Yokohama Publishers
ISSN 1349-8169 ONLINE JOURNAL

Pacific Journal of Optimization
Vol. 6, No. 1, pp. 57–64, January 2010

DISCRETE MODELING OF ECONOMIC EQUILIBRIUM PROBLEMS*

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Abstract: An application of discrete fixed point theorems to a Walrasian market equilibrium problem is presented, with a particular emphasis on the versatile notion of "direction preserving property" used in the theorems. Two variations of the model are also presented, one by using a new equilibrium concept and the other by introducing a numéraire, to overcome the potential lack of strict Walras law under the indivisibility of goods.

Key words: *integer lattice, discrete fixed point, indivisible goods, Walras law*

Mathematics Subject Classification: *90C10, 91B02, 91B50*

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1 Introduction

In this paper we present an application of discrete fixed point theorems ([4], [8]) to economic equilibrium problems. We will show, in particular, that the statement such that "the most demanded goods and the least demanded goods are different at close enough prices" can be interpreted as the *direction preserving* property of (aggregate) excess demand function, which guarantees the existence of an equilibrium.

We first consider a model of exchange economy with a finite number of agents, where all the goods are indivisible, the prices are integral, and there is no numéraire commodity. We will show a set of sufficient conditions for excess demand function to have a zero point, i.e., a (Walrasian) equilibrium. Markets consisting of the agents with unit demands and sufficient amount of behavioral heterogeneity (specified later) will satisfy our conditions, though this is stronger than is needed. Since our model does not satisfy Walras law strictly, we then propose an equilibrium concept that permits excess supply at equilibrium (with positive prices; differently from the free disposal). The introduction of numéraire, however, will guarantee the strict Walras law to hold, so we will see our model modified for such a situation too.

The direction preserving-ness of function is a key condition for the line of theorems we apply here. As we will see, and this is the point we want to make in this paper, this condition can be conceived of in many guises. We hope our presentation also serves as a hint for finding the direction preserving-ness of functions in other types of discrete equilibrium problems.

The paper is organized as follows. Section 2 will give some preliminaries. Section 3 will state and solve the equilibrium existence problem, with possible variations and remarks.

*This work was supported by a Grant-in-Aid for Scientific Research (C20530158) from Japan Society for the Promotion of Science (JSPS).

2 Preliminaries

We denote the n-dimensional integer lattice by $\mathbb{Z}^n$ and the n-dimensional Euclidean space by $\mathbb{R}^n$. We denote by e^i the vector whose ith component is one and all others zeros. The maximum norm of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is given by $\|x\|_\infty = \max_{i=1,\cdots,n} |x_i|$. A set spanned by $k+1$ affinely independent points in $\mathbb{R}^n$ is called a k-dimensional simplex ($0 \leq k \leq n$). Its subset spanned by some subset of the points is again a simplex, and is called a face of the simplex. We call the 0-dimensional faces of a simplex the vertices of the simplex. A simplex is integral if all of its vertices are integral, i.e., if they are in $\mathbb{Z}^n$. If $X \subset \mathbb{R}^n$ and if $\mathcal{S}$ is a finite collection of simplices such that (i) its union is X, (ii) all the faces of simplices in $\mathcal{S}$ are also in $\mathcal{S}$, and (iii) any intersection of pair of simplices in $\mathcal{S}$ is empty or a common face of both, then $\mathcal{S}$ is called a triangulation of X. A triangulation is integral if all of its simplices are integral.

