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CONVEX EXPECTED RESIDUAL MODELS FOR STOCHASTIC AFFINE VARIATIONAL INEQUALITY PROBLEMS AND ITS APPLICATION TO THE TRAFFIC EQUILIBRIUM PROBLEM*

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Abstract: The affine variational inequality problem (AVIP) is a wide class of problems which includes the quadratic programming problems and the linear complementarity problem. In this paper, we consider AVIP under uncertainty in order to present a more realistic view of real world problems. We call such a problem the stochastic affine variational inequality problem (SAVIP). Recently, a new approach called the expected residual (ER) method has been proposed to give a reasonable solution of SAVIP. The ER method regards a minimizer of an expected residual function for the AVIP as a solution of SAVIP. Previous studies on the ER method employed the "min" function or the Fischer-Burmeister (FB) function. Such functions however are nonconvex in general and hence we may not get a global solution. In this paper, we employ the regularized gap function and the D-gap function to define a residual in the ER model. We also show that our proposed ER models are convex under some conditions and hence a global solution can be obtained using existing solution methods. Finally we apply our proposed model to the traffic equilibrium problem under uncertainty using a sample network.

Key words: *stochastic affine variational inequality problem, expected residual method, regularized gap function, D-gap function, convexity, traffic equilibrium problem*

Mathematics Subject Classification: *65K10, 90C15, 90B20*

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1 Introduction

The affine variational inequality problem (AVIP) is to find $x \in S$ such that

$$\langle Mx + q, y - x\rangle \geq 0,\ \forall y \in S,$$

where $S = \{y \in R^n \mid Ay = b, y \geq 0\}$ with $A \in R^{m\times n}$ and $b \in R^m$, and $M \in R^{n\times n}$, $q \in R^n$. The AVIP is a wide class of problems which includes the quadratic programming problem and the linear complementarity problem.

In the real world, the coefficients M, q, A and b usually contain uncertainty. Hence it is more appropriate to take into account uncertainty in the formulation of the AVIP. Hence we consider the stochastic affine variational inequality problem (SAVIP) which is to find $x \in S(\omega)$ such that

$$\langle M(\omega)x + q(\omega), y - x\rangle \geq 0,\ \forall y \in S(\omega), \tag{1.1}$$

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where $S(\omega) = \{y \in R^n \mid A(\omega)y = b(\omega), y \geq 0\}$ with $A : \Omega \to R^{m\times n}$ and $b : \Omega \to R^m$, $M : \Omega \to R^{n\times n}$, $q : \Omega \to R^n$ and $(\Omega, \mathcal{F}, P)$ is a probability space with $\Omega \subseteq R^l$. When $S(\omega) \equiv R^n_+$, the problem is reduced to the stochastic linear complementarity problem (SLCP) [2].

There is no vector x satisfying (1.1) for all $\omega \in \Omega$ in general. We may consider two approaches in order to get a reasonable solution of SAVIP. One is the ***expected value*** (EV) method which formulates the problem as follows: Let $\bar{M} = E[M(\omega)]$, $\bar{q} = E[q(\omega)]$, $\bar{A} = E[A(\omega)]$ and $\bar{b} = E[b(\omega)]$, where E denotes the expectation. The EV formulation is to find a vector $x \in \bar{S} = \{x | \bar{A}x = \bar{b},\ x \geq 0\}$ such that

$$\langle \bar{M}x + \bar{q}, y - x\rangle \geq 0, \quad \forall y \in \bar{S}.$$

Another approach is the ***expected residual*** (ER) method which makes use of a residual function for AVIP. The ER method solves the following optimization problem:

$$\begin{array}{ll} \min & E[r(x,\omega)] \\ \text{s.t.} & x \in X, \end{array}$$

where $r(\cdot,\omega) : R^n \to R_+$ is a residual function for the variational inequality problem.

For the stochastic complementarity problem, the previous studies [2, 3, 5, 7] made use of an NCP function to formulate ER models. A function $\phi : R^2 \to R$ is called an *NCP function* if it has the property: $\phi(a,b) = 0 \Leftrightarrow a \geq 0, b \geq 0, ab = 0$. The two popular NCP functions are

"min" function $\phi(a,b) = \min(a,b)$

Fischer-Burmeister (FB) function $\phi(a,b) = a + b - \sqrt{a^2 + b^2}$.

All NCP functions are said to be equivalent in the sense that they can reformulate any complementarity problem as a system of nonlinear equations having the same solution set [2].

The ER model with the "min" function has been studied in [2, 3, 5] for the stochastic linear complementarity problem (SLCP). In particular, it is shown that, for a class of SLCPs, if the EV model has a bounded solution set, then the ER model also has a bounded solution set, but the converse is not true in general. Moreover, if $M(\omega)$ is a stochastic R_0 matrix, then the ER model has a bounded solution set. Recall that a stochastic matrix $M(\cdot)$ is a stochastic R_0 matrix [5] if $x \geq 0,\ M(\omega)x \geq 0,\ x^T M(\omega)x = 0$ *a.e.* implies that $x = 0$.

Thus we can expect to obtain a solution of the ER model by using existing solution methods. However, there is no guarantee that such a solution is a global optimal solution of the ER model. The following example shows the nonconvexity of the ER model with the natural residual function.

Example 1.1. Consider the SLCP with $G(x,\omega) = \left\{ \begin{array}{ll} 5x - 1 & \text{if } \omega = 1 \\ 2.7x - 0.9 & \text{if } \omega = 2 \end{array} \right.$ and $\Omega = \{1, 2\}$, $p(1) = p(2) = \frac{1}{2}$, and $r(x,\omega) = \min\ (x, G(x,\omega))^2$. Then the expected residual function $E[r(x,\omega)]$ is not convex as shown in Figure 1.

