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# DISTINGUISHING A GLOBAL MINIMIZER FROM LOCAL MINIMIZERS OF QUADRATIC MINIMIZATION WITH MIXED VARIABLES

V. Jeyakumar, G.M. Lee and S. Srisatkunarajah*

**Abstract:** We provide simple necessary, and sufficient conditions for a local minimizer to be a global minimizer of quadratic functions with mixed variables. We fully distinguish global minimizers from local minimizers in the case when the quadratic function is a sum of squares by providing a necessary and sufficient global optimality condition. We discuss examples to illustrate the significance of our conditions for identifying a global minimizer among local minimizers. Finally we apply our criteria for identifying global minimizers of a class of fractional programming problems.

**Key words:** *quadratic programs, weighted least squares, global optimality conditions, mixed variables*

**Mathematics Subject Classification:** *41A29, 90C20, 90C32*

## 1 Introduction

A common optimization problem in many real-world applications is to identify and locate a global minimizer of functions of several variables with bounds on the variables [6, 14]. Yet, identifying a global minimizer of a function of several variables, which may have several local minimizers that are not global, is inherently difficult [5, 9, 16]. The larger the number of local minimizers the more difficult the task of locating a global minimizer becomes.

On the other hand, locating a local minimizer of functions of several variables with bounds on the variables is well understood (see for instance [1]). In particular, complete characterizations of a minimizer are well known for a convex function with bounds on the variables, where a local minimizer is global. So, the question naturally arises: when is a local minimizer of a non-convex function of several variables with bounds on the variables a global minimizer? Answering this question is of significant practical value as our ability to distinguish local and global minimizers will lead to efficient methods for locating global minimizers. However, the development of mathematical criteria which completely characterize a local minimizer as global is known to be difficult even for a quadratic non-convex function. For recent developments of identifying global minimizers of quadratic minimization problems, see [2, 7, 8, 11, 12, 15].

In this paper, we provide simple necessary, and sufficient conditions for a local minimizer to be a global minimizer of quadratic functions with bounds on the mixed variables. These

*Work of this author was completed while he was at the Department of Applied Mathematics, University of New South Wales, Sydney, Australia necessary conditions, and sufficient conditions coincide for the weighted sum of squares function subject to bounds on the mixed variables.

Recent research saw the development of conditions necessary or sufficient for characterizing global minimizers of smooth functions with bounded mixed variables (see [10, 13] and other references therein). However, a drawback of this development is that the conditions were neither based on local optimality conditions nor expressed in terms of local minimizers. We provide an elementary proof for the necessary, and sufficient conditions by refining the method of proof, developed in [13], and by incorporating the local optimality conditions. We also obtain corresponding results for a unique global minimizer. We discuss examples to illustrate the significance of our conditions for identifying a global minimizer from the set of local minimizers. We apply our conditions for distinguishing local and global minimizers of fractional quadratic programs.

## 2 Characterizing Global Minimizers

In this section, we provide methods to characterize the global minimizers from the local minimizers of the following quadratic minimization problem over a box with mixed variables.

$$\begin{array}{lll} (P) & \min\limits_{x\in\mathbb{R}^n} & \frac{1}{2}x^TAx + a^Tx \\ & \text{subject to} & x_i \in [u_i, v_i],\ \text{if } i \in I \\ & & x_i \in \{u_i, v_i\},\ \text{if } i \in J \end{array}$$

where $I \cap J = \emptyset,\ I \cup J = \{1, 2, \ldots, n\},\ A = (a_{ij}) \in S_n,$ the set of all $n \times n$ symmetric matrices, $a = (a_i) \in \mathbb{R}^n$ and $u_i, v_i \in \mathbb{R}^n$ and $u_i < v_i,\quad i = 1, 2, \ldots, n$. Let $D = \{(x_1, \ldots, x_n)^T \in \mathbb{R}^n \big|\ x_i \in [u_i, v_i],\ \text{for } i \in I \text{ and } x_i \in \{u_i, v_i\},\ \text{for } i \in J\}$. First we begin with the necessary conditions for local optimality. For $i = 1, 2, \ldots, n$ define

$$\chi_i(\bar{x}) := \begin{cases} -1 & \text{if } \ \bar{x}_i = u_i \\ 1 & \text{if } \ \bar{x}_i = v_i \\ (A\bar{x} + a)_i & \text{if } \ \bar{x}_i \in (u_i, v_i). \end{cases}$$