A set $X \subset \mathbb{Z}^n$ is said to be *integrally convex* ([1], see also [7]) if

$$y \in \mathrm{co}(X) \implies y \in \mathrm{co}(X \cap N(y)), \tag{2.1}$$

where $\mathrm{co}(X)$ denotes the convex hull of X and so on, and $N(y) = \{x \in \mathbb{Z}^n : \|x - y\|_\infty < 1\}$ (integral neighborhood of y; note that the inequality is strict). In the following, we always assume nonempty-ness for integrally convex sets. Any finite integrally convex $X \subset \mathbb{Z}^n$ admits an integral triangulation $\mathcal{S}$ of its convex hull $\mathrm{co}(X)$ such that

$$y \in \mathrm{co}(X) \implies y \in \mathrm{co}(S_y \cap N(y)), \tag{2.2}$$

where $S_y \in \mathcal{S}$ is the smallest simplex containing y ([4, Lemma 1]). That is, $\mathrm{co}(X)$ is triangulated in such a way that, for every y in $\mathrm{co}(X)$, the integral vertices of the smallest simplex S_y containing y are elements of $N(y)$. Note that such a triangulation is not unique, but once $\mathcal{S}$ is given, every y in $\mathrm{co}(X)$ is uniquely expressed as a convex combination of the vertices of S_y. The following two types of integrally convex sets will be used in our application, whose convex hulls have well known types of triangulations satisfying (2.2).

1. $M_c = \{x \in \mathbb{Z}^n : \sum_{i=1}^n x_i = c,\ x_i \geq 0,\ i = 1, \cdots, n\}$ *with a positive integer c:* The simplicial $\mathrm{co}(M_c)$ has a triangulation $\mathcal{S}$ such that each $(n-1)$-dimensional simplex in $\mathcal{S}$ is spanned by some n points $x^0, x^1, \cdots, x^{n-1}$ with $x^0 \in \mathbb{Z}^n$, $x^k = x^{k-1} + e^{\pi_k + 1} - e^{\pi_k}$, $k = 1, \cdots, n-1$, where $(\pi_1, \cdots, \pi_{n-1})$ is a permutation of $(1, \cdots, n-1)$ (thus $\mathcal{S}$ is similar to the "regular triangulation of unit simplex $\mathrm{co}(\{e^1, \cdots, e^n\})$" [5]).

2. $R_b = \{x \in \mathbb{Z}^n : 0 \leq x_i \leq b_i,\ i = 1, \cdots, n\}$ *with a positive integers b_i:* The rectangular $\mathrm{co}(R_b)$ has a triangulation $\mathcal{S}$ such that each n-dimensional simplex in $\mathcal{S}$ is spanned by some $n+1$ points $x^0, x^1, \cdots, x^n$ with $x^0 \in \mathbb{Z}^n$, $x^k = x^{k-1} + e^{\pi_k}$, $k = 1, \cdots, n$, where $(\pi_1, \cdots, \pi_n)$ is a permutation of $(1, \cdots, n)$ (thus $\mathcal{S}$ is induced from the "standard (or Freudenthal) triangulation of $\mathbb{R}^n$" [2]).

We use the following form of discrete fixed point theorem for functions that appeared in [6, 9], or its corollary for zero.

Theorem 2.1. *Let $X \subset \mathbb{Z}^n$ be a finite integrally convex set and $f : X \to \mathrm{co}(X)$ a function. If $\mathcal{S}$ is a triangulation of $\mathrm{co}(X)$ satisfying (2.2) and $(f(x) - x) \cdot (f(x') - x') \geq 0$ for any vertices x, x' of any simplex in $\mathcal{S}$, then f has a fixed point; i.e., there is an $x \in X$ such that $f(x) = x$.*

Proof. See [9, Theorem 4.5]. □

Corollary 2.2. *Let $X \subset \mathbb{Z}^n$ be a finite integrally convex set and $g\colon X \to \mathbb{R}^n$ a function such that $x + t_x g(x) \in \mathrm{co}(X)$ for some $t_x > 0$ for each $x \in X$. If $\mathcal{S}$ is a triangulation of* $\mathrm{co}(X)$ *satisfying* (2.2) *and $g(x) \cdot g(x') \geq 0$ for any vertices x, x' of any simplex in $\mathcal{S}$, then g has a zero; i.e., there is an $x \in X$ such that $g(x) = \mathbf{0}$ (zero vector).*

Proof. Immediate by letting $f(x) = x + t_x g(x)$. □

Following [8], we call such $f\colon X \to \mathrm{co}(X)$ or $g\colon X \to \mathbb{R}^n$ *locally gross direction preserving (l.g.d.p.) (on the triangulation $\mathcal{S}$* of $\mathrm{co}(X)$).