Luo and Lin [8] considers the stochastic variational inequality problem (SVIP) which is to find a vector $x \in S \subseteq R^n$ such that

$$\langle G(x,\omega), y - x\rangle \geq 0,\ \forall y \in S, \tag{1.2}$$

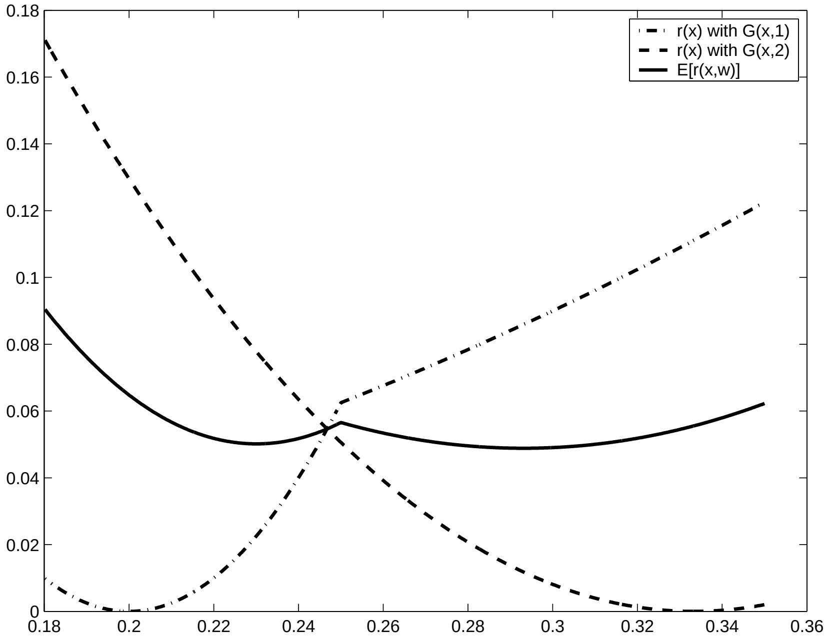

Figure 1: The ER function with the natural residual for Example 1.1.

where $G : R^n \times \Omega \to R^n$. The SVIP (1.2) is a generalization of the SLCP studied in [2, 3, 5]. In [8], the authors consider the ER model with the regularized gap function

$$g(x, \omega) = \max_{y \in S} \langle G(x, \omega), x - y \rangle - \frac{\alpha}{2} \|x - y\|^2,$$

where G is an affine function, that is, $G(x, \omega) = M(\omega)x + q(\omega)$. They establish the differentiability of this regularized gap function and the objective function $E[g(x, \omega)]$ of the ER model. They also establish the conditions for the level boundedness of $E[g(x, \omega)]$. They then propose a quasi-Monte Carlo method to solve the ER model for the SVIP by means of sequential approximation of $E[g(x, \omega)]$. The convergence properties of such an approximation method have also been established. However, they do not consider the convexity of the ER model.

In this paper, we consider the ER model for the stochastic affine variational inequality problem (SAVIP) based on the regularized gap function and the D-gap function for the AVIP. In particular, we establish convexity of both the regularized gap function and the D-gap function and show that the resulting ER models with the proposed residual functions are convex.

This paper is organized as follows. In the next section, we introduce the regularized gap function and the D-gap function for the AVIP, and establish the convexity results for those functions. The proposed ER models are then presented in Section 3. We also establish the convexity results for these models in this section. In Section 4, we discuss an important problem in which the main results of this paper can be applied − the traffic equilibrium problem (TEP) under uncertainty. Computational results for the TEP under uncertainty with the proposed ER models are given in Section 5. We give a brief conclusion in Section 6.

$\boxed{2}$ Convexity of the Regularized Gap Function and D-Gap Function for AVIP

In this section, we show that the regularized gap function and the D-gap function are convex when M is positive definite. The results are extensions of [10] where these functions are shown to be convex for LCP.

Let $G(x) = Mx+q$ and $S = \{x \in R^n | Ax = b, x \geq 0\}$. Then the regularized gap function $f_\alpha : R^n \to R$ and the D-gap function $g_\alpha : R^n \to R_+$ for the AVIP are defined, respectively, by

$$f_\alpha(x) = \max_{y \in S} \left\{ \langle G(x), x-y \rangle - \frac{1}{2\alpha} \|y-x\|^2 \right\} \tag{2.1}$$

and

$$g_\alpha(x) = f_\alpha(x) - f_{1/\alpha}(x),$$

where $\alpha > 1$ is a positive constant.

In what follows we show the main results of this section which are natural extensions of [10, Theorems 2.1 and 3.1].

Theorem 2.1. *Suppose that S is nonempty and M is positive definite. Then the following statements hold.*

(a) *The regularized gap function f_α is convex for all $\alpha \geq \frac{1}{\beta_{\min}}$, where $\beta_{\min} > 0$ is the minimum eigenvalue of $M+M^T$. Moreover, if $\alpha \geq \frac{1}{\beta_{\min}}(1+\beta)$ with a positive constant β, then f_α is strongly convex with modulus β.*

(b) *The D-gap function g_α is convex for all $\alpha \geq \bar{\alpha}$, where $\bar{\alpha}$ is given by*

$$\bar{\alpha} = \max_{\|x\|=1} \frac{1 + x^T M^T M x}{2x^T M x} > 0.$$

Moreover, g_α is strongly convex with modulus $\beta > 0$ for all $\alpha \geq \bar{\alpha} + \beta$.

Proof. (a) Suppose that $\alpha \geq \frac{1}{\beta_{\min}}(1+\beta)$ with a nonnegative constant β. Then $v^T(\alpha(M + M^T) - I)v \geq \beta\|v\|^2$ for all $v \in R^n$. It then follows that the maximand in (2.1) is convex in x for any y, and hence f_α is convex. Moreover, if $\beta > 0$, then the maximand is strongly convex with modulus β for every y, and hence f_α is also strongly convex with modulus β.

