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MINTY VARIATIONAL PRINCIPLE FOR SET-VALUED VARIATIONAL INEQUALITIES*

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Abstract: It is well known that a solution of a Minty scalar variational inequality of differential type is a solution of the related optimization problem, under lower semicontinuity assumption. This relation is known as "Minty variational principle".In the vector case, the Minty variational principle has been investigated by F. Giannessi [15] and subsequently by X. M. Yang, X. Q. Yang, K. L. Teo [22]. For a differentiable objective function $f : \mathbb{R}^n \to \mathbb{R}^m$ it holds only for pseudoconvex functions. In this paper we extend the Minty variational principle to set-valued variational inequalities with respect to an arbitrary ordering cone and non smooth objective function. As a special case of our result we get that of [22].

Key words: *set-valued variational inequalities, minty variational principle, generalized convexity, optimization*

Mathematics Subject Classification: *49J53, 49J52, 47J20*

1 Introduction

The classical *Minty variational principle* (MVP, for short) established in [20] asserts that if x^0 is a solution of the scalar (and smooth) differentiable Minty *variational inequality* (VI, for short) i.e.

$$f'(x)(x^0 - x) \le 0 \quad \forall\, x \in K,$$

then x^0 is a global minimizer of the lower semicontinuous function $f : K \to \mathbb{R}$ (here K is a convex subset of $\mathbb{R}^n$). This remarkable result is a subject to various generalizations.

Let X be a linear space and K be a convex subset of X. In [4] and [6] we studied the nonsmooth Minty-type VI of differential type

$$f'_-(x, x^0 - x) \le 0, \quad x \in K\,. \tag{1.1}$$

Here $f'_-(x,u)$ denotes the lower Dini directional derivative of $f : X \to \mathbb{R}$ in the direction $u \in X$ defined for $x \in X$ as an element of $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ by

$$f'_-(x,u) = \liminf_{t\to 0^+} \frac{1}{t}(f(x+tu) - f(x))\,.$$

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It has been shown in [4], [6] that, under certain regularity assumptions on f, if x^0 is a solution of (1.1), then it is also a solution of the minimization problem

$$\min f(x)\,, \quad x \in K\,. \tag{1.2}$$

In [15] smooth Minty vector VI of differential type is introduced in the form

$$\langle \nabla f\,(x)\,, x^+ - x\rangle \notin \operatorname{int} C.$$

Its solutions are related to those of a vector optimization problem, when $f : X \to \mathbb{R}^m$ and $\mathbb{R}^m$ is ordered by the cone $\mathbb{R}^m_+$. Under convexity of f, any solution x^0 of the VI is also a weak efficient solution of the primitive vector minimization problem. This result is extended to pseudoconvex vector functions in [22].

The MVP has been in the focus of the investigation of several publications of the present authors. Scalar VI have been considered in [4] and [6]. Various aspects of vector VI approached through scalarization are investigated in [8], [5], [7], [9] and [10]. The scalarization technique is generalized from vector to set-valued VI in [11]. Such a generalization cannot be considered as straightforward and leads to new concepts. In particular, motivated by possible generalizations of the MVP, we discover that the underlying set-valued optimization problems reveals two types of solutions, called in [11] point minimizers and set minimizers. In [12] we investigate the MVP for vector VI generalizing the results of [22]. The present paper continues this investigation with respect to set-valued VI. We make use of the extension of the image space with infinite elements introduced in [7] and the concepts for set-valued VI developed in [11].

Roughly speaking, we distinguish between two types of set-valued VI, namely a-VI and w-VI. The MVP relates to the a-VI the so called a-minimizers (absolutely efficient points) and to the w-VI the so called w-minimizers (weakly efficient points). In addition, we state that the solutions of the VI obey also appropriate increasing along rays (IAR, for short) property. We investigate for each of the considered type VI also two other type of results, namely when the IAR property with respect to x^0 implies that x^0 is a solution of the VI, and when the given *set-valued function* (svf, for short) obeys the IAR property with respect to each of its minimizers. In particular, dealing with w-VI, we show that the mentioned result in [22] is a simple corollary of our results (see Corollary 4.16). This is the general picture of the research in the paper. Let us still underline, that the type of the investigated VI and the type of the considered minimizers undergo more precise specifications during the discussion.

2 Preliminaries

In the sequel X denotes a real linear space and K is a convex subset of X. A direction $u \in X$ is said feasible for K at $x \in X$, if the set $R_{x,u} = \{t \in \mathbb{R}_+ \mid x + tu \in K\}$ has $t = 0$ as an accumulating point. The set of the feasible directions of K at x is denoted by $K(x)$. Further Y is a finite-dimensional normed space and $C \subset Y$ is a closed convex cone with nonempty interior. We denote by Y^* the topological dual of Y, by S the unit sphere in Y^*, and by $\langle \cdot, \cdot \rangle$ the dual paring on $Y^* \times Y$. Further $C' = \{\xi \in Y^* \mid \langle \xi, y\rangle \geq 0,\, \forall y \in C\}$ is the positive polar cone of C. Recall that a vector $\xi \in C'$ is said to be an extreme direction of C' when $\xi \in C' \setminus \{0\}$ and $\xi = \xi^1 + \xi^2$, ξ^1, $\xi^2 \in C'$, implies $\xi^1 = \lambda_1 \xi$, $\xi^2 = \lambda_2 \xi$ for some positive reals λ_1, λ_2. We denote by $\operatorname{extd} C'$ the set of extreme directions of C'.

Following [7], the space Y can be extended with infinite elements (this can be done for an arbitrary linear space Y). An element $v \in Y \setminus \{0\}$ generates an infinite element v_∞, and we identify two infinite elements $v^1_\infty = v^2_\infty$ if and only if the generating vectors satisfy the equality $v^2 = \lambda v^1$ for some $\lambda > 0$. The element v_∞ is interpreted as the infinite element in direction v. The set of the infinite elements of Y is denoted Y_∞. We put $\tilde{Y} = Y \cup Y_\infty$. Recall that in the literature on vector optimization, sometimes by analogy with the scalar problems, one or two-point extensions with infinite elements of the image space Y are considered. Such extensions are related to the specific cone-ordering of Y. The proposed extension $\tilde{Y}$ has a reacher structure and we find it more appropriate for building vector analogues of the scalar optimization theory. It does not depend on the ordering cone. Hence, it appears to be more natural when introducing concepts, which in principal should not depend on the ordering, as say Dini-type derivative.

A topology on $\tilde{Y}$ can be introduced in terms of local bases of neighbourhoods (this can be done for arbitrary topological linear space). If $y \in Y$ and $\mathcal{B}(y)$ is a local base of neighbourhoods of y in Y, we accept that $\mathcal{B}(y)$ is also a local base of neighbourhoods of y in $\tilde{Y}$. The family $\mathcal{B}(v_\infty) = \{(y + W) \cup W_\infty \mid v \in W,\ W \text{ open cone in } Y,\ y \in Y\}$ constitutes a local base of neighbourhoods of the infinite point v_∞ generated by v. Here $W_\infty = \{w_\infty \mid w \in W \setminus \{0\}\}$. Saying that W is an open cone in Y, we mean that W is an open set in Y such that $\lambda W \subset W$ for all $\lambda > 0$. The extended topological space $\tilde{Y}$ has the following important property.

Theorem 2.1 ([7]). *When Y is finite-dimensional, the space $\tilde{Y}$ is compact.*

When $\xi \in Y^*$ and $v_\infty \in Y_\infty$ is an infinite element determined by $v \in Y$, we extend the dual pairing putting $\langle \xi, v_\infty \rangle = +\infty$ if $\langle \xi, v \rangle > 0$, $\langle \xi, v_\infty \rangle = 0$ if $\langle \xi, v \rangle = 0$, and $\langle \xi, v_\infty \rangle = -\infty$ if $\langle \xi, v \rangle < 0$.

Since $\tilde{Y}$ is a topological space, we can apply topological operations on $\tilde{Y}$. Obviously $\operatorname{cl} C = \tilde{C} := C \cup C_\infty$ where $C_\infty = \{v_\infty \mid v \in C \setminus \{0\}\}$. We have also $\operatorname{int} \tilde{C} = \operatorname{int} C \cup C^\circ_\infty$ where $C^\circ_\infty = \{v_\infty \mid v \in \operatorname{int} C\}$. We can consider also limits in $\tilde{Y}$, since the limit is a topological operation.

