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LOCAL AND GLOBAL ERROR BOUNDS FOR PROPER FUNCTIONS

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Abstract: This paper studies local and global error bounds for an inequality defined by a proper function without assuming lower semicontinuity. The relationship between error bounds of the given function and its closure is established through "bridge" theorems. The equal closure property is introduced and shown to be critical in this establishment. Applying the bridge theorems to exterior characterizations valid for a lower semicontinuous function, one can obtain characterizations for a proper function.

Key words: *convex inequalities, equal closure property, local and global error bounds*

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1 Introduction

Let (X, d) be a metric space, and let f be a proper function defined on X, i.e., a function that nowhere has the value $-\infty$ and is not identically equal to $+\infty$. For the inequality $f \leq 0$, let $S_f = \{x \in X : f(x) \leq 0\}$ denote the solution set and $d_{S_f}(x) = \inf\{d(x, y) : y \in S_f\}$ denote the distance from x to S_f. Throughout this paper, it is assumed that $\emptyset \neq S_f \neq X$.

Let $f_+(x) = \max\{f(x), 0\}$. The inequality $f \leq 0$ is said to have a local error bound at a if there exist $\tau, \delta \in (0, \infty)$ such that $d_{S_f}(x) \leq \tau f_+(x)$ for all $x \in B(a, \delta)$, where $B(a, \delta)$ denotes the open ball centered at a with radius δ (cf. [7, 10, 18]). The inequality $f \leq 0$ is said to have a global error bound if there exists $\tau \in (0, \infty)$ such that $d_{S_f}(x) \leq \tau f_+(x)$ for all $x \in X$. The smallest global error bound for $f \leq 0$ is denoted by τ_f.

The study of an inequality defined by a non lower semicontinuous function arose from a broad class of outer approximation methods for convex optimization ([4] and references therein). It is also a theoretical interest to study error bounds without lower semicontinuity. Some commonly studied non lower semicontinuous functions include the indicator functions of nonclosed sets. For example, the feasible direction cones of a closed convex set may not be closed, thus their indicator functions may not be lower semicontinuous (cf. [3]).

Characterizations of local and global error bounds have been actively studied during the last two decades and references are too numerous to be completely listed. We can only select some that are closely related to the topics of this paper. An exterior characterization of error bounds is a condition on the exterior of the solution set. Such characterizations typically require f to be lower semicontinuous (e.g. [1, 2, 8, 13, 16, 17]). Without lower semicontinuity, they become not sufficient or not necessary (see Examples 2.1 and 2.2).

In this paper, we study local and global error bounds for an inequality defined by a proper function. The relation between error bounds of the given function and its closure is established through "bridge" theorems. The equal closure property is introduced and shown to be critical in this establishment. Applying the bridge theorems to any characterization of error bounds valid for lower semicontinuous functions, we can obtain a characterization of error bounds for proper functions. For example, an extensively studied exterior characterization is the norm of subgradients at all exterior points of the solution set being uniformly positive. Without lower semicontinuity, this condition is still necessary but not sufficient [13]. We can use the bridge theorems to extend this and other characterizations to proper functions.

The notation used in this paper is standard. For $A \subseteq X$, let $\mathrm{cl}(A)$, $\mathrm{bd}(A)$, $\mathrm{int}(A)$, $\mathrm{conv}(A)$, $\mathrm{cone}(A)$ denote the closure, boundary, interior, convex hull, and convex cone of A respectively. Let $d_A(x) = \inf\{d(x,a) : \ a \in A\}$ if $A \neq \emptyset$, otherwise, $d_A(x) = \infty$ by convention.

Let $\mathrm{dom} f = \{x \in X : \ f(x) < \infty\}$ and $\mathrm{epi} f = \{(x,y) \in X \times R : \ f(x) \leq y\}$. The closure of f is defined by $\mathrm{epi}(\mathrm{cl} f) = \mathrm{cl}(\mathrm{epi} f)$.

For a normed linear space $(X, \|\cdot\|)$, let $(X^*, \|\cdot\|_*)$ denote the topological dual space. For a convex set C and $x \in C$, let $N_C(x) = \{x^* \in X^* : \ \langle x^*, y-x\rangle \leq 0 \text{ for all } y \in C\}$ denote the normal cone of C at x, where $\langle x^*, y-x\rangle$ is the value of continuous linear functional x^* at $y-x$.