(b) First notice that $-f_{1/\alpha}(x)$ is the optimum value of the following convex quadratic programming problem:

$$\begin{array}{ll} \min_y & -\langle G(x), x-y \rangle + \frac{\alpha}{2}\|y-x\|^2 \\ \text{s.t.} & Ay = b \\ & y \geq 0. \end{array} \tag{2.2}$$

Then by direct calculation, the Lagrangian dual problem of (2.2) is formulated as

$$\begin{array}{ll} \max_{(\lambda,\mu)} & h(x,\lambda,\mu) \\ \text{s.t.} & \lambda \in R^m \\ & \mu \geq 0, \end{array}$$

where $\lambda \in R^m$ and $\mu \in R^n$ are the Lagrange multipliers of (2.2), and $h(x, \lambda, \mu)$ is given by

$$
\begin{aligned}
h(x,\lambda,\mu) &= -\frac{1}{2\alpha}\|A^T\lambda-\mu\|^2 - \langle b,\lambda\rangle - \frac{1}{\alpha}\langle A^T\lambda-\mu, Mx+q-\alpha x\rangle \\
&\quad -\frac{1}{2\alpha}\left\langle x, \left(M^TM-\alpha(M+M^T)+\alpha^2 I\right)x\right\rangle - \frac{1}{\alpha}\langle q,(M-\alpha I)x\rangle \\
&\quad -\frac{1}{2\alpha}\|q\|^2 + \frac{\alpha}{2}\|x\|^2 - \langle x, Mx\rangle - \langle q, x\rangle.
\end{aligned}
$$

By the duality theorem, we have $-f_{1/\alpha}(x) = \max\{h(x,\lambda,\mu) | \lambda \in R^m, \mu \geq 0\}$. Hence, the D-gap function is written as

$$
\begin{aligned}
g_\alpha(x) &= \max_{y\in S}\{\langle G(x), x-y\rangle - \frac{1}{2\alpha}\|y-x\|^2\} + \max_{\lambda\in R^m,\mu\geq 0} h(x,\lambda,\mu) \\
&= \max_{y\in S,\lambda\in R^m,\mu\geq 0}\left\{\langle G(x), x-y\rangle - \frac{1}{2\alpha}\|y-x\|^2 + h(x,\lambda,\mu)\right\} \\
&= \max_{y\in S,\lambda\in R^m,\mu\geq 0} p(x,y,\lambda,\mu),
\end{aligned}
$$

where

$$
p(x,y,\lambda,\mu) = \langle G(x), x-y\rangle - \frac{1}{2\alpha}\|y-x\|^2 + h(x,\lambda,\mu).
$$

Next we show that $p(\cdot, y, \lambda, \mu)$ is convex for every fixed (y, λ, μ). Note that

$$
\begin{aligned}
\nabla_x^2 p(x,y,\lambda,\mu) &= M+M^T - \frac{1}{\alpha}I - \frac{1}{\alpha}M^TM + M + M^T - \alpha I + \alpha I - M - M^T \\
&= M + M^T - \frac{M^TM+I}{\alpha}
\end{aligned}
$$

Then we can deduce that $M + M^T - \frac{M^TM+I}{\alpha}$ is positive semidefinite for any $\alpha \geq \bar{\alpha}$ in a way similar to the proof of [10, Theorem 3.1]. Therefore $p(\cdot, y, \lambda, \mu)$ is convex for all (y, λ, μ), and hence g_α is convex. □

Remark 2.2. In [10], the linear complementarity problem (LCP) with the general cone is considered. We can extend Theorem 2.1 to the AVIP with the general cone by assuming Slater's constraint qualification.

3 ER Models for SAVIP and Their Convexity

We formulate two ER models using the regularized gap function and the D-gap function. Let $G(x, \omega) = M(\omega)x + q(\omega)$. Then the regularized gap function and the D-gap function with random variable $\omega \in \Omega$ are defined by

$$
f_\alpha(x,\omega) = \max_{y\in S(\omega)}\left\{\langle G(x,\omega), x-y\rangle - \frac{1}{2\alpha}\|y-x\|^2\right\}
$$

and

$$
g_\alpha(x,\omega) = f_\alpha(x,\omega) - f_{1/\alpha}(x,\omega).
$$

Using these functions, we formulate the following two ER models:

$$\begin{array}{lll} \text{ER-R} & \min & E[f_\alpha(x,\omega) + \tau\|A(\omega)x - b(\omega)\|] \\ & \text{s.t.} & x \geq 0. \end{array}$$

$$\begin{array}{lll} \text{ER-D} & \min & E[g_\alpha(x,\omega)] \\ & \text{s.t.} & x \in R^n. \end{array}$$

The parameter $\tau > 0$ in ER-R is used for controlling the balance between the residual and the feasibility.

Let $\theta_\alpha^R(x)$ and $\theta_\alpha^D(x)$ be the objective functions of ER-R and ER-D, respectively, i.e.,

$$\begin{aligned} \theta_\alpha^R(x) &= E[f_\alpha(x,\omega) + \tau\|A(\omega)x - b(\omega)\|], \\ \theta_\alpha^D(x) &= E[g_\alpha(x,\omega)]. \end{aligned}$$

Now we investigate the conditions under which $\theta_\alpha^R(x)$ and $\theta_\alpha^D(x)$ are convex.

We call $M(\omega)$ uniformly positive definite with modulus β_0 if there exists a positive constant β_0 such that

$$\inf_{\omega\in\Omega, \|x\|=1} x^T M(\omega)x \geq \beta_0.$$

Theorem 3.1. *Suppose that $M(\omega)$ is uniformly positive definite with modulus β_0. Suppose also that $S(\omega)$ is nonempty for all $\omega \in \Omega$. Then the following statements holds.*

(a) *θ_α^R is convex for all $\alpha \geq \frac{1}{2\beta_0}$ and strongly convex with modulus $\beta > 0$ for all $\alpha \geq \frac{1}{2\beta_0}(1+\beta)$.*

(b) *Suppose that $M(\omega)$ is bounded on Ω. Then θ_α^D is convex for all $\alpha \geq \bar{\alpha}$, where $\bar{\alpha}$ is given by*

$$\bar{\alpha} = \sup_{w\in\Omega, \|x\|=1} \frac{1 + x^T M(\omega)^T M(\omega)x}{2x^T M(\omega)x}.$$

Moreover, θ_α^D is strongly convex with modulus $\beta > 0$ for all $\alpha \geq \bar{\alpha} + \beta$.

Proof. Since the sum of (strongly) convex functions is (strongly) convex, the statements (a) and (b) follow from Theorem 2.1.