Before we proceed to the next section, we briefly explain our strategy for the applications. Suppose we want to show the existence of a fixed point of $f\colon X \to \mathrm{co}(X)$, or, a zero of $g\colon X \to \mathbb{R}^n$ defined by $g(x) = f(x) - x$. Now, that g is l.g.d.p. is certainly a sufficient condition for the existence of a zero, but we have more: If there is $h\colon X \to \mathbb{R}^n$ such that

$$h(x) = \mathbf{0} \implies g(x) = \mathbf{0} \tag{2.3}$$

and h is l.g.d.p. then h has a zero and so does g.

An example is $h^1(x) = (\mathrm{sign}\, g_1(x), \cdots, \mathrm{sign}\, g_n(x)) \in \{-1, 0, +1\}^n$, where we have $h^1(x) = \mathbf{0} \iff g(x) = \mathbf{0}$. For $n = 1$, the conditions $h^1(x)h^1(x') \geq 0$ and $g(x)g(x') \geq 0$ are equivalent, but for $n = 2$, the former is implied by the latter and the converse is not true, so $h^1(x) \cdot h^1(x') \geq 0$ is more general for $n = 2$. They are independent if $n > 2$, which says that we can choose h^1 or g easier to use. Another example is $h^2(x) = (0, \cdots, \mathrm{sign}\, g_i(x), \cdots, 0)$ (at most one component is nonzero), choosing for each x an i such that $|g_i(x)| \geq |g_j(x)|$ for all $j = 1, \cdots, n$. Again we have $h^2(x) = \mathbf{0} \iff g(x) = \mathbf{0}$, and the conditions $h^2(x) \cdot h^2(x') \geq 0$ and $g(x) \cdot g(x') \geq 0$ are independent if $n \geq 2$.

We can think of such an h as a "view" of g (or of f). We can choose an appropriate view of g on the basis of application, and it seems that, for the high dimensional spaces, the condition is more likely to hold for function like h^2 than for g. Another advantage is semantic. Although it may be difficult to attach a meaning other than "geometrical" to the condition like $g(x) \cdot g(x') \geq 0$, the condition $h^1(x) \cdot h^1(x') \geq 0$ is interpreted such that "the signs of nonzero components are more (or no less) kept than reverted between x and x'", and $h^2(x) \cdot h^2(x') \geq 0$ such that, for functions having positive and negative components, "the maximum component and the minimum component are different at x and x'" (with the words of applications). In the next section we use such a view of excess demand function.

3 Existence of Equilibrium in Exchange Economy

3.1 The Model

We consider an exchange economy consisting of I agents $i = 1, \cdots, I$ and L types of indivisible commodities $l = 1, \cdots, L$. Since the quantities of the commodities are integers, the relative prices (the rates of exchange) are rational numbers and any system of relative prices can be represented by an integer vector. Let us first set our price space to $\mathbb{Z}^L_+ \setminus \{\mathbf{0}\}$ (the nonnegative orthant of $\mathbb{Z}^L$ excluding $\mathbf{0}$).