For a proper convex function f, let $\partial f(x) = \{x^* \in X^* : \ (x^*, -1) \in N_{\mathrm{epi} f}(x, f(x))\}$ denote the subdifferential of f at x. Let $f'(x;h)$ denote the classical directional derivative of f at x in the direction h, i.e.,

$$f'(x;h) = \lim_{t \to 0^+} \frac{f(x+th) - f(x)}{t}.$$

The strong slope of f at $x \in \mathrm{dom} f$ is defined by (e.g. [1, 2])

$$|\nabla f|(x) = \begin{cases} \limsup_{y \to x} \frac{f(x)-f(y)}{d(x,y)}, & if\ x\ is\ not\ a\ local\ minimum\ of\ f; \\ 0, & otherwise. \end{cases}$$

If $x \notin \mathrm{dom} f$, $|\nabla f|(x) = \infty$.

2 Error Bounds and Equal Closure Property

Characterizations of error bounds for inequalities have been actively studied in recent years. The following characterizations of global error bounds for $f \leq 0$, where f is a lower semicontinuous convex function in a Banach space, are well known and equivalent (cf. [1, 2, 5, 8, 13, 16, 17]).

$$\inf\{\|x^*\|_* : x^* \in \partial f(x), x \notin S_f\} \geq \tau^{-1}, \tag{2.1}$$

$$\partial f^*(x^*) \subseteq S_f, \ \forall x^* \in X^*, \|x^*\|_* < \tau^{-1}, \tag{2.2}$$

$$\inf\{|\nabla f|(x) : x \notin S_f\} \geq \tau^{-1}. \tag{2.3}$$

It is interesting to observe what may happen to these three characterizations when f is not lower semicontinuous. In the next two examples, we demonstrate that when lower semicontinuity is not present, (2.1) and (2.3) are no longer sufficient while (2.2) is no longer necessary.

Example 2.1. Let $D = \{(x,y) : x > 0, y \geq 0\} \cup \{(0,y) : 0 \leq y \leq 1\}$, and define $f(x,y) = I_D(x,y) + x$, where $I_D(x,y)$ is the indicator function of D. Note that f is proper convex but not lower semicontinuous, and $S_f = \{(x,y) : f(x,y) \leq 0\} = \{(0,y) : 0 \leq y \leq 1\}$ is compact.

First, we verify that (2.1) holds. If $(x, y) \notin D$, then $\partial f(x, y) = \emptyset$. If $(x, y) \in \text{int}(D)$, then $\partial f(x, y) = \partial I_D(x, y) + \{(1, 0)\} = \{(1, 0)\}$. On $\{(x, 0) : x > 0\}$, $\partial f(x, y) = N_D(x, y) + \{(1, 0)\} = \{(1, -t) : t \geq 0\}$. Therefore, $\inf\{\|(x^*, y^*)\|_* : (x^*, y^*) \in \partial f(x, y), (x, y) \notin S_f\} = 1$.

Next, we verify that (2.3) holds. We compute the strong slope for $(a, b) \in \text{dom} f \backslash S_f$. Let $a > 0$, $b \geq 0$, and $(a, b) \neq (x, y) \in \text{dom} f$, then

$$\frac{f(a, b) - f(x, y)}{\|(a, b) - (x, y)\|} = \frac{a - x}{\sqrt{(a - x)^2 + (b - y)^2}} \leq 1.$$

Choosing $0 < x < a$ and $y = b$ yields

$$\frac{f(a, b) - f(x, y)}{\|(a, b) - (x, y)\|} = 1.$$

Therefore, for any $(a, b) \in \text{dom} f \backslash S_f$,

$$|\nabla f|(a, b) = \limsup_{(x,y) \to (a,b)} \frac{f(a, b) - f(x, y)}{\|(a, b) - (x, y)\|} = 1.$$

Now we show that $f \leq 0$ does not have a global error bound. For any $t > 0$, let $(x_t, y_t) = (t, 1 + \sqrt{t})$. Then, $f(x_t, y_t) = t$ and

$$d_{S_f}(x_t, y_t) = d((t, 1 + \sqrt{t}), (0, 1)) = t\sqrt{1 + t^{-1}} = \sqrt{1 + t^{-1}} f(x_t, y_t).$$

$\sqrt{1 + t^{-1}} \to \infty$ as $t \to 0^+$. Therefore, $f \leq 0$ does not have a global error bound.