The theorem indicates that both ER-R and ER-D are convex programming problems, and hence we can obtain a global optimal solution using existing solution methods. These methods include the quasi-Newton methods and the interior point methods [9, 1].

We show the effects of α on $E[g_\alpha(x,\omega)]$ in the following example.

Example 3.2. Let $\Omega = \{1, 2\}$ and $p(1) = p(2) = \frac{1}{2}$. Let $G(x,\omega) = \begin{cases} 5x - 1 & \text{if } \omega = 1 \\ 2.7x - 0.9 & \text{if } \omega = 2, \end{cases}$ where $S(\omega) = \{x \in R \,|x \geq 0\}$. Figures 2 and 3 show that $E[g_\alpha(x,\omega)]$ becomes convex when α is large.

Theorem 3.3. *Suppose that $M(\omega)$ is uniformly positive definite. Suppose also that $S(\omega)$ is nonempty for all $\omega \in \Omega$. Then there exists a solution of ER-D. Moreover, if $\alpha \geq \frac{1}{2\beta_0}(1+\beta)$, then there exists a solution of ER-R.*

Proof. Since $M(\omega)$ is uniformly positive definite, $g_\alpha(x,\omega)$ is coercive for all $\omega \in \Omega$, see Proposition 10.3.9 in [4]. Therefore, θ_α^D is also coercive. Hence, ER-D has a solution.

Moreover, if $\alpha \geq \frac{1}{2\beta_0}(1+\beta)$, it follows from Theorem 3.1 (a) that θ_α^R is strongly convex. Thus, ER-R has a solution. □

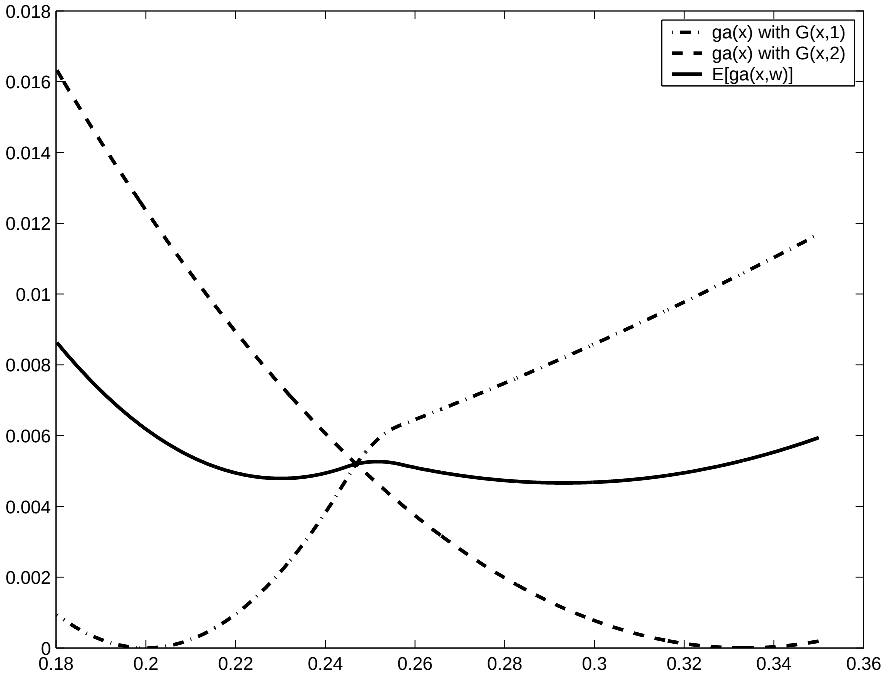

Figure 2: The ER-D function for Example 3.2 when $\alpha = 1.1$.

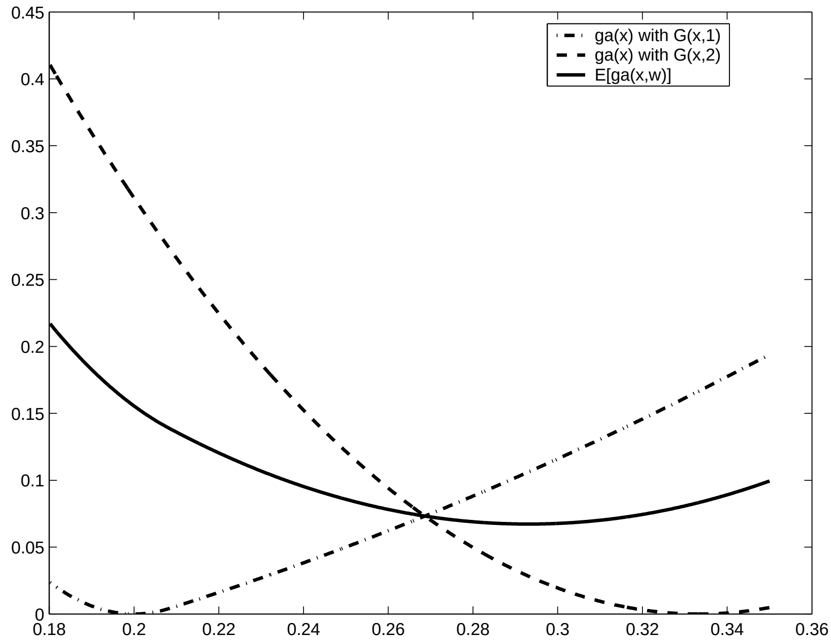

Figure 3: The ER-D function for Example 3.2 when $\alpha = 5$.

$\boxed{4}$ Application to the Traffic Equilibrium Problem

In this section, we apply the results obtained in the previous sections to the traffic equilibrium problem (TEP) under uncertainty.