**Lemma 2.1.** *If $\bar{x} \in D$ is a local minimizer of $(P)$ then the following optimality condition holds:*

$$\chi_i(\bar{x})(A\bar{x} + a)_i \le 0,\ \forall i \in I. \tag{2.1}$$

*Proof.* First we rewrite the problem $(P)$ as follows,

$$\begin{array}{llll} (RP) & \min\limits_{x\in\mathbb{R}^n} & \frac{1}{2}x^TAx + a^Tx & \\ & \text{subject to} & (x_i - u_i)(x_i - v_i) = 0, & \text{if } i \in J, \\ & & x_i \in [u_i, v_i], & \text{if } i \in I. \end{array}$$

Let $\Delta = \{(x_1, x_2, \ldots, x_n) \in I\!R^n \big|\ x_i \in [u_i, v_i],\ i \in I\}$. For $\lambda \in I\!R^{|J|}$, define the Lagrangian $L(\cdot, \lambda)$ by

$$L(x, \lambda) = \frac{1}{2}x^TAx + a^Tx + \sum_{i\in J} \lambda_i(x_i - u_i)(x_i - v_i).$$

If $\bar{x} \in D$ is a local minimizer of $(P)$ then obviously $\bar{x}$ is a local minimizer of $(RP)$. Then, by the necessary local optimality conditions [3] for $(RP)$, there exists $\lambda \in I\!R^{|J|}$ such that $\nabla_x L(x, \lambda)(x - \bar{x}) \ge 0,\ \forall x \in \Delta$. This indeed implies that, for each $i \in I$,

$$(A\bar{x} + a)_i(y - \bar{x}_i) \ge 0,\ \forall y \in [u_i, v_i]. \tag{2.2}$$

This is equivalent to $\chi_i(\bar{x})(A\bar{x} + a)_i \le 0,\ i \in I$. Hence the conclusion follows. $\square$

For a matrix $A \in S^n$, $A \succeq 0$, means that $A$ is positive semidefinite and $A \succ 0$, means that $A$ is positive definite. We now provide necessary condition and sufficient condition for a local minimizer to be a global minimizer of $(P)$.

**Theorem 2.2.** *Let $\bar{x} \in D$ be a local minimizer of $(P)$.*
(i) *If for some $i_0 \in \{1, 2, \ldots, n\}$*

$$a_{i_0 i_0} < \frac{2\chi_{i_0}(\bar{x})(A\bar{x} + a)_{i_0}}{(v_{i_0} - u_{i_0})} \tag{2.3}$$

*then $\bar{x}$ can not be a global minimizer of $(P)$.*
(ii) *If*

$$A - diag\left(\frac{2\chi_1(\bar{x})(A\bar{x} + a)_1}{v_1 - u_1}, \ldots, \frac{2\chi_n(\bar{x})(A\bar{x} + a)_n}{v_n - u_n}\right) \succeq 0, \tag{2.4}$$

*then $\bar{x}$ is a global minimizer of $(P)$.*
(iii) *If*

$$A - diag\left(\frac{2\chi_1(\bar{x})(A\bar{x} + a)_1}{v_1 - u_1}, \ldots, \frac{2\chi_n(\bar{x})(A\bar{x} + a)_n}{v_n - u_n}\right) \succ 0, \tag{2.5}$$

*then $\bar{x}$ is a unique global minimizer of $(P)$.*

*Proof.* Suppose that $\bar{x}$ is a global minimizer of $(P)$. Then

$$\frac{1}{2}x^T Ax + a^T x - \frac{1}{2}\bar{x}^T A\bar{x} - a^T\bar{x} \geq 0, \ \forall x \in D.$$

So,

$$\frac{1}{2}(x - \bar{x})^T A(x - \bar{x}) + (A\bar{x} + a)^T (x - \bar{x}) \geq 0, \ \forall x \in D.$$