Let $F : K \rightsquigarrow Y$ be a *set-valued function* (svf, for short). The Dini derivative of F at the point (x, y), where $x \in K$ and $y \in F(x)$, in the feasible direction $u \in K(x)$, is defined as

$$F'(x, y; u) = \operatorname*{Limsup}_{t \to 0^+} \frac{1}{t} (F(x + tu) - y)\,. \tag{2.1}$$

The upper set-limit here is taken in $\tilde{Y}$. In other words $\bar{y} \in F'(x, y; u)$ if $\bar{y} \in \tilde{Y}$, and there exist a sequence $t_k \to 0^+$ and points $y^k \in F(x + t_k u)$ such that $\bar{y} = \lim_k (1/t_k)(y^k - y)$, where the limit is taken in $\tilde{Y}$. When $f : K \to Y$ is a single-valued function, the Dini derivative is denoted $f'(x, u)$ instead of $f'(x, f(x); u)$.

Remark 2.2. Since we restrict to finite dimensional spaces Y, $\tilde{Y}$ is compact. Therefore $F'(x, y; u) \neq \emptyset$ for every feasible u and $y \in F(x)$.

As in the vector case, we do not have a unique way to extend the inequality sign. In this paper we focus on the following formulations, in terms of Dini derivatives, referring to these VI respectively as a-VI, w-VI:

$$\forall\, y \in F(x) : F'(x, y; x^0 - x) \cap (-\tilde{C}) \neq \emptyset, \quad x \in K\,, \tag{2.2}$$

$$\exists\, y \in F(x) : F'(x, y; x^0 - x) \not\subset \operatorname{int} \tilde{C}, \quad x \in K\,, \tag{2.3}$$

According to a Minty variational principle, a-VI and w-VI, under some continuity and (generalized) convexity assumptions, should imply some global solution to

$$\min\nolimits_C F(x)\,, \quad x \in K\,, \tag{2.4}$$

Several notions of solution to (2.4) for set-valued optimization problem have been recently introduced, by using some order definitions among sets (see e.g. [21]). However, usually, minimizers in a set-valued framework are defined as pairs (x^0, y^0), $x^0 \in K$ and $y^0 \in F(x^0)$. We call these type of minimizers point minimizers, since they are defined through the properties of the point y^0. We distinguish this kind of solution from set minimizers, a notion which exploit the properties of the set $F(x^0)$. The definitions of the point and set a-minimizers and w-minimizers are given respectively in Sections 3 and 4. Clearly also the solution of the variational inequality should respect the alternative formulation. Indeed, it is unlikely that solution such as in (2.2) or (2.3) can be related to some point minimizer.

When F is single-valued function $f : K \to Y$, (2.2) and (2.3) reduce to the following vector VI (respectively a-vi and w-vi)

$$f'(x, x^0 - x) \cap (-\tilde{C}) \neq \emptyset, \quad x \in K\,, \tag{2.5}$$

$$f'(x, x^0 - x) \not\subset \operatorname{int} \tilde{C}, \quad x \in K\,, \tag{2.6}$$

According to MVP, solutions of (2.5) and (2.5) should imply some global solution of the vector optimization problem

$$\min\nolimits_C f(x)\,, \quad x \in K\,. \tag{2.7}$$

We distinguish the following kinds of solutions:

Definition 2.3. The point $x^0 \in K$ is said an a-minimizer (absolute, or ideal, efficient point) for problem (2.7) if $f(x) \in f(x^0) + C$ for all $x \in K$.

Definition 2.4. The point $x^0 \in K$ is said a w-minimizer (weakly efficient point) for problem (2.7) if $f(x^0) \notin f(x) + \operatorname{int} C$ for all $x \in K$.

In the sequel C-convex and C-quasiconvex set-valued functions play an important role. We recall after [2] some basic definitions and characterizations.

The definition and characterization of C-convex svf in Theorem 2.6 generalizes that of Luc [19] for vector C-convex functions. The definition and characterization of C-quasiconvex svf in Theorem 2.8 generalizes that of Benoist, Borwein, Popovici [1] for vector C-quasiconvex functions. We adapt these results to a finite dimensional space Y (the result in Benoist, Borwein, Popovici [1] for instance is formulated for a Banach space Y).

Here and further to a given svf $F : K \rightsquigarrow Y$ and $\xi \in Y^*$ we associate the scalar function $\phi_\xi : K \to \mathbb{R}$ defined by $\phi_\xi(x) = \inf_{y \in F(x)} \langle \xi, y \rangle$. We put also $F_\xi(x) = \{y \in F(x) \mid \langle \xi, y \rangle = \phi_\xi(x)\}$ (possibly empty).

Definition 2.5. The svf $F : K \rightsquigarrow Y$ is said to be C-convex if for every $x^1, x^2 \in K$ and $t \in [0, 1]$ it holds $(1-t)F(x^1) + tF(x^2) \subseteq F((1-t)x^1 + tF(x^2)) + C$.

Theorem 2.6 ([2]). *The svf $F : K \rightsquigarrow Y$ is C-convex, with the assumption that C is closed and convex cone with possibly empty interior, if and only if the function ϕ_ξ is convex for every $\xi \in C'$.*

Definition 2.7. The svf $F : K \rightsquigarrow Y$ is said to be C-quasiconvex if for every $y \in Y$ the level set $\operatorname{lev}_y F = \{x \in K \mid F(x) \cap (y - C) \neq \emptyset\}$ is convex.

Theorem 2.8 ([2]). *The svf $F : K \rightsquigarrow Y$ is C-quasiconvex, with the assumption that C is closed and convex cone with nonempty interior, if and only if the function ϕ_ξ is quasiconvex for every $\xi \in \operatorname{extd} C'$.*

Remark 2.9. Theorem 2.8 is not true if the quasiconvexity of ϕ_ξ is required for all $\xi \in C'$ instead of $\xi \in \operatorname{extd} C'$. In opposite, when the cone C has nonempty interior, Theorem 2.6 remains true requiring the convexity of ϕ_ξ only to $\xi \in \operatorname{extd} C'$ instead of $\xi \in C'$.

The following example may be useful.

Example 2.10. Let $F : [0,1] \rightsquigarrow \mathbb{R}^2$, with $F(x) = \left[x, x^2\right] \times \left[-x^3, x^3\right]$. The ordering cone is $C = \mathbb{R}^2_+$. Functions $\phi_{(0,1)}$ and $\phi_{(1,0)}$ are quasiconvex, but $\phi_{(1,1)}$ is not quasiconvex.

We say that some property holds radially at $x^0 \in K$, if the property is satisfied along every feasible ray starting at x^0. The ray starting at x^0 with direction $u \in K(x^0)$ is denoted by $R_{x^0,u} = \{x \in K \mid x = x^0 + tu \text{ for some } t \in \mathbb{R}_+\}$. For instance, we say that the svf $F : K \rightsquigarrow Y$ is radially C-quasiconvex at $x^0 \in K$, and write $F \in C\text{-}RQC(K, x^0)$, if the function F restricted to any ray $R_{x^0,u}$, $u \in K(x^0)$, is C-quasiconvex. Obviously, the svf $F : K \rightsquigarrow Y$ is C-quasiconvex if and only if $F \in C\text{-}RQC(K, x)$ for all $x \in K$. The radial notions can be used to relax some hypotheses. Similarly we state the following radial notion of semicontinuity.

Definition 2.11. The scalar function $\phi : K \to \mathbb{R}$ is said to be radially lower semicontinuous along the rays starting at $x^0 \in K$, denoted by $\phi \in RLSC(K, x^0)$, if for every $x \in K$ the function $\psi : [0,1] \to Y$, $\psi(t) = \phi(x^0 + t(x - x^0))$ is lower semicontinuous.

The following proposition has an immediate proof and we omit it.

Proposition 2.12. *Given the svf $F : K \rightsquigarrow Y$, if $\operatorname{int} C' \neq \emptyset$, $\phi_\xi \in RLSC(K, x^0)$ for all $\xi \in C'$ if and only if $\phi_\xi \in RLSC(K, x^0)$ for all $\xi \in \operatorname{extd} C'$.*

3 Set-valued VI and a-minimizers

In this section we deal with the solutions of the a-VI (2.2) and relate them to the set a-minimizers of the set-valued minimization problem (2.4).

Definition 3.1. The pair (x^0, y^0), $x^0 \in K$ and $y^0 \in F(x^0)$, is said a point a-minimizer of (2.4) if $F(x) \subset y^0 + C$ for all $x \in K$.