Each agent i has an endowment vector $\omega^i \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\}$. Let $\sum_{i=1}^{I} \omega^i \in \mathbb{Z}^L_{++}$ (the positive orthant of $\mathbb{Z}^L$), without loss of generality. Given any price vector $p \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\}$, each i is assumed to choose a demand vector $x^i(p) \in \mathbb{Z}^L_+$ such that $p \cdot x^i(p) \leq p \cdot \omega^i$.† For each i, the

†In general, the demand of agent is given as a set of alternative demand vectors, so this assumption of demand function is a very strong condition if it is taken to mean that the alternatives are singletons. A somewhat milder interpretation is that the functions are selections from correspondences (set-valued functions). See also the remark at the end of this paper.

excess demand function $z^i\colon \mathbb{Z}^L_+ \setminus \{\mathbf{0}\} \to \mathbb{Z}^L$ is defined by $z^i(p) = x^i(p) - \omega^i$ for $p \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\}$. Clearly, the Weak Walras Law holds at the individual level:

(WWL) $\quad p \cdot z^i(p) \leq 0 \quad (p \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\})$.

We assume, for $p, p' \in \mathbb{Z}^L_{++}$, the so-called Weak Axiom (of Revealed Preferences)

(WA) $\quad p \cdot z^i(p') \leq 0,\ p' \cdot z^i(p) \leq 0 \implies z^i(p) = z^i(p') \quad (p, p' \in \mathbb{Z}^L_{++})$.

This is a consistency axiom, which is satisfied, e.g., if the agent is a "utility maximizer". Then z^i is homogeneous of degree zero in the sense that

(H0) $\quad z^i(p) = z^i(tp) \quad (p \in \mathbb{Z}^L_{++},\ t > 0,\ tp \in \mathbb{Z}^L_{++})$,

since if $p' = tp$ and $t > 0$, then $p \cdot z^i(p') \leq 0$ and $p' \cdot z^i(p) \leq 0$ by (WWL), so $z^i(p) = z^i(p')$ by (WA). The zero-homogeneity means the absence of money illusion.

The aggregate excess demand function $z\colon \mathbb{Z}^L_+ \setminus \{\mathbf{0}\} \to \mathbb{Z}^L$ then is defined by $z(p) = \sum_{i=1}^I z^i(p)$ for $p \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\}$, and inherits (WWL) and (H0):

(WWL′) $\quad p \cdot z(p) \leq 0 \quad (p \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\})$,

(H0′) $\quad z(p) = z(tp) \quad (p \in \mathbb{Z}^L_{++},\ t > 0,\ tp \in \mathbb{Z}^L_{++})$.

So far we have assumed (WWL) and (WA), and obtained aggregate conditions (WWL′) and (H0′). We call a price vector $p^* \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\}$ such that $z(p^*) = \mathbf{0}$ an equilibrium price vector.

3.2 Existence of Equilibrium

In this section we consider the existence of an equilibrium in our model. It is our basic assumption that our model satisfies (WWL′) and (H0′). First, using (H0′), let us restrict the domain of z to a set

$$M_c = \{p \in \mathbb{Z}^L\colon \sum_{l=1}^L p_l = c,\ p_l \geq 0,\ l = 1, \cdots, L\}, \tag{3.1}$$

where c is some positive integer. As we noted M_c is an integrally convex set, and $\mathrm{co}(M_c)$ has a triangulation $\mathcal{S}$ such that if p and p' are vertices of some $S \in \mathcal{S}$ then $\|p - p'\|_\infty \leq 1$. Let us call such $\mathcal{S}$ the regular triangulation of $\mathrm{co}(M_c)$. The larger the value of c, the behavior of z on M_c gets closer to that on the unit simplex by (H0′). We do not specify how large c should be, however. Also, since the rôle of (H0′) is only to provide this M_c, we will omit (H0′) from our list of assumptions when M_c is used; it should be understood, however, that (H0′) is assumed when M_c is used. Second, let us assume a boundary condition

(Bd) $\quad p_l = 0 \implies z_l(p) > 0 \quad (l = 1, \cdots, L,\ p \in M_c)$.

Note that (Bd) implies that any equilibrium price vector is a positive vector.