Hence,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{2}a_{ij}(x_i - \bar{x}_i)(x_j - \bar{x}_j) + \sum_{i=1}^{n}(A\bar{x} + a)_i(x_i - \bar{x}_i) \geq 0, \ \forall x \in D,$$

which, in turn, implies that, for each, $i = 1, 2, \ldots, n$,

$$\frac{1}{2}a_{ii}(x_i - \bar{x}_i)^2 + (A\bar{x} + a)_i(x_i - \bar{x}_i) \geq 0, \ (x_1, x_2, \ldots, x_n) \in D. \tag{2.6}$$

To see this implication suppose that

$$\frac{1}{2}a_{i_0 i_0}(x_{i_0} - \bar{x}_{i_0})^2 + (A\bar{x} + a)_{i_0}(x_{i_0} - \bar{x}_{i_0}) < 0,$$

for some $i_0 \in \{1, 2, \ldots, n\}$. Then by taking
$\tilde{x} = (\bar{x}_1, \ldots, \bar{x}_{i_0 - 1}, x_{i_0}, \bar{x}_{i_0+1}, \ldots, \bar{x}_n)$, we see that $\tilde{x} \in D$ and

$$\frac{1}{2}\tilde{x}^T A\tilde{x} + a^T\tilde{x} - \frac{1}{2}\bar{x}^T A\bar{x} - a^T\bar{x} = \frac{1}{2}a_{i_0 i_0}(x_{i_0} - \bar{x}_{i_0})^2 + (A\bar{x} + a)_{i_0}(x_{i_0} - \bar{x}_{i_0}) < 0.$$

This contradicts that $\bar{x}$ is a global minimizer.

We now show that (2.6) holds if and only if

$$a_{ii} \geq \frac{2\chi_i(\bar{x})(A\bar{x} + a)_i}{(v_i - u_i)}, \ i = 1, 2, \ldots, n, \tag{2.7}$$

holds, by considering the following three cases.
**Case–1.** Let $\bar{x}_i = u_i$. If $i \in I$ local optimality condition (2.1) implies $(A\bar{x}+a)_i \geq 0$, and (2.6) is equivalent to

$$a_{ii}(x_i - u_i) + 2(A\bar{x}+a)_i \geq 0, \ \forall \, x_i \ \in (u_i, v_i]. \tag{2.8}$$

By taking $x_i = v_i$, we see that (2.8) readily implies (2.7). Conversely, if (2.7) holds then, $a_{ii}(v_i - u_i) + 2(A\bar{x}+a)_i \geq 0$. Thus, if $a_{ii} < 0$ then, for each, $x_i \ \in (u_i, v_i]$,

$$a_{ii}(x_i - u_i) + 2(A\bar{x}+a)_i \geq a_{ii}(v_i - u_i) + 2(A\bar{x}+a)_i \geq 0.$$

Hence, (2.8) holds. On the other hand, if $a_{ii} \geq 0$ then (2.8) holds as $(A\bar{x}+a)_i \geq 0$,. So, (2.7) is equivalent to (2.8).
If $i \in J$ then (2.6) is equivalent to

$$a_{ii}(x_i - u_i) + 2(A\bar{x}+a)_i \geq 0, \ \forall \, x_i \neq u_i.$$

i.e., (2.6) holds if and only if

$$a_{ii}(v_i - u_i) + 2(A\bar{x}+a)_i \geq 0.$$

So, (2.6) holds if and only if (2.7) holds.
**Case–2.** Let $\bar{x}_i = v_i$. If $i \in I$ local optimality condition (2.1) implies $(A\bar{x}+a)_i \leq 0$, and (2.6) is equivalent to

$$a_{ii}(x_i - v_i) + 2(A\bar{x}+a)_i \leq 0, \ \forall \, x_i \ \in [u_i, v_i). \tag{2.9}$$

By taking $x_i = u_i$, (2.9) implies (2.7).
Conversely, if (2.7) holds then in this case $a_{ii}(v_i - u_i) - 2(A\bar{x}+a)_i \geq 0$. Thus, if $a_{ii} < 0$ then, for each, $x_i \ \in (u_i, v_i]$,

$$a_{ii}(x_i - v_i) + 2(A\bar{x}+a)_i \leq a_{ii}(u_i - v_i) + 2(A\bar{x}+a)_i \leq 0.$$

Hence, (2.9) holds. On the other hand, if $a_{ii} \geq 0$ then trivially, (2.9) holds. So, (2.7) is equivalent to (2.9).
If $i \in J$ then (2.6) is equivalent to

$$a_{ii}(x_i - v_i) + 2(A\bar{x}+a)_i \leq 0, \ \forall \, x_i \neq v_i.$$

i.e., (2.6) holds if and only if

$$a_{ii}(v_i - u_i) - 2(A\bar{x}+a)_i \geq 0.$$

So, (2.6) holds if and only if (2.7) holds.