On the other hand, $\text{cl} f(x, y) = I_{\text{cl}(D)}(x, y) + x$, and $S_{\text{cl} f} = \{(0, y) : y \geq 0\}$. Direct verification shows that $\tau = 1$ is a global error bound for $\text{cl} f \leq 0$. Observe in this case, $S_f = \text{cl}(S_f) \subset S_{\text{cl} f}$, and $\text{cl}(S_f) \cap B((0, 1), r) \subset S_{\text{cl} f} \cap B((0, 1), r)$ for any $0 < r < 1$.

Example 2.2. Let $D = \{(x, y) : x \geq 0, y \geq 0\} \backslash \{(0, 0)\}$, and $f(x, y) = I_D(x, y) + x$. Note that f is proper convex but not lower semicontinuous, $S_f = \{(0, y) : y > 0\}$ is nonclosed, and $\text{cl}(S_f) = S_{\text{cl} f} = \{(0, y) : y \geq 0\}$. It is easy to verify directly that $f \leq 0$ has a global error bound $\tau = 1$. Next, we show that condition (2.2) fails to hold.

$$\begin{aligned} f^*(x^*, y^*) &= \sup\{\langle (x^*, y^*), (x, y) \rangle - f(x, y) : (x, y) \in R^2\} \\ &= \sup\{\langle (x^* - 1, y^*), (x, y) \rangle - I_D(x, y) : (x, y) \in R^2\} \\ &= \sup\{\langle (x^* - 1, y^*), (x, y) \rangle : (x, y) \in D\} \\ &\geq 0. \end{aligned} \tag{2.4}$$

For any $0 < a^* < 1$, we claim that $(0, 0) \in \partial f^*(a^*, 0)$. Indeed, because $a^* - 1 < 0$, $f^*(a^*, 0) = 0$ by (2.4). Hence, $f^*(x^*, y^*) - f^*(a^*, 0) = f^*(x^*, y^*) \geq 0$ for all $(x^*, y^*) \in R^2$, namely, $(0, 0) \in \partial f^*(a^*, 0)$. Note that $(0, 0) \notin S_f$, we conclude that condition (2.2) fails to hold.

Examples 2.1 and 2.2 indicate that the closedness of the solution set S_f is not necessary for the existence of a global error bound, while the condition $\text{cl}(S_f) = S_{\text{cl} f}$ seems to be critical. In addition, the existence of a global error bound for $\text{cl} f \leq 0$ is not sufficient to ensure the existence of a global error bound for $f \leq 0$. These observations lead to the following useful topological condition, and motivate the research for the relation between error bounds of $f(x) \leq 0$ and error bounds of $\text{cl} f(x) \leq 0$.

Definition 2.3. The inequality $f \le 0$ is said to have the equal closure property (ECP) if it satisfies $\mathrm{cl}(S_f) = S_{\mathrm{cl}f}$.

Obviously, if f is lower semicontinuous, then f satisfies ECP. Example 2.2 shows that ECP is indeed weaker than lower semicontinuity. We'll prove that $\mathrm{cl}(S_f) = S_{\mathrm{cl}f}$ is a necessary condition for the existence of a global error bound. Before presenting the main result of this section, we need the following convenient lemma.

Lemma 2.4. *Let (X, d) be a metric space. Then, for all $x \in X$, we have* $\mathrm{cl}(f_+)(x) = (\mathrm{cl}f)_+(x)$.

Proof. We calculate the epigraphs of the functions.

$$\mathrm{epi}(\mathrm{cl}(f_+)) = \mathrm{cl}(\mathrm{epi}(f_+)) = \mathrm{cl}(\mathrm{epi}f \cap \mathrm{epi}0).$$

$$\mathrm{epi}((\mathrm{cl}f)_+) = \mathrm{epi}(\mathrm{cl}f) \cap \mathrm{epi}0 = \mathrm{cl}(\mathrm{epi}f) \cap \mathrm{epi}0.$$

By [7, Lemma 2.3 (i)], $\mathrm{cl}(\mathrm{epi}f \cap \mathrm{epi}0) = \mathrm{cl}(\mathrm{epi}f) \cap \mathrm{epi}0$, therefore Lemma 2.4 holds. Note that [7, Lemma 2.3 (i)] is stated for a proper function in a Banach space, but the same proof is valid in a metric space. □