Proposition 3.1. *Suppose $z\colon M_c \to \mathbb{Z}^L$ satisfies (WWL′) and (Bd). Suppose also*

(ED) if $z_l(p) < 0$ for some l, then $z_m(p) > 0$ for some m $(p \in M_c)$, and

(DP) if $z(p) \neq \mathbf{0}$, $z(p') \neq \mathbf{0}$, and $p, p' \in S$ for some $S \in \mathcal{S}$, then $\{m\colon z_m(p) \geq z_n(p),\ n = 1, \cdots, L\} \cap \{l\colon z_l(p') \leq z_n(p'),\ n = 1, \cdots, L\} = \emptyset$ $(p, p' \in M_c)$.

Then there is $p^* \in M_c$ such that $z(p^*) = \mathbf{0}$.

The condition (ED) says that "if some goods are supplied then some goods must be demanded". (DP) says that "the most demanded goods and the least demanded goods are different at close enough prices".

Proof. (WWL′) implies that if $z_m(p) > 0$ for some m then $z_l(p) < 0$ for some l, so, with (ED), there are m and l such that $z_m(p) > 0$ and $z_l(p) < 0$ if $z(p) \neq \mathbf{0}$. Define $h\colon M_c \to \{-1, 0, +1\}^L$ by

$$h(p) = \begin{cases} \mathbf{0} & \text{if } z(p) = \mathbf{0}, \\ e^m - e^l & \text{otherwise, where } z_m(p) \geq z_n(p),\ z_l(p) \leq z_n(p),\ n = 1, \cdots, L. \end{cases} \tag{3.2}$$

Clearly $h(p) = \mathbf{0} \iff z(p) = \mathbf{0}$. By (Bd), we have $p + h(p) \in M_c$ for all $p \in M_c$. The condition (DP) implies that $h(p) \cdot h(p') \geq 0$ if p, p' are vertices of a simplex in the regular triangulation of $\mathrm{co}(M_c)$. Hence, by Corollary 2.2, there is $p^* \in M_c$ such that $h(p^*) = \mathbf{0}$, i.e., $z(p^*) = \mathbf{0}$. □

Remark 3.2. The condition (DP) holds, in particular, if $\|z(p') - z(p)\|_\infty \leq 1$ whenever $\|p' - p\|_\infty \leq 1$.

Example 3.3. *Heterogeneous agents with unit demands:* Suppose that every agent demands each commodity at most one, and they have a sufficient amount of behavioral heterogeneity in the sense that if two price vectors p and p' are such that $\|p - p'\|_\infty \leq 1$ then at most one agent changes his demand. Then the excess demand function satisfies the condition in the previous remark. This type of market thus has an equilibrium under the conditions (WWL), (WA), (Bd), and (ED).

Remark 3.4. The condition (ED) is imposed in order to overcome the weakness of (WWL′). At the individual level, i.e., for each individual excess demand function z^i, it is possible to have a similar property if we assume a desirability condition such as

(M) $\quad z' \gneq z^i(p) \implies p \cdot z' > 0 \quad (z' \in \mathbb{Z}^L,\ p \in \mathbb{Z}^L_{++}),$

since $\mathbf{0} \gneq z^i(p)$ is impossible then ($x \gneq y$ denotes $x \geq y$ and $x \neq y$). But this of course is not inherited to the aggregate. We will consider other possible resolutions for this in the next section.

3.3 Some Variations

One of difficulties for our model to have an equilibrium is the lack of (strict) Walras law

(WL) $\quad p \cdot z(p) = 0 \quad (p \in \mathbb{Z}^L_+ \setminus \{\mathbf{0}\}).$

We only have its weaker form (WWL′) even if we assume desirability (or monotonicity) like (M) for all the agents. This is so, because each agent i's purchasing power $p \cdot \omega^i$ may not all be spent due to the indivisibility of goods (if he spends $p \cdot x^i(p) < p \cdot \omega^i$ and if $p \cdot (x^i(p) + e^l) > p \cdot \omega^i$ for all $l = 1, \cdots, L$, then he has $p \cdot \omega^i - p \cdot x^i(p) = -p \cdot z^i(p) > 0$ unusable purchasing power). This in particular leaves the possibility of $z(p) \lneq \mathbf{0}$, the overall excess supply.