**Case–3.** Let $\bar{x}_i \in (u_i, v_i)$. In this case, $i \in I$ and local optimality condition (2.1) implies, $(A\bar{x}+a)_i = 0$ and (2.6) holds if and only if $a_{ii} \geq 0$. Also, $a_{ii} \geq 0$ if and only if (2.7) holds. Therefore, the conclusion of (i) follows from the above three cases.

We now prove (ii). Let, $x \in D$ and $q_i = \dfrac{2\chi_i(\bar{x})(A\bar{x}+a)_i}{(v_i - u_i)}$, $i = 1, 2, \ldots, n$. Then,

$$\begin{aligned}
\frac{1}{2}x^T Ax + a^T x - \frac{1}{2}\bar{x}^T A\bar{x} - a^T\bar{x} &= \frac{1}{2}(x-\bar{x})^T A(x-\bar{x}) + (A\bar{x}+a)^T(x-\bar{x}) \\
&= \frac{1}{2}(x-\bar{x})^T(A - diag(q_1,\ldots,q_n))(x-\bar{x}) \\
&\quad + \frac{1}{2}(x-\bar{x})^T diag(q_1,\ldots,q_n)(x-\bar{x}) + (A\bar{x}+a)^T(x-\bar{x}) \\
&= \frac{1}{2}(x-\bar{x})^T(A - diag(q_1,\ldots,q_n))(x-\bar{x}) \\
&\quad + \sum_{i=1}^{n} \frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i).
\end{aligned}$$

Condition (2.4) means that $A - diag(q_1, \ldots, q_n) \succeq 0$ and so,

$$\frac{1}{2}(x-\bar{x})^T(A - diag(q_1,\ldots,q_n))(x-\bar{x}) \geq 0,\ \forall x \in D.$$

We now claim that local optimality condition (2): $\chi_i(\bar{x})(A\bar{x}+a)_i \leq 0,\ i \in I$ implies that,

$$\sum_{i=1}^{n} \frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i) \geq 0,\ \forall (x_1, x_2, \ldots, x_n) \in D. \tag{2.10}$$

To see this , we consider the following cases.
**Case 1:** Let $\bar{x}_i = u_i$. If $i \in I$ then (2) implies $(A\bar{x}+a)_i \geq 0$. So, $(x_1, x_2, \ldots, x_n) \in D$,

$$\begin{aligned}
\frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i) &= (x_i-u_i)(A\bar{x}+a)_i\left(1 - \frac{x_i-u_i}{v_i-u_i}\right) \\
&\geq 0.
\end{aligned}$$

If $i \in J$ then

$$\begin{aligned}
\frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i) &= (x_i-u_i)(A\bar{x}+a)_i\left(1 - \frac{x_i-u_i}{v_i-u_i}\right) \\
&= 0,\ \text{as } x_i \in \{u_i, v_i\}.
\end{aligned}$$

**Case 2:** Let $x_i = v_i$. Then (2) implies $(A\bar{x}+a)_i \leq 0$. So, $(x_1, x_2, \ldots, x_n) \in D$,

$$\begin{aligned}
\frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i) &= (x_i-v_i)(A\bar{x}+a)_i\left(1 - \frac{v_i-x_i}{v_i-u_i}\right) \\
&\geq 0.
\end{aligned}$$

If $i \in J$ then

$$\begin{aligned}
\frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i) &= (x_i-v_i)(A\bar{x}+a)_i\left(1 + \frac{x_i-v_i}{v_i-u_i}\right) \\
&= 0,\ \text{as } x_i \in \{u_i, v_i\}.
\end{aligned}$$

