{d}_1 = (1,5)$, $\boldsymbol{d}_2 = (3,3)$, $\boldsymbol{d}_3 = (4,0)$, $\boldsymbol{d}_4 = (0,1)$ and $m_1 = 3$, $m_2 = 4$, $m_3 = m_4 = 2$, and consider the following FMCP:

$$\max_{\boldsymbol{x} \in \mathbb{R}^2} (\mu_1(\|\boldsymbol{x} - \boldsymbol{d}_1\|_1), \mu_2(\|\boldsymbol{x} - \boldsymbol{d}_2\|_1), \mu_3(\|\boldsymbol{x} - \boldsymbol{d}_3\|_1), \mu_4(\|\boldsymbol{x} - \boldsymbol{d}_4\|_1))$$

where each μ_i, $i \in I$ is any function defined on $\mathbb{R}$ satisfying that $\mu_i(x) = 0$ for $x \in (-\infty, 0]$ and $\mu_i(m_i) = 1$ and that μ_i is strictly increasing on $[0, m_i]$ and strictly decreasing on $[m_i, \infty)$. Checking all subregions, edges and corners by using Proposition 3.1, 3.3 and 3.4, we have all locally strictly efficient, alternately efficient and quasiefficient solutions of FMCP illustrated in Fig.4.

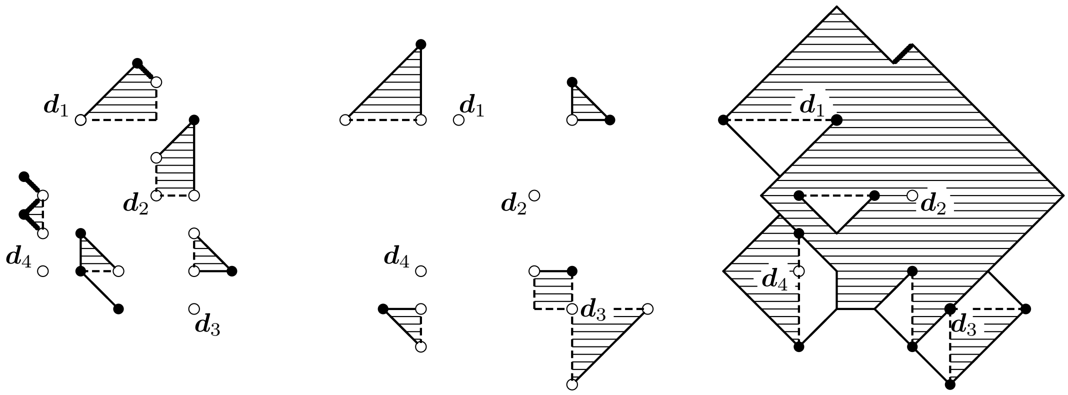


Figure 4-1: FLSE(D). Figure 4-2: FLAE(D). Figure 4-3: FLQE(D).

Figure 4: Locally strictly efficient, alternately efficient and quasiefficient solutions of FMCP.

4 Conclusions

We dealt with a fuzzy multicriteria location problem (FMCP) with rectilinear norm on the plane. First, as Proposition 2.1-2.3, we gave some properties of efficient and locally efficient solutions of FMCP. Next, we introduced the concept of the summary diagram, and gave characterizations of locally efficient solutions of FMCP as Proposition 3.1. Such characterizations were then used to readily find all locally efficient solutions of FMCP by Proposition 3.3 and 3.4.

References

[1] L. G. Chalmet, R.L. Francis and A. Kolen, Finding efficient solutions for rectilinear distance location problems efficiently, *European J. Oper. Res.* 6 (1981) 117–124.

[2] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I: Fundamentals*, Springer-Verlag, Berlin, 1993.

[3] M. Kon, On fuzzy multicriteria location problems, in *Proceedings of the Third International Conference on Nonlinear Analysis and Convex Analysis*, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Japan, 2004, pp. 227–232.

[4] B. Pelegrin and F.R. Fernandez, Determination of efficient points in multiple-objective location problems, *Naval Res. Logist. Quart.* 35 (1988) 697–705.

[5] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, N. J., 1970.

[6] J.E. Ward and R.E. Wendell, Characterizing efficient points in location problems under the one-infinity norm, in *Locational Analysis of Public Facilities*, J.-F. Thisse and H.G. Zoller (eds.), North Holland, Amsterdam, 1983, pp. 413-429.

[7] L.A. Zadeh, Fuzzy sets, *Inform. and Control* 8 (1965) 338-353.

Manuscript received 17 December 2007
revised 8 January 2009
accepted for publication 10 September 2009

Masamichi Kon
Graduate School of Science and Technology, Hirosaki University
3 Bunkyo, Hirosaki 036-8561, Japan
E-mail address: masakon@cc.hirosaki-u.ac.jp