Theorem 2.5. *Suppose that (X, d) is a metric space, f is a proper function, and $\tau \in (0, \infty)$. The following statements are equivalent.*
(i) $d_{S_f}(x) \le \tau f_+(x)$ *for all* $x \in X$.
(ii) $\mathrm{cl}(S_f) = S_{\mathrm{cl}f}$ *and* $d_{S_{\mathrm{cl}f}}(x) \le \tau(\mathrm{cl}f)_+(x)$ *for all* $x \in X$.
In this case, the least global error bound for $f \le 0$ is the same as that for $\mathrm{cl}f \le 0$, i.e., $\tau_f = \tau_{\mathrm{cl}f}$.

Proof. (i) ⇒ (ii). From (i), we have $\tau^{-1} d_{S_f}(x) \le f_+(x)$ for all $x \in X$. Since $\tau^{-1} d_{S_f}(x)$ is continuous and $\mathrm{cl}(f_+)$ is the greatest lower semicontinuous function majorized by f_+, by Lemma 2.4 we have that

$$\tau^{-1} d_{S_f}(x) \le \mathrm{cl}(f_+)(x) = (\mathrm{cl}f)_+(x) \le f_+(x) \ \forall x \in X. \tag{2.5}$$

Since $\mathrm{cl}f \le f$ and $S_{\mathrm{cl}f}$ is closed, $S_f \subseteq \mathrm{cl}(S_f) \subseteq S_{\mathrm{cl}f}$. This implies that

$$d_{S_{\mathrm{cl}f}}(x) \le d_{S_f}(x) \ \forall x \in X. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$d_{S_{\mathrm{cl}f}}(x) \le d_{S_f}(x) \le \tau(\mathrm{cl}f)_+(x) \ \forall x \in X. \tag{2.7}$$

It remains to show $S_{\mathrm{cl}f} \subseteq \mathrm{cl}(S_f)$. Let $x \in S_{\mathrm{cl}f}$. Then $\mathrm{cl}f(x) \le 0$, which implies that $(\mathrm{cl}f)_+(x) = 0$. By (2.7), $d_{S_f}(x) = 0$, which implies that $x \in \mathrm{cl}(S_f)$. Thus $S_{\mathrm{cl}f} \subseteq \mathrm{cl}(S_f)$.

(ii) ⇒ (i). By (ii) and Lemma 2.4,

$$d_{S_f}(x) = d_{\mathrm{cl}(S_f)}(x) = d_{S_{\mathrm{cl}f}}(x) \le \tau(\mathrm{cl}f)_+(x) = \tau\mathrm{cl}(f_+)(x) \le \tau f_+(x) \ \forall x \in X.$$

□

Theorem 2.5 is a useful bridge. Applying Theorem 2.5 to characterizations valid for a lower semicontinuous function, one can obtain characterizations for a proper function. In particular, for a convex inequality in a Banach space, we immediately have the following extension of (2.1), (2.2) and (2.3).

Theorem 2.6. *Suppose that X is a Banach space, f is a proper convex function, and $\tau \in (0,\infty)$. The following statements are equivalent.*
(i) $d_{S_f}(x) \le \tau f_+(x)$ *for all* $x \in X$.
(ii) $\mathrm{cl}(S_f) = S_{\mathrm{cl}f}$ *and* $\inf\{\|x^*\|_* : x^* \in \partial(\mathrm{cl}f)(x), x \notin S_{\mathrm{cl}f}\} \ge \tau^{-1}$.
(iii) $\mathrm{cl}(S_f) = S_{\mathrm{cl}f}$ *and* $\partial f^*(x^*) \subseteq S_{\mathrm{cl}f}$, *for all* $x^* \in X^*$ *satisfying* $\|x^*\|_* < \tau^{-1}$.
(iv) $\mathrm{cl}(S_f) = S_{\mathrm{cl}f}$ *and* $\inf\{|\nabla \mathrm{cl}f|(x) : x \notin S_{\mathrm{cl}f}\} \ge \tau^{-1}$.

It was known that condition (2.1) remains necessary even when lower semicontinuity is not present (cf. [13]). This necessity is easily seen from Theorem 2.6 (ii) because $\partial f(x) \subseteq \partial(\mathrm{cl}f)(x)$. In order to derive a local version of Theorem 2.5, we need the following lemma.