Definition 3.2. The point $x^0 \in K$ is said a set a-minimizer of (2.4) if $F(x) \subset F(x^0) + C$ for all $x \in K$.

One can easily notice that, if $\left(x^0, y^0\right)$ is a point a-minimizer, then x^0 is a set a-minimizer. However the converse is not necessarily true.

Example 3.3. Let $F : [0,1] \to \mathbb{R}^2$ be such that $F(x) = [0, x] \times [0, x]$. The ordering cone is $C = \left\{y \in \mathbb{R}^2_+ \, : \, y_2 \leq 2y_1\right\} \cap \left\{y \in \mathbb{R}^2_+ \, : \, y_2 \geq \frac{1}{2} y_1\right\}$, which is closed, convex, pointed and with not-empty interior.
It can be proved that $x^0 = 1$ is a set a-minimizer, although there is no $y^0 \in F\left(x^0\right)$ such that $\left(x^0, y^0\right)$ is a point a-minimizer.

In the case of a single-valued function $F = f$ the set a-minimizer coincides with the a-minimizer from Definition 2.3. The property $F(x) \subset F(x^0) + C$ from Definition 3.2 shows a similarity with the property $f(x) \in f(x^0) + C$ from Definition 2.3. Therefore it seems that the set a-minimizer is a good candidate as a generalization of the notion of a-minimizer from vector to set-valued problems. Nevertheless, the following notion of a minimizer, given in

Definition 3.4 and referred as *set A-minimizer*, is more appropriate to relate to the set-valued VI considered in this paper.

Given $\xi \in Y^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ we will denote by $H_+(\xi,\alpha)$ the set $H_+(\xi,\alpha) = \{y \in Y \mid \langle \xi, y\rangle \geq \alpha\}$.

Definition 3.4. We say that the point $x^0 \in K$ is a set A-minimizer of the set-valued minimization problem (2.4) if

$$F(x) \subset \bigcap \left\{ H_+(\xi, \phi_\xi(x^0)) \mid \xi \in \operatorname{extd} C' \cap \Gamma \right\} \quad \text{for all} \quad x \in K \setminus \left\{x^0\right\} . \tag{3.1}$$

Here Γ is a base of C'. Since $H_+(\xi,\alpha) = H_+(\lambda\xi, \lambda\alpha)$ for $\lambda > 0$, condition (3.1) does not depend on the choice of Γ.

Theorem 3.5. *For the set-valued minimization problem (2.4) every set a-minimizer is also a set A-minimizer.*

Proof. Let $x^0 \in K$ be a set a-minimizer of (2.4). Fix $x \in K$ and $y \in F(x)$. Since $y \in F(x) \subset F(x^0) + C$, we have $\langle \xi, y\rangle \geq \phi_\xi(x^0)$ and hence $y \in H_+(\xi, \phi_\xi(x^0))$ for all $\xi \in \operatorname{extd} C' \cap \Gamma$. Therefore it holds (3.1). □

The next example shows that in Theorem 3.5 the converse is not true.

Example 3.6. Let $X = K = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and let the svf $F : K \rightsquigarrow Y$ be given by

$$F(x) = \begin{cases} [(1,\,0),\,(0,\,1)]\,, & x = 0\,, \\ \{(0,\,0)\}\,, & x \in \mathbb{R}\setminus\{0\}\,. \end{cases}$$

Then $x^0 = 0$ is a set A-minimizer but not a set a-minimizer.

In the sequel we will consider the following VI referred as A-VI:

$$\forall\, \xi \in \operatorname{extd} C' : \forall y \in F_\xi(x) : \langle \xi, F'(x,y;x^0 - x)\rangle \cap (-\overline{\mathbb{R}}_+) \neq \emptyset\,, \quad x \in K\,. \tag{3.2}$$

Obviously, the validity of (3.2) does not change if we confine the choice of ξ to $\xi \in \operatorname{extd} C' \cap \Gamma$ where Γ is a base of C'.

When $F = f$ is single-valued, the A-VI (3.2) as the a-VI (2.2) coincides with a-vi (2.5).

Theorem 3.7. *Let the svf $F : K \rightsquigarrow Y$ be compact-valued. Then any solution x^0 of the a-VI (2.2) is also a solution of the A-VI (3.2).*

Proof. Fix $x \in K$. Take $\xi \in \operatorname{extd} C'$ and let $y \in F_\xi$. Since x^0 is a solution of (2.2), there exists $z \in F'(x,y;x^0 - x)$ such that $z \in -\tilde{C}$. The latter gives $\langle \xi, z\rangle \leq 0$ which shows that $\langle \xi, F'(x,y;x^0 - x)\rangle \cap (-\overline{\mathbb{R}}_+) \neq \emptyset$. □

The following example shows that the solutions of A-VI (3.2) need not be solutions of a-VI (2.2).

Example 3.8. Let $X = \mathbb{R}$, $K = [0,\,1/2]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$. Define the svf $F : K \rightsquigarrow Y$ by $F(x) = \{(1-x,\,-1+x),\,(-1+x,\,1-x)\}$. Then $x^0 = 0$ is a solution of A-VI (3.2) but is not so for a-VI (2.2).

To show in this example that $x^0 = 0$ is a solution of A-VI (3.2) it is enough to check it for $\xi^1 = (1, 0)$ and $\xi^2 = (0, 1)$. Fix $x \in K$. Now $F_{\xi^1}(x) = \{(-1+x, 1-x)\}$ and $F_{\xi^2}(x) = \{(1-x, -1+x)\}$.

For $\xi^1 = (1, 0)$, and $y^1 = (-1+x, 1-x) \in F_{\xi^1}(x)$ we have $F'(x, y^1; x^0 - x) = \{(-x, x), (1, -1)_\infty\}$. Now $\langle \xi^1, F'(x, y^1; x^0 - x)\rangle = \{-x, +\infty\}$ has a common point $-x$ with $-\mathbb{R}_+$. At the same time $F'(x, y^1; x^0 - x) \cap (-\tilde{C}) = \emptyset$.

For $\xi^2 = (0, 1)$ and $y^2 = (1-x, -1+x) \in F_{\xi^2}(x)$ we have $F'(x, y^2; x^0 - x) = \{(x, -x), (-1, 1)_\infty\}$. Now $\langle \xi^2, F'(x, y^2; x^0 - x)\rangle = \{-x, +\infty\}$ has a common point $-x$ with $-\mathbb{R}_+$. At the same time $F'(x, y^2; x^0 - x) \cap (-\tilde{C}) = \emptyset$.

The following theorem states the MVP for A-VI and generalizes a similar statement for vector VI proved in [7]. A simplified set-valued variant of this theorem can be found in [11].

Theorem 3.9 (MVP for A-VI). *Let the svf $F : K \rightsquigarrow Y$ be compact-valued, $x^0 \in K$, and $\phi_\xi \in RLSC(K, x^0)$ for all $\xi \in C'$. Let x^0 be a solution of A-VI (3.2) which satisfies the condition: for $\xi \in \operatorname{extd} C'$ and $y \in F_\xi(x)$ it holds $\langle \xi, F'(x, y; x^0 - x)\rangle \cap (-\operatorname{int} \overline{\mathbb{R}}_+) \neq \emptyset$ whenever $\langle \xi, F'(x, y; x^0 - x)\rangle \cap (-\mathbb{R}_+) = \emptyset$. Then the following properties have place:*

1^0 *(A-IAR property). For $u \in K(x^0)$ and $0 \le t_1 < t_2$ such that $x^0 + t_2 u \in K$ it holds*

$$F(x^0 + t_2 u) \subset \bigcap \big\{ H_+(\xi, \phi_\xi(x^0 + t_1 u)) \mid \xi \in \operatorname{extd} C' \cap \Gamma \big\} \,. \tag{3.3}$$

2^0 *(A-MIN property). The point x^0 is a set A-minimizer of problem (2.4).*

Proof. 1^0. We obtain the A-IAR property as a consequence of the A-MIN property formulated in point 2^0 and proved below. Define the svf function $F^0 : K_0 \rightsquigarrow Y$ where K_0 is the segment $K_0 = [x^0 + t_1 u,\, x^0 + t_2 u]$ and F^0 is the restriction of F on K_0. Consider A-VI (3.2) with F replaced by F^0 and K replaced by K_0. We claim $x^1 = x^0 + t_1 u$ is a solution of this VI. Indeed for any $x = x^0 + tu$, $t_1 \le t \le t_2$, by the positive homogeneity of the Dini derivative with respect to the direction we have

$$F'(x, y; x^1 - x) = F'(x, y; (1 - \tfrac{t_1}{t})(x^0 - x)) = (1 - \tfrac{t_1}{t}) F'(x, y; x^0 - x)\,. \tag{3.4}$$

Let $\xi \in \operatorname{extd} C'$ and $y \in F_\xi(x)$. Since x^0 is a solution of A-VI (3.2), there exists $z \in F'(x, y; x^0 - x)$ such that $\langle \xi, z\rangle \le 0$. Now $(1 - t_1/t)z \in F'(x, y; x^1 - x)$ and $\langle \xi, (1 - t_1/t)z\rangle = (1 - t_1/t)\langle \xi, z\rangle \le 0$. Thus, x^1 is a solution of the restricted A-VI and the function F^0 satisfies the hypotheses of the theorem.