We consider a network $\mathcal{G} = (\mathcal{A}, \mathcal{N})$, where $\mathcal{A}$ is the set of arcs (with cardinality $n_{\mathcal{A}}$) and $\mathcal{N}$ is the set of nodes (with cardinality $n_{\mathcal{N}}$). We denote by W the set of origin-destination (OD) pairs in $\mathcal{G}$ (with cardinality n_{W}). For every OD pair $\mathrm{w} \in \mathrm{W}$, there corresponds the set R_{w} of routes connecting the OD pair w. We denote by R the set of all routes (with cardinality n_R), i.e., $R = \bigcup_{\mathrm{w}\in\mathrm{W}} R_{\mathrm{w}}$. We assume that the network $\mathcal{G}$ is connected, that is, there exists a route between each pair of nodes. The cost experienced by a person using route r is denoted by C_r. In general, route costs can be a function of the entire vector of route flows. The travel demand associated with each OD pair w, denoted by D_{w}, is a function of the vector of minimum OD travel costs.

The TEP is to find a vector pair (F,u) of route flows and minimum route costs satisfying the Wardrop user equilibrium conditions [11] which can be represented as the variational inequality problem [4]

$$\langle C(F^*), F - F^* \rangle \geq 0, \ \forall F \in S = \{F \in R^{n_R} | F \geq 0, \ \Gamma^T F = D(u)\}. \tag{4.1}$$

Here, $F \in R_+^{n_R}$ is the vector of route flows F_r, u_{w} is the minimal route cost for the OD pair w, $u \in R_+^{n_{\mathrm{W}}}$ is the vector with components u_{w} and Γ is the route-OD pair incidence matrix whose entries are given by

$$\Gamma_{r\mathrm{w}} = \begin{cases} 1 & \text{if} \quad r \in R_{\mathrm{w}} \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we are concerned with the special case where the travel demands do not depend on the route costs, which is the case of fixed travel demands.

Let Ω denote the sample space of factors contributing to the uncertainty in the traffic network, such as weather and accidents. For each event $\omega \in \Omega$, we assign an occurrence probability p. Let

u_{w} :	minimal route cost for OD pair $\mathrm{w} \in \mathrm{W}$,
u :	vector with components u_{w},
$D_{\mathrm{w}}(\omega)$:	travel demand under uncertainty for OD pair $\mathrm{w} \in \mathrm{W}$,
$D(\omega)$:	vector with components $D_{\mathrm{w}}(\omega)$,
$C(F,\omega)$:	vector of route cost functions $C_r(F,\omega)$.

The traffic equilibrium problem with uncertainty can be written as the following stochastic variational inequality problem (SVIP): Find F such that

$$\langle C(F^*,\omega), F - F^* \rangle \geq 0, \ \forall F \in S(\omega) = \{F | F \geq 0, \Gamma^T F = D(\omega)\}. \tag{4.2}$$

The route cost function C_r is defined by

$$C_r(F,\omega) = \sum_{a\in\mathcal{A}} \kappa_{ar} t_a((KF),\omega), \tag{4.3}$$

where $K = (\kappa_{ar})$ is the arc-route incidence matrix with elements

$$\kappa_{ar} = \begin{cases} 1 & \text{if route } r \text{ passes through arc } a \\ 0 & \text{otherwise,} \end{cases}$$

and t_a is the travel time with uncertainty on arc a.

Note that the TEP with uncertainty (4.2) – (4.3) may also be written as the following stochastic mixed complementarity problem (SMCP):

$$\begin{aligned} &0 \le F \perp C(F,\omega) - \Gamma u \ge 0, \\ &\Gamma^T F - D(\omega) = 0, \end{aligned} \tag{4.4}$$

where $x \perp y$ means vector x and y are perpendicular to each other. However, we observe that

$$M(\omega) = \begin{pmatrix} \nabla_F C(F,\omega) & -\Gamma \\ \Gamma^T & 0 \end{pmatrix}$$

is not a positive definite matrix, even if C is affine with respect to F and $\nabla_F C(F,\omega)$ is positive definite. Hence, we cannot apply Theorems 3.3 and 3.1 to the SMCP formulation.

Remark 4.1. When the travel time $t_a(\cdot,\omega)$ is an affine function for each a and any ω, the SVIP (4.2) – (4.3) becomes the SAVIP. In (4.2), $M(\omega) = \nabla_F C(F,\omega)$ is positive definite under some conditions such as that t_a is an increasing function of the link flows. Hence, by Theorems 3.1 and 3.3, the convexity of the proposed ER models (ER-R and ER-D) guarantees that we can obtain a global solution of the ER model for the AVIP formulation of the TEP with uncertainty.

In the following, we give a particular example to illustrate the meaning of the solutions obtained by the ER-R and ER-D models.

Example 4.2. Consider the case where there are two events, ω_1 and ω_2 that can happen, say, $\omega_1 = fine\ day$ and $\omega_2 = rainy\ day$. The TEP without uncertainty only considers the case when the traffic users know the exact weather of the day. That is, where either ω_1 happens with probability 1 or ω_2 happens with probability 1. However, in reality, nobody can exactly predict the weather, and the available weather information such as the weather forecast cannot be trusted completely. The TEP with uncertainty considers the case when the occurrence probability of ω_1 is, say, 0.6 and that of ω_2 is, say, 0.4. The solution obtained by the ER model is regarded as the traffic flow pattern that satisfies the equilibrium condition on average.

5 Numerical Experiments

In this section, we present our computational results. In the numerical experiments, we solve the TEP with uncertainty (4.2) – (4.3) using the ER-D model proposed in Section 3. We solve the problem using the solver **fminunc** in the Optimization Toolbox of Matlab. We employ the quadratic programming solver **quadprog** of Matlab to compute $g_\alpha(x,\omega)$ for each $\omega \in \Omega$. The TEP under uncertainty is solved using different values of the parameter α to find its influence on the solution obtained. Moreover, we consider the case where $D(\omega)$ is fixed for all $\omega \in \Omega$ and the case where there is also uncertainty in $D(\omega)$. We also solve the MCP formulation (4.4) of the TEP under uncertainty using the ER method with Fischer-Burmeister (FB) function and compare the solutions obtained with those of the ER-D method.