**Case 3:** Let $u_i < \bar{x}_i < v_i$. Then $i \in I$ and (2) implies $(A\bar{x}+a)_i = 0$. So, $(x_1, x_2, \ldots, x_n) \in D$,

$$\frac{1}{2}q_i(x_i-\bar{x}_i)^2 + (A\bar{x}+a)_i(x_i-\bar{x}_i) = 0.$$

by combining the above all three cases, (2.10) holds. Hence the conclusion of (ii) follows. Moreover, if (2.5) holds then $A - diag(q_1, \ldots, q_n) \succ 0$. So,

$$\frac{1}{2}x^T Ax + a^T x - \frac{1}{2}\bar{x}^T A\bar{x} - a^T\bar{x} > 0, \text{ for } x \in D \setminus \{\bar{x}\}.$$

Therefore the uniqueness result follows. □

**Example 2.3.** Consider the following nonconvex minimization problem:

$$(E1) \qquad \min\left\{ -x_1^2 - x_2^2 - x_1x_2 + x_1 \;\middle|\; \begin{array}{c} -1 \le x_1 \le 1 \\ x_2 \in \{-1, 1\} \end{array} \right\}.$$

Local minimiers of E(1) are (1,1), (-1,1) and (-1,-1). Direct calculation shows that (2.3) is satisfied at $(-1, 1)$ with $i_0 = 2$ and not satisfied at $(-1, -1)$ and $(1, 1)$ for $i = 1, 2$. Hence, first of all $(-1, 1)$ can not be a global minimizer. Further, among the remaining local minimizers $(1, 1)$ and $(-1, -1)$, (2.5) holds at $(-1, -1)$. The point $(-1, -1)$ is indeed the unique global minimizer.

It is noted that if $A$ is a diagonal matrix, that is, in the case of minimization of the sum of weighted squares, (2.7) and (2.4) coincide with each other and become necessary and sufficient conditions. The following Corollary provides an answer to the question "When is a local minimizer of weighted sum of squares with variable bounds to be a global minimizer?"

**Corollary 2.4.** *For $(P)$, let $a_{ij} = 0$ for all $i \neq j$ and let $\bar{x} \in D$ be a local minimizer. Then,*
*$(i)$ $\bar{x}$ is a global minimizer of $(P)$ if and only if*

$$a_{ii}(v_i - u_i) - 2\chi_i(\bar{x})(a_{ii}\bar{x}_i + a_i) \ge 0, \ \forall i = 1, 2, \ldots, n. \tag{2.11}$$

*$(ii)$ $\bar{x}$ is the unique global minimizer of $(P)$ if and only if*

$$a_{ii}(v_i - u_i) - 2\chi_i(\bar{x})(a_{ii}\bar{x}_i + a_i) > 0, \ \forall i = 1, 2, \ldots, n. \tag{2.12}$$

*Proof.* The conclusion of (i) immediately follows from Theorem 2.2, as (2.7) and (2.4) collapse to (2.11) when $A$ is a diagonal matrix.
We now prove (ii). The point $\bar{x}$ is the unique global minimizer of $(P)$ if and only if $\frac{1}{2}x^T Ax + a^T x - \frac{1}{2}\bar{x}^T A\bar{x} - a^T\bar{x} > 0$, $\forall x \in D \setminus \{\bar{x}\}$, which is equivalent to, for each, $i = 1, 2 \ldots, n$,

$$\frac{1}{2}a_{ii}(x_i - \bar{x}_i)^2 + (a_{ii}\bar{x}_i + a_i)(x_i - \bar{x}_i) > 0, \ \ \forall (x_1, x_2, \ldots, x_n) \in D \setminus \{\bar{x}\}. \tag{2.13}$$

Now, by considering the three cases as in the proof of Theorem 2.2, we see that (2.13) holds if and only if (2.12) holds. Hence the conclusion of (ii) follows, □

**Example 2.5.** Consider the following nonconvex problem:

$$(E2) \qquad \min\left\{ -x_1^2 + x_2^2 - x_1 \;\middle|\; \begin{array}{c} x_1 \in \{-1, 1\} \\ -1 \le x_2 \le 1 \end{array} \right\}.$$

It is easy to check that both $(-1, 0)$ and $(1, 0)$ are the local minimizers of $(E2)$. The condition (2.12) is clearly satisfied at $\bar{x} = (1, 0)$ and $\bar{x}$ is indeed the unique global minimizer of $(E2)$.