According to the A-MIN property x^1 is a set A-minimizer of F^0 on K_0, which entails (3.3) when applied to x^1 and x^2.

2^0. Let x^0 be a solution of A-VI (3.2) and $\xi \in \operatorname{extd} C'$. We claim x^0 solves the scalar VI

$$(\phi_\xi)'_-(x, x^0 - x) \le 0\,, \quad x \in K\,. \tag{3.5}$$

Indeed fix $x \in K$ and take $y \in F_\xi(x)$. Since x^0 is a solution of A-VI (3.2), there exists $z \in F'(x, y; x^0 - x)$ such that $\langle \xi, z\rangle \le 0$. We may assume that $z = (1/t_k)(y^k - y)$, where $t_k \to 0^+$ and $y^k \in F(x + t_k(x^0 - x))$.

The following cases may arise:

a) $\langle \xi, F'(x, y; x^0 - x)\rangle \cap (-\mathbb{R}_+) \neq \emptyset$. Hence we can choose $z \in Y$. Now

$$\begin{aligned}(\phi_\xi)'_-(x, x^0 - x) &= \liminf_{t \to 0^+} \frac{1}{t}\big(\phi_\xi(x + t(x^0 - x)) - \phi_\xi(x)\big)\\ &\le \lim_k \frac{1}{t_k}\big(\langle \xi, y^k\rangle - \langle \xi, y\rangle\big) = \langle \xi, z\rangle \le 0\,.\end{aligned}$$

b) $\langle \xi, F'(x, y; x^0 - x)\rangle \cap (-\mathbb{R}_+) = \emptyset$. Then according to the hypotheses we may take $z \in \tilde{Y} \backslash Y$ such that $\langle \xi, z\rangle < 0$, and consequently $\langle \xi, z\rangle = -\infty$. In such a case $\langle \xi, (1/t_k)(y^k - y)\rangle < 0$ for all sufficiently large k. Therefore again

$$(\phi_\xi)'_-(x, x^0 - x) \le \liminf_k \langle \xi, \frac{1}{t_k}\left(z^k - y\right)\rangle \le 0 .$$

Thus, x^0 solves the scalar VI (3.5) with $\phi_\xi : K \to \mathbb{R}$ such that $\phi_\xi \in RLSC(K, x^0)$. According to [4, Theorem 2.2] the point x^0, being a solution of the scalar VI (3.5), is a global minimizer of ϕ_ξ, that is

$$\phi_\xi(x^0) \le \phi_\xi(x) , \quad \forall\, x \in K . \tag{3.6}$$

Hence for arbitrary $y \in F(x)$ it holds $\langle \xi, y\rangle \ge \phi_\xi(x) \ge \phi_\xi(x^0)$, whence $y \in H_+(\xi, \phi_\xi(x^0))$. Consequently we get (3.1) establishing that x^0 is a set A-minimizer of problem (2.4). □

Theorem 3.9 establishes besides the MVP for A-VI (that is that the hypotheses imply point 2^0), but also property (3.3) stating that F increases in some sense along the rays starting at x^0. Usually the MVP, like here, is accompanied by an *increasing along rays* (IAR, for short) property. The IAR property is associated to the considered type of minimizers. Property (3.3) is called A-IAR, since it is associated to the set A-minimizers. The notation $F \in A\text{-}IAR(K, x^0)$ will denote that the svf $F : K \to Y$ has the A-IAR property at the rays starting at x^0.

For short, we call also *A-MIN property* the statement that x^0 *is a set A-minimizer of problem (2.4)*, and *A-VI property* the statement that x^0 *is a solution of the A-VI (3.2)*. Similarly, the *a-MIN property* is the statement that x^0 *is a set a-minimizer of problem (2.4)*, and *a-VI property* is the statement that x^0 *is a solution of the a-VI (2.2)*.

Unfortunately, Theorem 3.9 do not imply x^0 is an $a - minimizer$.

Example 3.10. Let $X = \mathbb{R}$, $K = [0,\,1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and let $F : K \rightsquigarrow Y$ be given by $F(x) = [(x,\,0),\,(0,\,x)]$. Then $x^0 = 1$ is a solution of A-VI (3.2), but does not solve the a-VI (2.2). The point $x^0 = 1$ is a set A-minimizer of (2.4), but not a set a-minimizer.

Generally the IAR property associate to certain type of minimizers can be obtained in the following way. We say that the IAR property holds at x^0 if for any $u \in K(x^0)$ and $0 \le t_1 < t_2$ with $x^0 + t_2 u \in K$, if the point $x^0 + t_1 u$ is the minimal (in sense of the accepted notion of minimizer) between $x^0 + t_1 u$ and $x^0 + t_2 u$. For instance, with the notion of a set a-minimizer we associate the a-IAR property (and write $F \in a\text{-}IAR(K, x^0)$) determined by

$$F(x^0 + t_2 u) \subset F(x^0 + t_1 u) + C . \tag{3.7}$$

We, actually, failed to prove a Minty Variational Principle for a-type solutions. We leave the following conjecture as an open problem.

Conjecture 3.11 (MVP for a-VI). Let the svf $F : K \rightsquigarrow Y$ be compact-valued, $x^0 \in K$, and $\phi_\xi \in RLSC(K, x^0)$ for all $\xi \in C'$. Let x^0 be a solution of a-VI (3.2) which satisfies the condition: $F'(x, y; x^0 - x) \cap (-\mathrm{int}\, \tilde{C}) \ne \emptyset$ whenever $F'(x, y; x^0 - x) \cap (-C) = \emptyset$. Then x^0 is a set a-minimizer of (2.4) (if this is true, then it can be proved easily that also $F \in a\text{-}IAR(K, x^0)$).

Still, we can define a suitable IAR property which enhance optimality as well as solution to the a-VI.

Theorem 3.12. *Let for the svf $F : K \rightsquigarrow Y$ there exists a point $x^0 \in K$ such that $F \in$ a-$IAR(K, x^0)$. Then the following properties have place:*

1^0 *(a-MIN property). The point x^0 is a set a-minimizer of F on K.*

2^0 *(a-VI property). The point x^0 is a solution of the a-VI (2.2).*

Proof. 1^0. Take $x \in K$ arbitrary, and put in (3.7) $u = x - x^0$, $t_1 = 0$ and $t_1 = 1$. We get $F(x) \subset F(x^0) + C$ which shows that x^0 is a set a-minimizer of F.

2^0. Let $x \in K$ and $y \in F(x)$. Put in (3.7) $u = x - x^0$, $t_1 = 1 - t$ with $0 < t \le 1$, and $t_2 = 1$. We get $y \in F(x) \subset F(x + t(x^0 - x))$, and consequently $\frac{1}{t}\big(F(x + t(x^0 - x)) - y\big) \cap (-C) \ne \emptyset$. Since $F'(x, y; x^0 - x) \ne \emptyset$ in $\tilde{Y}$, and $\tilde{C}$ is the closure in $\tilde{Y}$ of C, we get from here $F'(x, y; x^0 - x) \cap \tilde{C} \ne \emptyset$. □

Theorem 3.9 remains true, if instead of the a-VI (2.2) we consider the finite a-VI, that is the VI

$$F'(x, y; x^0 - x) \cap (-C) \ne \emptyset\,. \tag{3.8}$$

The statement even simplifies, since the requirements on the infinite points are dropped: the existence of a solution x^0 of (3.8) implies that $F \in a$-$IAR(K, x^0)$, and x^0 is a set a-minimizer of F. Theorem 3.12 however fails with respect to point 2^0 when we confine to finite a-VI. In other words, $F \in a$-$IAR(K, x^0)$ does not imply that x^0 is a solution of VI (3.8). This remark underlines the advantage to deal with the a-VI (2.2) instead with the finite a-VI (3.8), and hence the advantage to have extended the space Y with infinite elements to $\tilde{Y}$. Within the a-VI we may say (provided Conjecture 3.11 is true) that the property $F \in a$-$IAR(K, x^0)$ is (nearly) equivalent to the property that x^0 is a solution of the a-VI. Similar equivalence within the finite a-VI does not have place.