Consider, then, our exchange economy *plus* the following rule:

(*) If $z(p) \lneq \mathbf{0}$, every agent donates his unusable purchasing power $-p \cdot z^i(p) \geq 0$ to a fictitious agent 0 (possibly an auctioneer), who uses the donations to buy all the remaining excess supply commodities.

Since the nominal purchasing power is useless when there is no goods to buy, the donation of useless powers may not harm the agents.[‡] Also, those who plan to sell the excess supply goods at positive prices can do so because the agent 0 will back up the buying (total amount $-p \cdot z(p) > 0$ is necessary and sufficient for buying $-z(p) \gneq \mathbf{0}$ under $p \in \mathbb{Z}^L_{++}$). This may be a collective rather than a decentralized way of disposing excess supply commodities, by which we will be able to call $p^* \in \mathbb{Z}^L_{++}$ such that $z(p^*) \lneq \mathbf{0}$ an equilibrium price vector. A zero of $h\colon M_c \to \{-1, 0, +1\}^L$ defined by

$$h(p) = \begin{cases} \mathbf{0} & \text{if } z(p) \leq \mathbf{0}, \\ e^m - e^l & \text{otherwise, where } z_m(p) \geq z_n(p),\ z_l(p) \leq z_n(p),\ n = 1, \cdots, L, \end{cases} \tag{3.3}$$

is obtained without (ED), with almost the same proof as before. Since $h(p^*) = \mathbf{0} \iff z(p^*) \leq \mathbf{0}$, we have the following

Proposition 3.5. *Suppose $z\colon M_c \to \mathbb{Z}^L$ satisfies (WWL′), (Bd), and (DP). Then there is $p^* \in M_c$ such that $z(p^*) \leq \mathbf{0}$. p^* is an equilibrium price vector under the rule (*).*

The example model of the behaviorally heterogeneous agents with unit demands thus has an equilibrium under (WWL), (WA), and (Bd), if the rule (*) is employed.

Another way of overcoming the lack of Walras law is to install it by introducing a numéraire (or money), whose value per piece is sufficiently small compared to that of the other goods. Let the Lth commodity be such a numéraire and consider z restricted on the price space $\mathbb{Z}^{L-1}_+ \times \{1\}$ (the price of numéraire is one). This makes no loss of generality under (H0′). For brevity, let $\hat{z}(q) = z(q, 1)$, where $q \in \mathbb{Z}^{L-1}_+$ is a price vector of goods $l = 1, \cdots, L-1$ (those prices are listed in terms of the quantity of numéraire). It is easy to see that if the agents have no satiation for the numéraire then the Weak Walras Law in the form of $q \cdot \hat{z}(q) + z_L(q, 1) \leq 0$ is strengthened to the strict Walras Law of

(WL′) $\quad q \cdot \hat{z}(q) + z_L(q, 1) = 0 \quad (q \in \mathbb{Z}^{L-1}_+)$,

consistently with the indivisibility of goods and integrality of prices. The typical model of exchange among agents with quasi-linear utilities (with sufficient amount of money) satisfies (WL′), in particular.

In this case, let

$$R_b = \{q \in \mathbb{Z}^{L-1}\colon 0 \leq q_l \leq b_l,\ l = 1, \cdots, L-1\}, \tag{3.4}$$

where b_l are some (large) positive integers, and assume a boundary condition

(Bd′) $\quad [q_l = 0 \implies \hat{z}_l(q) > 0],\ [q_l = b_l \implies \hat{z}_l(q) < 0] \quad (l = 1, \cdots, L-1,\ q \in R_b)$.