**Example 2.6.** Consider the problem:

$$(E3) \qquad \min\left\{ -x_1^2 + x_2^2 - x_2 \;\middle|\; \begin{array}{c} x_1 \in \{-1, 1\} \\ -1 \le x_2 \le 1 \end{array} \right\}.$$

The points $(-1, \frac{1}{2})$ and $(1, \frac{1}{2})$ are both local and global minimizers of $(E3)$ and the condition (2.11) is also satisfied at both points. However, the uniqueness condition (2.12) is not satisfied.

## $\boxed{3}$ Fractional Quadratic Programs

In this Section, we apply the results of the previous section to distinguish the local and global minimizers of the following fractional quadratic minimization model problems:

$$\begin{array}{lll} (FP) & \min\limits_{x\in\mathbb{R}^n} & \dfrac{\frac{1}{2}x^TA_1x+a_1^Tx}{\frac{1}{2}x^TA_2x+a_2^Tx} \\ & \text{subject to} & x_i\in[u_i,v_i],\ \text{if } i\in I \\ & & x_i\in\{u_i,v_i\},\ \text{if } i\in J, \end{array}$$

where $I\cap J=\emptyset,\ I\cup J=\{1,2,\ldots,n\}$, $A_1=(a_{ij}^1)$ and $A_2=(a_{ij}^2)$ are in $S_n,$ the set of all $n\times n$ symmetric matrices, $a_1=(a_i^1)$ and $a_2=(a_i^2)$ are in $\mathbb{R}^n,\ u_i,v_i\in\mathbb{R}^n$ and $u_i<v_i,\quad i=1,2,\ldots,n$ and for each $x\in D=\{(x_1,\ldots,x_n)^T\in\mathbb{R}^n\big|\ x_i\in[u_i,v_i],\ \ i\in I$ and $x_i\in\{u_i,v_i\},\ i\in J\},\ \dfrac{1}{2}x^TA_2x+a_2^Tx>0$. For $x\in D$ define

$$s(x)=\frac{\frac{1}{2}x^TA_1x+a_1^Tx}{\frac{1}{2}x^TA_2x+a_2^Tx}.$$

Let $\bar{x}\in D$. For $x\in D$ define,

$$f(x)=\frac{1}{2}x^TA_1x+a_1^Tx-s(\bar{x})\left(\frac{1}{2}x^TA_2x+a_2^Tx\right).$$

Then we note that $f(\bar{x})=0$ and

$$f(x)-f(\bar{x})=\left(\frac{1}{2}x^TA_2x+a_2^Tx\right)(s(x)-s(\bar{x})),\ \forall x\in D.$$

Hence $\bar{x}$ is a (local) global minimizer of $(FP)$ if and only if $\bar{x}$ is a (local) global minimizer of the following quadratic minimization problem:

$$\begin{array}{lll} (FQP) & \min\limits_{x\in\mathbb{R}^n} & f(x):=\dfrac{1}{2}x^TA_1x+a_1^Tx-s(\bar{x})\left(\dfrac{1}{2}x^TA_2x+a_2^Tx\right) \\ & \text{subject to} & x_i\in[u_i,v_i],\ \text{if } i\in I \\ & & x_i\in\{u_i,v_i\},\ \text{if } i\in J. \end{array}$$

By employing the approach developed in the previous section to $(FQP)$, we distinguish the local and global minimizers of $(FP)$.