Theorem 3.13. *Let the svf $F : K \rightsquigarrow Y$ be compact-valued and there exists a point $x^0 \in K$ such that $F \in A$-$IAR(K, x^0)$. Then the following properties have place:*

1^0 *(A-MIN property). The point x^0 is a set A-minimizer of F on K.*

2^0 *(A-VI property). The point x^0 is a solution of the A-VI (3.2).*

Proof. 1^0. Take $x \in K$ arbitrary, and put in (3.3) $u = x - x^0$, $t_1 = 0$ and $t_1 = 1$. We get (3.1)which shows that x^0 is a set A-minimizer of F.

2^0. Let $x \in K$ and $\xi \in \operatorname{extd} C'$. Put in (3.3) $u = x - x^0$, $t_1 = 1 - t$ with $0 < t \le 1$, and $t_2 = 1$. We get $F(x) \subset H_+(\xi, \phi_\xi(x + t(x^0 - x)))$. Therefore for $y \in F_\xi(x)$ we have $\langle \xi, y\rangle \ge \phi_\xi(x + t(x^0 - x))$. Let $t_k \to 0^+$ and $y^k \in F_\xi(x + t_k(x^0 - x))$. Passing to a subsequence, we may assume that $(1/t_k)(y^k - y) \to z \in F'(x, y; x^0 - x)$ (here the compactness of $\tilde{Y}$ is used). Now

$$\langle \xi, z\rangle = \lim_k \langle \xi, \frac{1}{t_k}\left(y^k - y\right)\rangle = \lim_k \frac{1}{t_k}\left(\phi_\xi(x + t_k(x^0 - x)) - \langle \xi, y\rangle\right) \le 0\,.$$

Therefore x^0 is a solution of A-VI (3.2). □

Remark 3.14. Theorem 3.9 remains true, if instead of the A-VI (3.2) we consider the finite A-VI, that is the VI

$$\forall\, \xi \in \operatorname{extd} C' : \forall y \in F_\xi(x) : \langle \xi, F'(x, y; x^0 - x)\rangle \cap (-\mathbb{R}_+) \ne \emptyset\,, \quad x \in K\,. \tag{3.9}$$

The statement even simplifies, since the requirements on the infinite points are dropped. Theorem 3.13 however fails with respect to point 2^0 when we confine to the finite A-VI (3.9), since the compactness of $\tilde{Y}$ is essential.

Remark 3.15. Without the assumption $F \in RLSC(K, x^0)$, Theorem 3.9 and Conjecture 3.11 are not true. Compare for instance with [4, Example 2.1]. However Theorems 3.12 and 3.13 hold true without assuming the RLSC property on F.

Theorem 3.9 cannot be reverted without further assumptions (see e.g. [4, Example 4.1]). As in the scalar case, we need to assume some convexity on F.

Theorem 3.16. *Let the point $x^0 \in K$ be a set A-minimizer of problem (2.4) with a svf $F : K \rightsquigarrow Y$. Suppose that F is C-quasiconvex. Then the following properties have place:*

1^0 *(A-IAR property). For $u \in K(x^0)$ and $0 \le t_1 < t_2$ such that $x^0 + t_2 u \in K$ the inclusion (3.3) holds.*

2^0 *(A-VI property). The point x^0 is a solution of the A-VI (3.2).*

Proof. 1^0. Since x^0 is a set A-minimizer, for $x \in K$ and $\xi \in \operatorname{extd} C'$ we have $F(x) \subset H_+(\xi, \phi_\xi(x^0))$, which gives $\phi_\xi(x^0) \le \phi_\xi(x)$. The function ϕ_ξ is quasiconvex on the base of Theorem 2.8 and attains its minimum at x^0. Therefore it is increasing along the rays starting at x^0. This gives that for $u \in K(x^0)$ and $0 \le t_1 < t_2$ with $x^0 + t_2 u \in K$ it holds $\phi_\xi(x^0 + t_1 u) \le \phi_\xi(x^0 + t_2 u)$, whence $F(x^0 + t_2 u) \subset H_+(\xi, \phi_\xi(x^0 + t_1 u))$. Since this is true for all $\xi \in \operatorname{extd} C'$, we get (3.3).

2^0. It follows from 1^0 and Theorem 3.13. □

Remark 3.17. Theorem 3.16 remains true, if the hypothesis that F is C-quasiconvex is relaxed to $F \in C\text{-}RCQ(K, x^0)$.

4 Set-valued VI and w-minimizers

Now we consider other kind of solutions to (2.4).

Definition 4.1. The pair (x^0, y^0), $x^0 \in K$ and $y^0 \in F(x^0)$, is said a point w-minimizer of (2.4) if $F(x) \cap (y^0 - \operatorname{int} C) = \emptyset$ for all $x \in K$.

Definition 4.2. The point $x^0 \in K$ is said a set w-minimizer of (2.4) if for each $x \in K$ there exists $y^0 \in F(x^0)$ such that $F(x) \cap (y^0 - \operatorname{int} C) = \emptyset$. Equivalently, x^0 is a set w-minimizer if for each $x \in K$ it holds $F(x^0) \not\subset F(x) + \operatorname{int} C$.

When $F = f$ single valued, both definitions collapse onto Definition 2.3. However, as the variational inequality implies a singleton as its solution, we prefer to focus on Definition 4.2.

Nevertheless, the following notion of a minimizer, given in Definition 4.3 and referred as *set W-minimizer*, is more appropriate to relate to the set-valued VI considered in this paper.

Given $\xi \in Y^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ we will denote by $H_+^\circ(\xi, \alpha)$ the half-space $H_+^\circ(\xi, \alpha) = \{y \in Y \mid \langle \xi, y \rangle > \alpha\}$.

Definition 4.3. We say that the point $x^0 \in K$ is a set W-minimizer of the set-valued minimization problem (2.4) if

$$F(x^0) \not\subset \bigcap \left\{ H_+^\circ(\xi, \phi_\xi(x)) \mid \xi \in \operatorname{extd} C' \cap \Gamma \right\} \quad \text{for all} \quad x \in K \setminus \left\{x^0\right\} . \tag{4.1}$$

Here Γ is a base of C'. Since $H_+^\circ(\xi, \alpha) = H_+^\circ(\lambda\xi, \lambda\alpha)$ for $\lambda > 0$, condition (4.1) does not depend on the choice of Γ.

Theorem 4.4. *For the set-valued minimization problem (2.4) every set W-minimizer is also a set w-minimizer.*

Proof. Let $x^0 \in K$ be a set W-minimizer of (2.4). Fix $x \in K$. Then there exists $y^0 \in F(x^0)$ and $\xi \in \operatorname{extd} C' \cap \Gamma$ such that $y^0 \notin H^\circ_+(\xi, \phi_\xi(x))$. Then also $y^0 \notin F(x) + \operatorname{int} C$, since otherwise we would have $y^0 = y + c$ for some $y \in F(x)$ and $c \in \operatorname{int} C$, and in consequence $\langle \xi, y^0 \rangle = \langle \xi, y \rangle + \langle \xi, c \rangle > \langle \xi, y \rangle \geq \phi_\xi(x)$. Therefore it holds $F\left(x^0\right) \not\subset F(x) + C$. $\square$

The next example shows that in Theorem 4.4 the converse is not true.

Example 4.5. Let $X = K = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and let the svf $F : K \rightsquigarrow Y$ be given by

$$F(x) = \begin{cases} \{(1/3,\, 1/3)\}\,, & x = 0\,, \\ [(1,\, 0),\, (0,\, 1)]\,, & x \in \mathbb{R} \setminus \{0\}\,. \end{cases}$$

Then $x^0 = 0$ is a set w-minimizer but not a set W-minimizer.