The sample network shown in Figure 4 is used in our experiment. The attributes of this sample network are given in Table 1. We use the linear link cost function given by $t_a(f,\omega) = H(\omega)f + k(\omega)$, where $f = KF$ is the vector of link flows f_a, $k_i(\omega)$ represents the

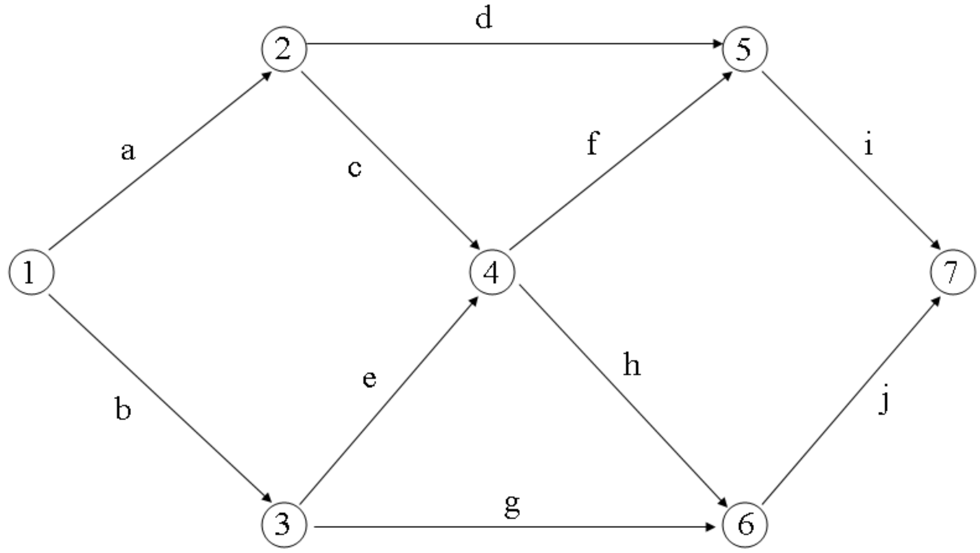

Figure 4: A sample traffic network.

Table 1: OD pair, routes and links of the sample network.

O-D pair	Routes	Links
1-7	1	{a,d,i}
	2	{a,c,f,i}
	3	{a,c,h,j}
	4	{b,e,f,i}
	5	{b,e,h,j}
	6	{b,g,j}

free travel cost of link i and $H_{ij}(\omega)$ represents the magnitude of the effect of flows on link j to the link cost of link i. The corresponding values of $H(\omega)$ and $k(\omega)$ are as follows:

$$H(\omega) = \begin{pmatrix} 22 & 0 & 2 & 2 & 4 & 1 & 2 & 0 & 4 & 5 \\ 0 & 15 & 0 & 0 & 1 & 2 & 0 & 3 & 5 & 3 \\ 2 & 0 & 14 & 0 & 2 & 0 & 1 & 3 & 2 & 3 \\ 2 & 0 & 0 & 16+50\omega & 0 & 2 & 3 & 1 & 2 & 4 \\ 4 & 1 & 2 & 0 & 12 & 0 & 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 10 & 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & 3 & 2 & 0 & 11 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 2 & 0 & 0 & 14 & 0 & 1 \\ 4 & 5 & 2 & 2 & 0 & 1 & 0 & 0 & 16+50\omega & 0 \\ 5 & 3 & 3 & 4 & 0 & 2 & 0 & 1 & 0 & 20 \end{pmatrix}$$

and $k(\omega) = [50, 30, 40, 40 + 60\omega, 30, 50, 20, 60, 40 + 40\omega, 70]^T$.

Note that in this sample network, only the costs of links $d = (2, 5)$ and $i = (5, 7)$ depend on the random variable ω. We assume that ω is uniformly distributed in the interval

$[\frac{1}{2}-\delta, \frac{1}{2}+\delta]$. Hence, the expectation of ω is $\frac{1}{2}$ and its variance is $\frac{\delta^2}{3}$.

In our experiments, we choose L samples of ω from the interval $[\frac{1}{2}-\delta, \frac{1}{2}+\delta]$ to approximate the actual continuous distribution. Hence, the occurrence probability of each event ω_i is $p_i = \frac{1}{L}$. We set $L = 21$. The value of α have been arbitrarily chosen in our numerical experiments.

5.1 Comparison of Link Flows for Different Values of α

In this experiment, we look at the influence of α. Here we set $\delta = 0.1$ and assume the fixed demand $D(\omega) = 200$ for all $\omega \in \Omega$. We present the results for various values of α in Tables 2 and 3. Table 2 shows that some route flows obtained are negative for small values of α. However, as the value of α becomes large, the route flows obtained become all positive. In Table 3, it can be seen that as the value of α increases, the link flows on links $d = (2, 5)$ and $i = (5, 7)$ increase correspondingly. It is also interesting to observe that as the value of α becomes larger, the corresponding route flows obtained get closer to satisfying the demand, as shown in Figure 5. Moreover, the increase in the total route flow is small when α is large, so the effect of α to a solution is small for large α.

Table 2: Route flows for different values of α when $D(\omega) = 200$.

α	Route Flows
3.3	(35.85, 19.85, 8.47, 17.72, -2.28, 148.07)
5	(35.76, 19.81, 8.46, 17.68, -2.28, 147.77)
18	(29.39, 21.20, 2.45, 8.57, 3.40, 125.25)
50	(29.90, 23.63, 0.39, 6.58, 5.58, 127.24)
100	(30.38, 24.24, 0.13, 6.43, 5.93, 129.13)
1000	(31.00, 18.67, 6.14, 12.56, 0.06, 131.53)

5.2 The Case of Travel Demand with Uncertainty

In our experiments, we also consider the case where the travel demand is subject to uncertainty. Here, we set $\delta = 0.1$ and assume that the travel demand is given by $D(\omega) = 500\omega - 100$. The results are shown in Table 4.

It can be seen from Table 4 that some route flows obtained are negative when α is small. It is also observed that, similar to the case where the demand is fixed as $D(\omega) = 200$, the total route flow increases as the value of α increases. It can be seen from Figure 6 that as α becomes large, the total route flow approaches 150. Moreover, the increase in the total route flow is small when α is large. Hence, the effect of α to a solution is small when α is large.

Remark 5.1. Note that even if ER-D is not convex, i.e., when α is small, we may still obtain a reasonable solution. However, it can be seen from Table 2 and Table 4 that the solutions tend to be infeasible when α is small.