**Theorem 3.1.** *Let $\bar{x}\in D$ be a local minimizer of $(FP)$.*
(i) *If for some $i_0\in\{1,2,\ldots,n\}$*

$$a_{i_0i_0}^1-s(\bar{x})a_{i_0i_0}^2<\frac{2\chi_{i_0}(\bar{x})((A_1-s(\bar{x})A_2)\bar{x}+a_1-s(\bar{x})a_2)_{i_0}}{(v_{i_0}-u_{i_0})} \tag{3.1}$$

*then $\bar{x}$ can not be a global minimizer of $(FP)$.*
(ii) *If*

$$A_1-s(\bar{x})A_2- \\ diag\left(\tfrac{2\chi_1(\bar{x})((A_1-s(\bar{x})A_2)\bar{x}+a_1-s(\bar{x})a_2)_1}{v_1-u_1},\ldots,\tfrac{2\chi_n(\bar{x})((A_1-s(\bar{x})A_2)\bar{x}+a_1-s(\bar{x})a_2)_n}{v_n-u_n}\right)\succeq 0,$$

*then $\bar{x}$ is a global minimizer of $(FP)$.*
(iii) *If*

$$A_1-s(\bar{x})A_2-diag\left(\frac{2\chi_1(\bar{x})((A_1-s(\bar{x})A_2)\bar{x}+a_1-s(\bar{x})a_2)_1}{v_1-u_1},\dots,\frac{2\chi_n(\bar{x})((A_1-s(\bar{x})A_2)\bar{x}+a_1-s(\bar{x})a_2)_n}{v_n-u_n}\right)\succ 0,$$

*then $\bar{x}$ is a unique global minimizer of $(FP)$.*

*Proof.* The conclusion follows from applying Theorem 2.2 to $(FQP)$. □

Now, consider a special case of $(FP)$ which has the form:

$$\begin{array}{lll}(FPS) & \min\limits_{x\in\mathbb{R}^n} & \dfrac{\frac{1}{2}\sum\limits_{i=1}^{n}\alpha_i x_i^2+\sum\limits_{i=1}^{n} r_i x_i}{\frac{1}{2}\sum\limits_{i=1}^{n}\beta_i x_i^2+\sum\limits_{i=1}^{n} t_i x_i} \\ & \text{subject to} & x_i\in[u_i,v_i],\ \text{if } i\in I \\ & & x_i\in\{u_i,v_i\},\ \text{if } i\in J,\end{array}$$

where $\alpha_i,\ r_i,\ \beta_i,\ t_i,\ u_i,\ v_i\in\mathbb{R}^n,\ i=1,2,\dots,n,$ and, for each $x\in D$, $\frac{1}{2}\sum\limits_{i=1}^{n}\beta_i x_i^2+\sum\limits_{i=1}^{n} t_i x_i>0$. For $x\in D$, define

$$p(x)=\frac{\frac{1}{2}\sum\limits_{i=1}^{n}\alpha_i x_i^2+\sum\limits_{i=1}^{n} r_i x_i}{\frac{1}{2}\sum\limits_{i=1}^{n}\beta_i x_i^2+\sum\limits_{i=1}^{n} t_i x_i}.$$

**Theorem 3.2.** *For $(FPS)$, let $\bar{x}\in D$ be a local minimizer. Then,*
(i) *$\bar{x}$ is a global minimizer of $(FPS)$ if and only if*

$$(\alpha_i-p(\bar{x})\beta_i)(v_i-u_i)-2\chi_i(\bar{x})(\alpha_i-p(\bar{x})\beta_i)\bar{x}_i+r_i-p(\bar{x})t_i)\geq 0,\ \forall i=1,2,\dots,n.$$

(ii) *$\bar{x}$ is the unique global minimizer of $(FPS)$ if and only if*

$$(\alpha_i-p(\bar{x})\beta_i)(v_i-u_i)-2\chi_i(\bar{x})(\alpha_i-p(\bar{x})\beta_i)\bar{x}_i+r_i-p(\bar{x})t_i)>0,\ \forall i=1,2,\dots,n.$$

*Proof.* The conclusion follows from applying Corollary 2.4 to the corresponding quadratic minimization problem analogous to $(FPS)$. □

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V. Jeyakumar
Department of Applied Mathematics, University of New South Wales
Sydney 2052, Australia
E-mail address: v.jeyakumar@unsw.edu.au

G.M. Lee
Department of Applied Mathematics, Pukyong National University
Pusan 608-737, Korea
E-mail address: gmlee@pknu.ac.kr

S. Srisatkunarajah
Department of Mathematics and Statistics, University of Jaffna, Thirunelvely
Jaffna, Sri Lanka
E-mail address: srisatku@yahoo.com