Investigating the MVP for w-VI we adopt an approach similar to the one for A-VI from the previous section. However, while the A-VI show similarities with the scalar case, the w-VI mark differences. In general the MVP for w-VI can be established only for special classes of functions. The following example shows that the MVP for w-VI is not valid even for vector VI with C-quasiconvex functions.

Example 4.6 ([12]). Let $X = \mathbb{R}$, $K = [0,\, 2]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $f : K \to Y$ given by

$$f(x) = \begin{cases} \left(0,\, (x-1)^2 - 1\right)\,, & x \in [0, 1]\,, \\ \left(-(x-1)^2,\, -1\right)\,, & x \in (1, 2]\,. \end{cases}$$

The function f is C-quasiconvex and the point $x^0 = 0$ is a solution of the vector VI (2.6), but it is not a w-minimizer of problem (2.7).

As it was pointed out, the MVP for special vector VI is proved in Giannessi [15] under C-convexity hypotheses and is generalized in [22] under C-pseudoconvexity hypotheses. Since the class of C-pseudoconvex functions is intermediate between the classes of C-convex and C-quasiconvex functions, and Example 4.6 shows that the MVP for w-VI fails for C-quasiconvex functions, we will work under hypotheses of pseudoconvexity.

Recall that the scalar function $\phi : K \to \mathbb{R}$ is said pseudoconvex if for all $x^1,\, x^2 \in K$, the inequality $\phi(x^1) > \varphi(x^2)$ implies $\phi'_-(x^1, x^2 - x^1) < 0$. We propose the following of pseudoconvexity for set-valued functions

Definition 4.7. We say that the svf $F : K \rightsquigarrow Y$ is C-pseudoconvex if all the scalar functions ϕ_ξ, $\xi \in \operatorname{extd} C'$, are pseudoconvex.

Remark 4.8. As a possibility for further generalization, let us do the following remark. The class of the (scalar) higher-order pseudoconvex functions introduced in [17] is more general than the class of pseudoconvex functions and less general than the class of quasiconvex functions. Therefore, it remains an open problem, whether the MVP proved in Theorem 4.12 admits a further generalization replacing the hypothesis that F is C-pseudoconvex with the more general hypothesis that the functions ϕ_ξ, $\xi \in \operatorname{extd} C' \cap \Gamma$, are higher-order pseudoconvex.

The notions of pseudoconvexity can be relaxed to radial notions as follows. For a given point $x^0 \in K$ the svf $F : K \rightsquigarrow Y$ is said radially C-pseudoconvex at x^0 if the restriction of F to the rays starting at x^0 is C-pseudoconvex. The class of radially C-pseudoconvex svf is denoted by $RPC(K, x^0)$.

Obviously, the notion of pseudoconvexity from Definition 4.7 reduces to the usual notion of pseudoconvexity when F is a single-valued scalar function. Let us however mention, that when F is a differentiable vector function, it does not coincide with the notion of C-pseudoconvexity for such functions introduced in [3] and generalized in [16]. For such functions the class of C-pseudoconvex functions in sense of Definition 4.7 is contained in the class of C-pseudoconvex functions in sense of Cambini [3] (see e.g. [12]).

Consider a continuous svf $F : K \rightsquigarrow Y$ for which the values $F(x)$ have non empty interiors. Let $x^0 \in K$ be arbitrary. For each $x \in K$ choose $y^x \in \operatorname{int} F(x)$. Now $F(x, y^x; x^0 - x) = \tilde{Y}$, whence x^0 is a solution of the $w-VI$ (2.3). This observation shows that to find a reasonable generalization of the MVP associated to set w-minimizers we need to modify the VI. Here we propose to consider the following VI referred in the sequel as W-VI:

$$\exists \xi \in \operatorname{extd} C' : \exists y \in F_\xi(x) : \langle \xi, F'(x, y; x^0 - x)\rangle \not\subset \operatorname{int} \overline{\mathbb{R}}_+ , \quad x \in K . \tag{4.2}$$

Obviously, the validity of (4.2) does not change if we confine the choice of ξ to $\xi \in \operatorname{extd} C' \cap \Gamma$ where Γ is a base of C'.

When F is single-valued, the W-VI (4.2) reduces to the w-vi (2.6). Any solution of W-VI (4.2) is also a solution of w-VI (2.3).

The following lemmas will be useful. To prove each of them we need to recall the following Mean-Value Theorem.

Theorem 4.9 (Diewert Mean Value Theorem [14]). *Let $\phi : [a, b] \to \mathbb{R}$ ($a < b$ reals) be a lsc functions. Then there exists a point $c \in [a, b)$ such that*

$$\phi'_-(c) \geq \frac{\phi(b) - \phi(a)}{b - a} .$$

Lemma 4.10. *Let $\phi : (a, b) \to \mathbb{R}$ ($a < b$ reals) be a lsc functions with negative lower directional derivative*

$$\phi'_-(t) = \liminf_{t \to 0^+} \frac{1}{h} \left(\phi(x + h) - \phi(x)\right) < 0 , \quad \forall x \in (a, b) .$$

Then ϕ is strictly decreasing.

Proof. Assume on the contrary that $\phi(t_1) \leq \phi(t_2)$ for some $t_1 < t_2$. From Theorem 4.9 below there exists a point $t_0 \in [t_1, t_2)$ such that

$$\phi'_-(t_0) \geq \frac{\phi(t_2) - \phi(t_1)}{t_2 - t_1} \geq 0 ,$$

a contradiction. □

The next Lemma recalls the structure of the lsc pseudoconvex functions and can be proved applying the Diewert Mean Value Theorem and Lemma 4.10. This result is well known and can be obtained by the structure of the quasiconvex lsc functions [13]. Some rigorous proof is given in [17].

Lemma 4.11. *Let $\phi : [a,\, b] :\to \mathbb{R}$ ($a < b$ reals) be a lsc pseudoconvex function. Denote by M the points in $[a,\, b]$ where ϕ attains its minimum. Then M is a closed interval $M = [\alpha,\, \beta]$ and ϕ is strictly decreasing on the interval $[a,\, \alpha]$ and strictly increasing on the interval $[\beta,\, b]$.*

The next theorem establishes the MVP for W-VI. We work under the hypothesis that the cone C is polyhedral. It is an open problem wether it can be extended to arbitrary cones.

Theorem 4.12 (MVP for W-VI). *Assume that the cone C is polyhedral. Let the svf $F : K \rightsquigarrow Y$ be compact-valued, and $x^0 \in K$ be a solution of W-VI (4.2). Suppose that F is C-pseudoconvex, and $\phi_\xi \in RLSC(K, x^0)$ for $\xi \in \operatorname{extd} C' \cap \Gamma$, where Γ is a base of C'. Then the following properties have place:*

1^0 *(W-IAR property). For $u \in K(x^0)$ and $0 \le t_1 < t_2$ such that $x^0 + t_2 u \in K$ it holds*

$$F(x^0 + t_1 u) \not\subset \bigcap \left\{ H_+^\circ(\xi, \phi_\xi(x^0 + t_2 u)) \mid \xi \in \operatorname{extd} C' \cap \Gamma \right\} . \tag{4.3}$$

2^0 *(W-MIN property). The point x^0 is a set W-minimizer of problem (2.4).*

Proof. 1^0. We obtain the W-IAR property as a consequence of the W-MIN property formulated in point 2^0 and proved below. Define the svf function $F^0 : K_0 \rightsquigarrow Y$ where K_0 is the segment $K_0 = [x^0 + t_1 u,\, x^0 + t_2 u]$ and F^0 is the restriction of F on K_0. Consider the W-VI (4.2) with F replaced by F^0 and K replaced by K_0. The point $x^1 = x^0 + t_1 u$ is a solution of this VI. To check this we take the point $x = x^0 + tu$, $t_1 \le t \le t_2$, and observe that the positive homogeneity of the Dini derivative with respect to the direction gives, as calculated in (3.4), $F'(x, y; x^1 - x) = (1 - t_1/t) F'(x, y; x^0 - x)$. Let $\xi \in \operatorname{extd} C'$, $y \in F_\xi(x)$ and $z \in F'(x, y; x^0 - x)$ be such that $\langle \xi, z \rangle \le 0$. Now $(1 - t_1/t) z \in F'(x, y; x^1 - x)$ and $\langle \xi, (1 - t_1/t) z \rangle = (1 - t_1/t) \langle \xi, z \rangle \le 0$. Thus, x^1 is a solution of the restricted W-VI and the function F^0 satisfies the hypotheses of the theorem. According to the W-MIN property x^1 is a set W-minimizer of F^0 on K_0, which in particular comparing the points x^1 and x^2 gives (4.3).