5.3 Comparison of ER-D Model with Another ER Model

In this experiment, we also compare the ER-D model proposed in this paper with another ER model which is based on the MCP formulation (4.4) and uses the Fischer-Burmeister

Table 3: Link flows for different values of α when $D(\omega) = 200$.

α	Link Flows
18	(53.04, 137.22, 23.65, 29.39, 11.97 29.77, 125.25, 5.85, 59.16, 131.10)
50	(53.91, 139.40, 24.01, 29.90, 12.16 30.21, 127.24, 5.96, 60.11, 133.20)
75	(54.45 , 140.73, 24.24, 30.21, 12.27, 30.50, 128.44, 6.03, 60.70, 134.47)
100	(54.75, 141.49, 24.37, 30.38, 12.36, 30.66, 129.13, 6.07, 61.04, 135.20)
500	(55.59, 143.59, 24.72, 30.87, 12.56, 31.11, 131.03, 6.17, 61.98, 137.20)
5000	(55.80, 144.11, 24.81, 30.99, 12.61, 31.22, 131.50, 6.20, 62.22, 137.70)
9000	(55.82, 144.14, 24.82, 31.00, 12.61, 31.23, 131.52, 6.20, 62.23, 137.72)
10000	(55.82, 144.14, 24.82, 31.00, 12.61, 31.23, 131.53, 6.20, 62.23, 137.73)

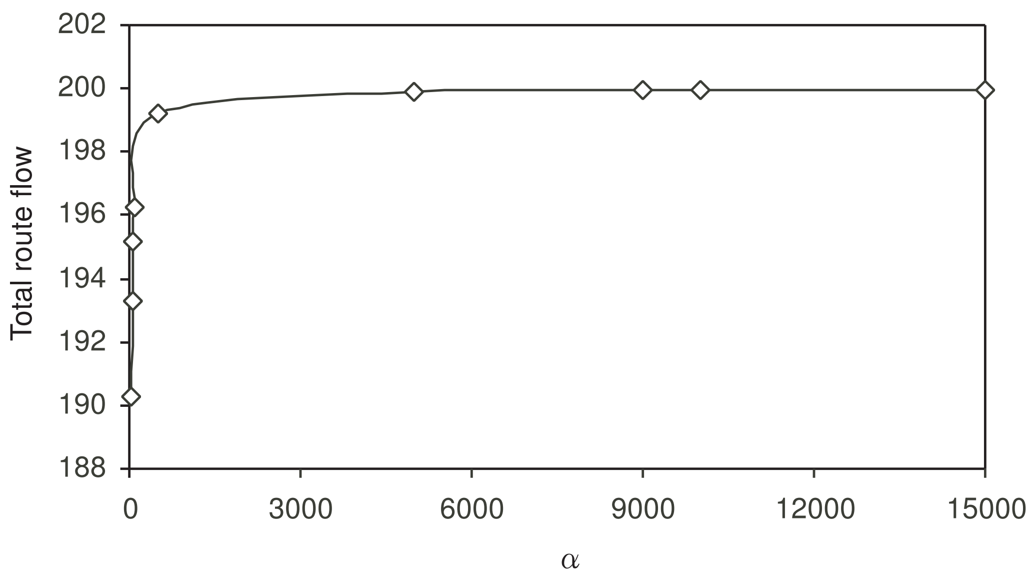

Figure 5: Total route flow for different values of α when $D(\omega) = 200$.

Table 4: Route flows for different values of δ when $D(\omega) = 500\omega - 100$.

α	Route Flows
10	(25.6609, 14.1777, 5.9815, 12.7606, -2.0497, 107.0979)
50	(22.4198, 19.0718, -1.2081, 3.6273, 5.1809, 96.2403)
145	(22.8787, 18.2127, -0.0121, 4.9095, 4.0867, 98.0357)
150	(22.8883, 18.1652, 0.0425, 4.9659, 4.0342, 98.0733)
500	(23.0951, 17.1067, 1.2529, 6.2162, 2.8694, 98.8816)
1000	(23.1423, 15.8479, 2.5464, 7.5188, 1.5863, 99.0661)
5000	(23.1808, 15.9142, 2.5084, 7.4884, 1.6328, 99.2165)
10000	(23.1857, 15.9310, 2.4952, 7.4760 1.6471, 99.2355)

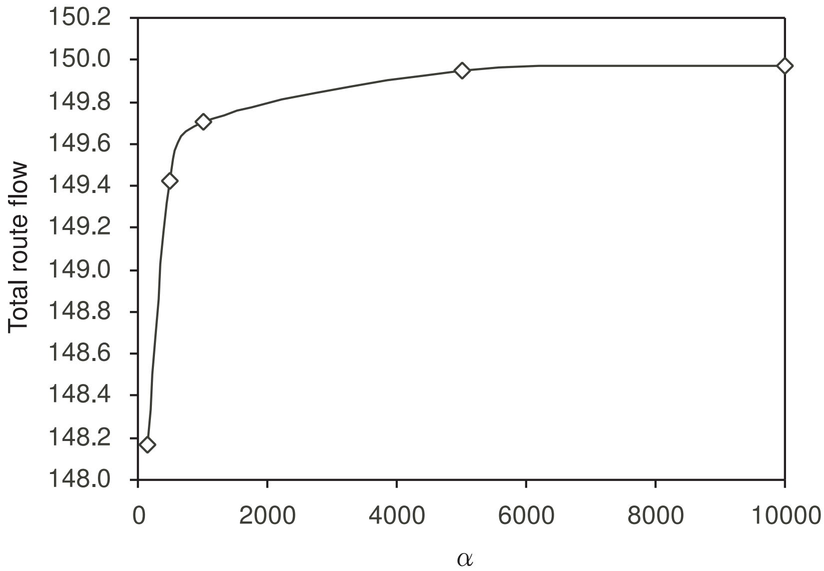

Figure 6: Total route flow for different values of α when $D(\omega) = 500\omega - 100$.