As we noted R_b is integrally convex and $\mathrm{co}(R_b)$ has a triangulation induced from the standard triangulation of $\mathbb{R}^{L-1}$, which we will call the standard triangulation of $\mathrm{co}(R_b)$. So, assume a "direction preserving" condition such as

(DP′) if $\hat{z}(q) \neq \mathbf{0}$, $\hat{z}(q') \neq \mathbf{0}$, and $q \leq q'$ or $q \geq q'$, $\|q - q'\|_\infty \leq 1$, then $\{m\colon \hat{z}_m(q) \geq \hat{z}_n(q),\ n = 1, \cdots, L-1\} \cap \{l\colon \hat{z}_l(q') \leq \hat{z}_n(q'),\ n = 1, \cdots, L-1\} = \emptyset \quad (q, q' \in R_b)$.

[‡]Another way of looking at the "donations" is that there is a fiat money issued by the agent 0, which itself is useless, and the moneys not spent are returned when $z(p) \lneq \mathbf{0}$.

Then a function $h\colon R_b \to \{-1, 0, +1\}^{L-1}$ defined as

$$h(q) = \operatorname{sign} \hat{z}_k(q) e^k, \text{ where } k \text{ is one of } |\hat{z}_k(q)| \geq |\hat{z}_n(q)|,\ n = 1, \cdots, L-1, \tag{3.5}$$

is l.g.d.p. on the standard triangulation of $\mathrm{co}(R_b)$ by (DP′), and $q + h(q) \in R_b$ for any $q \in R_b$ by (Bd′). Hence there is $q^* \in R_b$ such that $h(q^*) = \mathbf{0}$ due to Corollary 2.2. Note that $h(q^*) = \mathbf{0}$ is equivalent to $\hat{z}(q^*) = \mathbf{0}$, and $z_L(q^*, 1) = 0$ follows from $\hat{z}(q^*) = \mathbf{0}$ under (WL′). We thus have the following proposition.

Proposition 3.6. *Suppose $z\colon R_b \times \{1\} \to \mathbb{Z}^L$ satisfies (WL′), (Bd′), and (DP′). Then there is $q^* \in R_b$ such that $z(q^*, 1) = \mathbf{0}$.*

We repeat that (WL′) is easily established from (WWL′) under the existence of numéraire. The example model of heterogeneous agents with unit demands also satisfies (DP′), so, if the numéraire is introduced, (WL′) follows from the nonsatiation for it, and essentially (WA) and (Bd′) constitute the sufficient conditions for there to be an equilibrium.

Remark 3.7. Our equilibrium price vector p^* is an exact solution for $z(p) = \mathbf{0}$ (not an approximate one). We can conceive of a tâtonnement process defined by $p^{t+1} = p^t + h(p^t)$ using any h appeared in the propositions, since $p^t \in M_c$ (or R_b) for all $t = 0, 1, \cdots$ (t designates time). Its convergence is, however, another issue, which should be interesting.

Remark 3.8. The rôle of behavioral heterogeneity and unit demands in the example model may best be understood if we imagine a textbook style partial equilibrium diagram of excess demand function of such a market. The excess demand function then is “contiguous” if we use the terminology of [3].

Remark 3.9. In the field of “Discrete Convex Analysis”, our M_c is known as an “M-convex” set, and R_b is known to be both “M$^\natural$-convex” and “L$^\natural$-convex” (see [7]). All are subclasses of the integral convexity.

Remark 3.10. We have applied a discrete fixed point theorem for functions (actually its zero point corollary). A similar theorem is possible for correspondences whose images are “hole free” in that all the integral points in the convex hull of an image belong to the image, as in [4]. Aggregate excess demand set is hole free if all the individual excess demand sets are M$^\natural$-convex, for example, since M$^\natural$-convex sets are hole free and their Minkowski sum is also M$^\natural$-convex (see [7]). In this paper, however, we took a route free of this sort of convexity arguments.

Acknowledgments

The author thanks useful comments from anonymous referees.

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Manuscript received 28 March 2008
revised 11 October 2008
accepted for publication 15 October 2008

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