2^0. Suppose to the contrary that x^0 is not a set W-minimizer. Then for some $x^1 \in K$ and all $\xi \in \operatorname{extd} C' \cap \Gamma$ it holds $F(x^0) \subset H_+^\circ(\xi, \phi_\xi(x^1))$. This gives $\phi_\xi(x^0) > \phi_\xi(x^1)$. Put $u = x^1 - x^0$ and $x(t) = x^0 + tu$. The function $\phi_\xi(x(t))$ is pseudoconvex in t by the hypotheses. Because of the structure of the pseudoconvex functions (see Lemma 4.11), there exists $\delta_\xi \in (0,\, 1)$ such that $\phi_\xi(x(t))$ is strictly decreasing for $t \in [0,\, \delta_\xi]$. The set $\operatorname{extd} C' \cap \Gamma$ is finite, since the cone C is polyhedral. Put $\delta = \min\{\delta_\xi \mid \xi \in \operatorname{extd} C' \cap \Gamma\}$. Then $\delta > 0$ and all the functions $\phi_\xi(x(t))$, $\xi \in \operatorname{extd} C' \cap \Gamma$, are strictly decreasing on $[0,\, \delta]$. Since they are pseudoconvex, applying the positive homogeneity of the lower directional derivative we get $(\phi_\xi)'_-(x(t), x^1 - x(t)) = (1 - t)(\phi_\xi)'_-(x(t), x^1 - x^0) < 0$ for $t \in [0,\, \delta]$. Therefore $(\phi_\xi)'_-(x(t), x^1 - x^0) < 0$ for $t \in [0,\, \delta]$. The function ϕ_ξ being monotone is differentiable almost everywhere on $[0,\, \delta]$ (the proof of this result known as Theorem of Lebesgue can be found e. g. in [18, page 321]). Since the set $\operatorname{extd} C' \cap \Gamma$ is finite, there exists $\bar{t} \in (0,\, \delta)$ such that at $\bar{t}$ all the functions $\phi_\xi(x(t))$, $\xi \in \operatorname{extd} C' \cap \Gamma$, are differentiable. Put $\bar{x} = x(\bar{t})$. This gives

$$(\phi_\xi)'_-(\bar{x}, x^1 - x^0) = \phi'_\xi(\bar{x}, x^1 - x^0) = -\phi'_\xi(\bar{x}, x^0 - x^1) < 0 .$$

Thus, we have $\phi'_\xi(\bar{x}, x^0 - x^1) > 0$ for all $\xi \in \operatorname{extd} C' \cap \Gamma$. Now take arbitrary $y \in F_\xi(\bar{x})$ and $z \in F'(x, y; x^0 - \bar{x})$. We have $z = (1/t_k)(y^k - y)$ for some $t_k \to 0^+$ and $y^k \in F(\bar{x} + t_k(x^0 - \bar{x}))$. Therefore

$$\begin{aligned} \langle \xi, z \rangle &= \lim_k \langle \xi, \frac{1}{t_k} \left(y^k - y \right) \rangle = \lim_k \frac{1}{t_k} \left(\langle \xi, y^k \rangle - \phi_\xi(\bar{x}) \right) \\ &\ge \lim_k \frac{1}{t_k} \left(\phi_\xi(\bar{x} + t_k(x^0 - \bar{x})) - \phi_\xi(\bar{x}) \right) = \phi'_\xi(\bar{x}, x^0 - \bar{x}) > 0 . \end{aligned}$$

This is a contradiction with the hypothesis that x^0 is a solution of $W-VI$ (4.2). $\square$

Remark 4.13. The assumption of C-pseudoconvexity can be relaxed to $F \in C\text{-}RPC(K, x^0)$.

Theorem 4.12 remains true, if instead of W-VI (4.2) we consider the finite W-VI, that is

$$\exists\, \xi \in \operatorname{extd} C' : \exists\, y \in F_\xi(x) : \langle \xi, F'(x, y; x^0 - x)\rangle \not\subset \operatorname{int} \mathbb{R}_+ , \quad x \in K . \tag{4.4}$$

where the limit in (2.1) defining the Dini derivative is taken in Y instead of $\tilde{Y}$.

Since from Theorem 4.4 every set W-minimizer is also a set w-minimizer, weakening the thesis of Theorem 4.12 we get immediately the following result.

Theorem 4.14 (weak MVP for W-VI). *Assume that the cone C is polyhedral. Let the svf $F : K \rightsquigarrow Y$ be compact-valued, and $x^0 \in K$ be a solution of W-VI (4.2). Suppose that F is C-pseudoconvex (or more generally $F \in C\text{-}RPC(K, x^0)$), and $\phi_\xi \in RLSC(K, x^0)$ for $\xi \in \operatorname{extd} C' \cap \Gamma$, where Γ is a base of C'. Then the following properties have place:*

1^0 *(w-IAR property). For $u \in K(x^0)$ and $0 \le t_1 < t_2$ such that $x^0 + t_2 u \in K$ it holds*

$$F(x^0 + t_1 u) \not\subset F(x^0 + t_2 u) + \operatorname{int} C . \tag{4.5}$$

2^0 *(w-MIN property). The point x^0 is a set w-minimizer of problem (2.4).*

When $F = f$ is a single-valued function the W-VI (4.2) reduces to the vector w-vi (2.6). On this base from Theorem 4.12 we obtain for the vector VI the following result.

Corollary 4.15 (MVP for w-vi). *Let the cone C be polyhedral and $x^0 \in K$ be a solution of w-vi (2.6). Suppose that $f : K \to Y$ is C-pseudoconvex (or more generally $f \in C\text{-}RPC(K, x^0)$), and $\phi_\xi \in RLSC(K, x^0)$ for $\xi \in \operatorname{extd} C' \cap \Gamma$, where Γ is a base of C'. Then the point x^0 is a w-minimizer of the vector problem (2.7).*

As a special case we obtain also the following result.

Corollary 4.16 (X. M. Yang, X. Q. Yang, K. L. Teo [22]). *Let X be a normed space, $Y = \mathbb{R}^m$ and $C = \mathbb{R}^m_+$. Let the function $f : K \to Y$ be of class C^1. Put $f = (f_1, \ldots, f_m)$ and assume that the coordinate functions f_i, $i = 1, \ldots, m$, are pseudoconvex. If $x^0 \in K$ is such that for every $x \in K$ there exists an index i for which $f_i'(x)(x^0 - x) \le 0$, then x^0 is a w-minimizer for problem (2.7).*

Now, similarly to Theorem 3.13, we prove that the W-IAR property at x^0 implies that x^0 is a solution of the W-VI. Let us underline, that the cone C now is not polyhedral, and that the result uses essentially the extension of Y with infinite elements to $\tilde{Y}$.

Theorem 4.17. *Let the svf $F : K \rightsquigarrow Y$ be compact-valued, and such that there exists a point $x^0 \in K$ where F has the W-$IAR(K, x^0)$ property. Then the following properties have place:*

1^0 *(W-MIN property). The point x^0 is a set W-minimizer (and hence a set w-minimizer) of problem (2.4).*

2^0 *(W-VI property). The point x^0 is a solution of W-VI (4.2) (and hence of w-VI (2.3)).*

Proof. 1^0. Take $x \in K$ arbitrary, and put in (4.3) $u = x - x^0$, $t_1 = 0$ and $t_1 = 1$. Now (4.3) reduces to (4.1) which shows that x^0 is a set W-minimizer of (2.4).