(FB) function. This ER model is referred to as ER-FB and is defined as follows:

$$\begin{array}{ll} \min & E[\Psi(F,u,\omega)] \\ \text{s.t.} & \Gamma^T F - \bar{D} = 0, \end{array} \tag{5.1}$$

where $\Psi(F,u,\omega) = \|\Phi(F,u,\omega)\|^2$ with

$$\Phi(F,u,\omega) = \begin{pmatrix} \phi(F_1, (C(F,\omega) - \Gamma u)_1) \\ \vdots \\ \phi(F_{n_R}, (C(F,\omega) - \Gamma u)_{n_R}) \end{pmatrix},$$

and the travel demand is assumed to be fixed at $\bar{D} = 200$.

Here, we consider the effect of δ, which defines the interval $\Omega = [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$, on the feasibility of the solutions obtained by the two ER methods. Link flows obtained for different values of α and different values of δ are shown in Table 5. It can be seen from the table that ER-FB and ER-D obtained the same solution when δ is very small. However, the results vary when δ and α become larger.

Moreover, as shown in Figure 7, as δ increases, that is, as the variance of ω becomes larger, the obtained route flows tend to violate the demand condition. More specifically, the bigger the value of δ, the smaller the total route flow. However, as seen from the figure, the decrease in the total route flow for the ER-FB model is more significant than the ER-D models with large α. Thus the solutions obtained by the ER-D models with larger values of α are more stable than the solutions obtained by the ER-FB model when the variance of random variable ω becomes large.

Remark 5.2. Note that when we consider ER-FB for the SAVIP, we need to convert SAVIP into SMCP. Hence, the ER-FB model may lose some properties of the SAVIP. The difference between the solutions obtained by the ER-D model and the ER-FB model can be seen in the numerical results above, where the solution of ER-FB tends to violate the conditions more than the solution of ER-D.

6 Conclusion

In this paper, we have proposed two new ER models, the *ER-R model* which uses the regularized gap function and the *ER-D model* which uses the D-gap function for the stochastic affine variational inequality problem (SAVIP). Sufficient conditions for the models to be convex have been established. One of the ER models proposed in this paper, the ER-D model, is then applied to the traffic equilibrium problem under uncertainty. In the numerical experiment, we compare the ER-D model with the MCP-based ER model with the Fischer-Burmeister function (ER-FB).

The numerical results show that, when the demand $D(\omega)$ is fixed ($D(\omega) = 200$), the proposed ER-D model with large α can obtain more reasonable solutions since the obtained route flows tend to satisfy the demand condition. Moreover, the demand condition is not greatly affected by the increase in the variance of ω, that is, in the change in δ, as compared to the ER-FB model.

In this paper, the values of α used in the numerical experiments were only chosen arbitrarily. Determining how large the value of α (and $\bar{\alpha}$) based on Theorem 3.1 and investigating its effect on the feasibility of the solutions would be an interesting topic to consider in the future.

Table 5: Link flows for ER-D and ER-FB for different values of δ when $D(\omega) = 200$.

δ	Link Flows				
	ER-D ($\alpha = 100$)	ER-D ($\alpha = 500$)	ER-D ($\alpha = 1000$)	ER-D ($\alpha = 5000$)	ER-FB
0.0001	54.85	54.85	54.85	54.85	54.85
	145.15	145.15	145.15	145.15	145.15
	26.60	26.60	26.60	26.60	26.60
	28.25	28.25	28.25	28.25	28.25
	11.78	11.78	11.78	11.78	11.78
	31.59	31.59	31.59	31.59	31.59
	133.37	133.37	133.37	133.37	133.37
	6.79	6.79	6.79	6.79	6.79
	59.85	59.85	59.85	59.85	59.85
	140.15	140.15	140.15	140.15	140.15
0.01	54.75	55.60	55.71	55.80	54.61
	141.49	143.59	143.88	144.11	142.20
	24.37	24.72	24.77	24.81	24.02
	30.38	30.87	30.94	30.99	30.59
	12.36	12.56	12.59	12.61	15.50
	30.66	31.11	31.18	31.22	32.05
	129.14	131.03	131.29	131.50	126.68
	6.07	6.17	6.19	6.20	7.50
	61.05	61.98	62.11	62.22	62.64
	135.20	137.20	137.48	137.70	134.20
0.02	53.67	55.61	55.88	56.11	55.66
	137.62	142.47	143.15	143.70	138.40
	22.79	23.61	23.72	23.81	21.60
	30.88	32.01	32.17	32.30	34.07
	12.07	12.54	12.61	12.66	16.40
	28.82	29.82	29.96	30.08	33.16
	125.54	129.93	130.54	131.04	121.97
	6.05	6.32	6.36	6.40	4.80
	59.70	61.83	62.13	62.37	67.29
	131.59	136.26	136.90	137.43	126.80

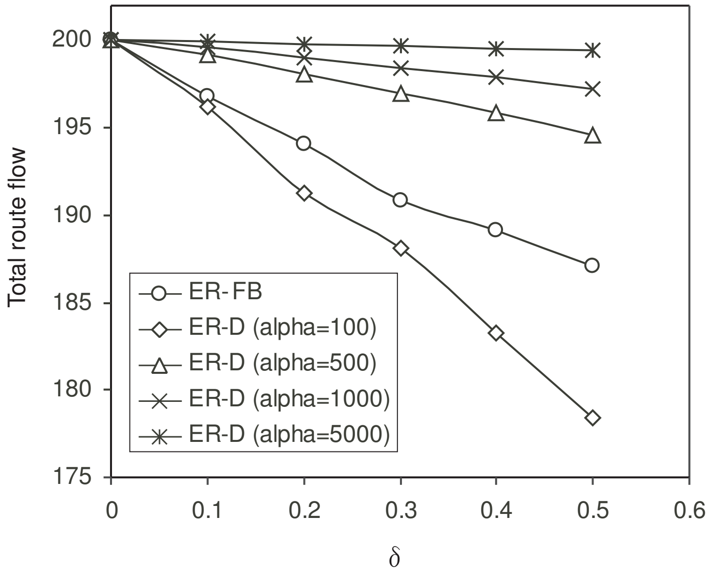

Figure 7: Total route flow for different values of δ when $D(\omega) = 200$.

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