Lemma 2.7. *Let (X,d) be a metric space. If $a \in A \subseteq X$ and $r \in (0,\infty)$, then $d_A(x) = d_{A\cap B(a,r)}(x)$ for all $x \in B(a,r/2)$.*

Proof. Let $x \in B(a,r/2)$. Since $A \cap B(a,r) \subseteq A$ implies that $d_A(x) \le d_{A\cap B(a,r)}(x)$, we only need to show that $d_{A\cap B(a,r)}(x) \le d_A(x)$. For all $n = 1,2,\ldots$, choose $a_n \in A$ such that $d(x,a_n) < d_A(x) + (1/n)$. We have

$$d(a_n,a) \le d(a_n,x) + d(x,a) < d_A(x) + (1/n) + (r/2).$$

Since $d(x,a) < (r/2)$, there exists $N > 0$ such that $d(x,a) + (1/n) < (r/2)$ for all $n \ge N$. Therefore, for all $n \ge N$ we have

$$d(a_n,a) < d_A(x) + (1/n) + (r/2) \le d(x,a) + (1/n) + (r/2) < r,$$

which implies that $a_n \in A \cap B(a,r)$ for all $n \ge N$. Consequently, $d_{A\cap B(a,r)}(x) \le d(x,a_n) < d_A(x) + (1/n)$ for all $n \ge N$, and thus $d_{A\cap B(a,r)}(x) \le d_A(x)$. □

Now we can establish a bridge theorem for local error bounds.

Theorem 2.8. *Suppose that (X,d) is a metric space, f is a proper function, and $a \in \mathrm{cl}(S_f) \subseteq S_{\mathrm{cl}f}$. The following statements are equivalent.*
(i) *There exist $\tau, \delta \in (0,\infty)$ such that $d_{S_f}(x) \le \tau f_+(x)$ for all $x \in B(a,\delta)$.*
(ii) *There exist $\tau, \gamma \in (0,\infty)$ such that $\mathrm{cl}(S_f) \cap B(a,\gamma) = S_{\mathrm{cl}f} \cap B(a,\gamma)$ and $d(x, S_{\mathrm{cl}f}) \le \tau(\mathrm{cl}f)_+(x)$ for all $x \in B(a,\gamma)$.*

Proof. (i) ⇒ (ii). From (i), we have $\tau^{-1} d_{S_f}(x) \le f_+(x)$ for all $x \in B(a,\delta)$. Define a function $H(x)$ by

$$H(x) = \begin{cases} f_+(x), & if x \in B(a,\delta); \\ \tau^{-1} d_{S_f}(x), & otherwise. \end{cases}$$

Note that $\tau^{-1} d_{S_f}(x)$ is continuous and $\tau^{-1} d_{S_f}(x) \le H(x)$ for all $x \in X$, we have $\tau^{-1} d_{S_f}(x) \le \mathrm{cl}H(x) \le H(x)$ for all $x \in X$. Since $H(x) = f_+(x)$ for all $x \in B(a,\delta)$, $\mathrm{cl}H(x) = \mathrm{cl}(f_+)(x)$ in $B(a,\delta)$. Thus,

$$\tau^{-1} d_{S_f}(x) \le \mathrm{cl}(f_+)(x) = (\mathrm{cl}f)_+(x) \ \forall x \in B(a,\delta),$$

which implies that

$$d_{S_{\mathrm{cl}f}}(x) \le d_{S_f}(x) \le \tau(\mathrm{cl}f)_+(x) \ \forall x \in B(a,\delta). \tag{2.8}$$

Now we show that $\mathrm{cl}(S_f) \cap B(a,\delta) = S_{\mathrm{cl}f} \cap B(a,\delta)$. As $\mathrm{cl}(S_f) \subseteq S_{\mathrm{cl}f}$, we only need to show that $S_{\mathrm{cl}f} \cap B(a,\delta) \subseteq \mathrm{cl}(S_f) \cap B(a,\delta)$. If $x \in S_{\mathrm{cl}f} \cap B(a,\delta) = S_{(\mathrm{cl}f)_+} \cap B(a,\delta)$, then $(\mathrm{cl}f)_+(x) = 0$. By (2.8), $d_{S_f}(x) = 0$, thus $x \in \mathrm{cl}(S_f) \cap B(a,\delta)$.