2^0. Take $x \in K$ arbitrary and put in (4.3) $u = x - x^0$, $t_1 = 1 - t$ with $0 < t \le 1$, and $t_2 = 1$. We get $F(x + t(x^0 - x)) \not\subset H_+^\circ(\xi(t), \phi_{\xi(t)})$ for some $\xi(t) \in \operatorname{extd} C' \cap \Gamma$. Therefore there exist $\xi(t) \in \operatorname{extd} C' \cap \Gamma$ such that

$$\langle \xi(t), F(x + t(x^0 - x)) \rangle \not\subset \langle \xi(t), F(x) \rangle + \operatorname{int} \mathbb{R}_+ \,.$$

Since Γ is compact, there exist a sequence $t_k \to 0^+$ and $\xi^0 \in \operatorname{extd} C' \cap \Gamma$, such that $\xi(t_k) \to \xi^0$. Now

$$\langle \xi^0, F(x + t_k(x^0 - x)) \rangle \not\subset \langle \xi^0, F(x) \rangle + \operatorname{int} \mathbb{R}_+$$

for all sufficiently large k. Take $y^0 \in F_{\xi^0}(x)$ and let $y^k \in F(x + t_k(x^0 - x))$ be such that

$$\langle \xi^0, y^k \rangle \notin \langle \xi^0, F(x) \rangle + \operatorname{int} \mathbb{R}_+ = \phi_{\xi^0}(x) + \operatorname{int} \mathbb{R}_+ = \langle \xi^0, y^0 \rangle + \operatorname{int} \mathbb{R}_+ \,.$$

This gives

$$\langle \xi^0, \frac{1}{t_k}(y^k - y^0) \rangle \notin \operatorname{int} \mathbb{R}_+ \,.$$

Because $\tilde{Y}$ is compact, passing to a subsequence, we may assume that $(1/t_k)(y^k - y^0) \to z \in \tilde{Y}$. Now we have $z \in F'(x, y^0, x^0 - x)$, $y^0 \in F_{\xi^0}(x)$, and $\langle \xi^0, z \rangle \notin \operatorname{int} \overline{\mathbb{R}}_+$. This shows that x^0 is a solution of W-VI (4.2) (the lack of compactness of Y makes this property not true for the finite W-VI (4.4)). Further, since any solution of the W-VI is a solution also of the corresponding w-VI, x^0 is also a solution of w-VI (2.3). □

After Theorem 4.17 it is natural to pose the question whether under pseudoconvexity conditions, the set W-minimizers of problem (2.4) are also solutions of the W-VI. The next Theorem 4.18 gives an affirmative answer. It states even more, that this result is valid for C-quasiconvex functions. In connection with this, let us recall the structure of the scalar quasiconvex functions (see e. g. [13, Section 3]): Let $\phi : [a,\, b] \to \mathbb{R}$ be quasiconvex. Then there exists $t \in [a,\, b]$ so that, either ϕ is non increasing on $[a,\, t]$ and nondecreasing on $(t,\, b]$, or ϕ is non increasing on $[a,\, t)$ and nondecreasing on $[t,\, b]$.

Theorem 4.18. *Consider W-VI (4.2). Let the svf $F : K \rightsquigarrow Y$ be C-quasiconvex. Suppose that the point $x^0 \in K$ is a set W-minimizer of problem (2.4). Then the following properties hold.*

1^0 *(W-IAR property). For $u \in K(x^0)$ and $0 \le t_1 < t_2$ such that $x^0 + t_2 u \in K$ the relation (4.3) is true.*

2^0 *(W-VI property). If in addition F is compact-valued, then x^0 is a solution of W-VI (4.2).*

Proof. 1^0. Assume to the contrary, that the W-IAR property does not hold. Then for some $u \in K(x^0)$, $0 < t_1 < t_2$, and for all $\xi \in \operatorname{extd} C' \cap \Gamma$, we would have $F(x^0 + t_1 u) \subset H_+^\circ(\xi, \phi_\xi(x^0 + t_2 u))$, or equivalently $\phi_\xi(x^0 + t_2 u) < \phi_\xi(x^0 + t_1 u)$. Since ϕ_ξ is pseudoconvex, this inequality implies $\phi_\xi(x^0 + t_1 u) \le \phi_\xi(x^0)$. Consequently $\phi_\xi(x^0 + t_2 u) < \phi_\xi(x^0)$ for all $\xi \in \operatorname{extd} C' \cap \Gamma$. These inequalities give

$$F(x^0) \subset \bigcap \left\{ H_+^\circ(\xi, \phi_\xi(x^0 + tu)) \mid \xi \in \operatorname{extd} C' \cap \Gamma \right\} \,,$$

a contradiction the assumption x^0 is a set W-minimizer.

2^0. We proved that the hypotheses imply the W-IAR property, which on the base of Theorem 4.17 implies the W-VI property. □

Remark 4.19. The assumption of C-quasiconvexity can be relaxed to $F \in C\text{-}RQC(K, x^0)$.

Let us underline that Theorem 4.18 in opposite to Theorem 4.18 does not assume neither that the cone C is polyhedral, nor $\phi_\xi \in RLSC(K, x^0)$ for $\xi \in \operatorname{extd} C' \cap \Gamma$.

The following example shows that, dealing with set w-minimizers, Theorem 4.18 is not true. Even more, it shows that a C-convex svf $F : K \rightsquigarrow Y$ having $x^0 \in K$ as a set w-minimizer need not have the w-IAR property (4.5).

Example 4.20. Let $X = \mathbb{R}$, $K = [0, \beta]$ with $0 < \beta \le 1/4$, $Y = \mathbb{R}^3$, $C = \mathbb{R}^3_+$. We consider Y as the Euclidean space, then Y^* is identified with Y and the dual pairing with the scalar product $\langle \xi, x \rangle = \xi^1 x_1 + \xi^2 x_2 + \xi^3 x_3$. Define the svf $F : K \rightsquigarrow Y$ by

$$F(x) = \begin{cases} \text{co}\left\{(x, \frac{x+1}{2}, \frac{x+1}{2}), (\frac{x+1}{2}, x, \frac{x+1}{2}), (\frac{x+1}{2}, \frac{x+1}{2}, x)\right\}, & 0 \le x < \beta, \\ \text{co}\left\{(1, x, x), (x, 1, x), (x, x, 1)\right\}, & x = \beta. \end{cases}$$

Then F is compact-valued and convex-valued and it is C-convex, the point $x^0 = \beta$ is a set w-minimizer of F (hence set W-minimizer), but the w-IAR propert (4.5) fails.

To explain the example put $\xi^1 = (1, 0, 0)$, $\xi^2 = (0, 1, 0)$, $\xi^3 = (0, 0, 1)$. The svf F is C-convex because the functions $\phi_{\xi^i}(x) = x$, $x \in [0, \beta]$, are convex for $i = 1, 2, 3$.

The point $x^0 = \beta$ is a set w-minimizer because $F(\beta) \not\subset F(x) + \operatorname{int} C$ for $0 \le x < \beta$. Indeed, any point $y = (y_1, y_2, y_3) \in F(x)$ has at least two coordinates greater than $1/4$. To show this, rearranging the coordinates of y, we can find numbers $\lambda \ge \mu \ge \nu$, $\lambda + \mu + \nu = 1$, and $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, such that the coordinates of y are

$$\begin{aligned} \frac{\lambda+\mu}{2} + \left(\frac{\lambda+\mu}{2} + \nu\right) t + c_1 &> \frac{\lambda+\mu}{2}\,, \\ \frac{\lambda+\nu}{2} + \left(\frac{\lambda+\nu}{2} + \mu\right) t + c_2 &> \frac{\lambda+\nu}{2}\,, \\ \frac{\mu+\nu}{2} + \left(\frac{\mu+\nu}{2} + \lambda\right) t + c_3 &> \frac{\mu+\nu}{2}\,. \end{aligned}$$

Since $1 = \lambda + \mu + \nu \ge 3\nu$ and $1 = \lambda + \mu + \nu \ge 2\mu$ we get $\nu \le 1/3$ and $\mu \le 1/2$ whence

$$\frac{\lambda+\mu}{2} = \frac{1}{2} - \frac{\nu}{2} \ge \frac{1}{3} > \frac{1}{4}\,, \qquad \frac{\lambda+\nu}{2} = \frac{1}{2} - \frac{\mu}{2} \ge \frac{1}{4}\,.$$

At the same time any of the points $(1, \beta, \beta)$, $(\beta, 1, \beta)$, $(\beta, \beta, 1)$, being vertices of $F(\beta)$, has at least two coordinates not exceeding $1/4$.

The svf F does not satisfy the w-IAR property, since for $0 < t < \beta$ it holds evidently $F(t) \subset F(0) + \operatorname{int} C$.

5 List of Abbreviations

MVP :	Minty Variational Principle
VI :	Variational Inequality
IAR :	Increasing along rays
svf :;	set-valued function
$C - RQC\left(K, x^0\right)$:	radially C-quasiconvex at $x^0 \in K$
$RLSC\left(K, x^0\right)$:	radially lower semicontinuous along rays starting at $x^0 \in K$.

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