(ii) $\Rightarrow$ (i). Let $a \in \text{cl}(S_f) \subseteq S_{\text{cl}f}$ and $x \in B(a, \gamma/2)$.

$$\begin{aligned} d_{S_f}(x) &= d_{\text{cl}(S_f)}(x) \\ &= d_{\text{cl}(S_f)\cap B(a,\gamma)}(x) \text{ (Lemma2.7)} \\ &= d_{S_{\text{cl}f}\cap B(a,\gamma)}(x) \\ &= d_{S_{\text{cl}f}}(x) \text{ (Lemma2.7)} \\ &\le \tau(\text{cl}f)_+(x) \\ &\le \tau f_+(x). \end{aligned}$$

Therefore, (i) and (ii) are equivalent. □

We have seen that ECP is a necessary condition for the existence of a global error bound for an inequality in a metric space. Now we show that for a convex inequality $f \le 0$ in a Banach space, ECP is also necessary for the existence of local error bounds on the entire $\text{cl}(S_f)$. First, we clarify some technical facts.

Fact 2.1. Suppose that C, D are convex sets in a normed linear space X, and $a \in C \subseteq D$. If there exists $r \in (0, \infty)$ such that $C \cap B(a, r) = D \cap B(a, r)$, then $N_C(a) = N_D(a)$.

Indeed, since $C \subseteq D$ implies $N_D(a) \subseteq N_C(a)$, we only need to verify that $N_C(a) \subseteq N_D(a)$. If $a^* \in N_C(a)$, then $\langle a^*, x - a\rangle \le 0$ for all $x \in C$. Let $y \in D$ and choose $t \in (0, 1)$ sufficiently small such that $a + t(y - a) \in B(a, r)$. By the convexity of D, $a + t(y - a) = (1 - t)a + ty \in D \cap B(a, r) = C \cap B(a, r) \subseteq C$. Define $z = a + t(y - a) \in C$, we have $\langle a^*, y - a\rangle = (1/t)\langle a^*, z - a\rangle \le 0$. Since the above inequality holds for all $y \in D$, we conclude that $a^* \in N_D(a)$.

Fact 2.2. Suppose that X is a Banach space, C, D are nonempty convex sets, C is closed, and $C \subseteq D$. If $N_C(x) = N_D(x)$ for all $x \in C$, then $C = D$.

Without loss of generality, we may assume that $X \ne \{0\}$. If $C \ne D$, then pick $b \in D\backslash C$ and let $A = \{b\}$. By the strong separation Theorem (cf. [11, Theorem 2.2.28]), C and A can be strongly separated. Now applying the Bishop-Phelps Theorem (cf. [11, Theorem 2.11.12]) to A and C, there exist a unit $a^* \in X^*$ and $a \in C$ such that

$$\langle a^*, a\rangle = \sup\{\langle a^*, x\rangle : x \in C\} < \langle a^*, b\rangle. \tag{2.9}$$

It follows that $a^* \in N_C(a)$. By the assumption $N_C(a) = N_D(a)$, we have $a^* \in N_D(a)$. Since $b \in D$, $\langle a^*, b - a\rangle \le 0$, which contradicts (2.9). Therefore, we must have $C = D$.

Now we are ready to show that ECP is necessary for the existence of local error bounds on $\text{cl}(S_f)$.

Theorem 2.9. *Suppose that f is a proper convex function in a Banach space. If $f \le 0$ has a local error bound for all $x \in \text{cl}(S_f)$, then $\text{cl}(S_f) = S_{\text{cl}f}$.*

Proof. Let $a \in \text{cl}(S_f) \subseteq S_{\text{cl}f}$. By Theorem 2.8, $f \le 0$ has a local error bound at a implies that there exists $r \in (0, \infty)$ such that $\text{cl}(S_f) \cap B(a, r) = S_{\text{cl}f} \cap B(a, r)$. By Fact 2.1, $N_{\text{cl}(S_f)}(a) = N_{S_{\text{cl}f}}(a)$. Since the above equation holds for all $a \in \text{cl}(S_f)$, by Fact 2.2 we conclude that $\text{cl}(S_f) = S_{\text{cl}f}$. □

From Theorem 2.9, we can derive some simple sufficient conditions for the equal closure property when f is a proper convex function in a Banach space. Indeed, if f satisfies the Slater condition, then $f \le 0$ has a local error bound for all $x \in \text{cl}(S_f)$ (cf. [14]). By Theorem 2.9, we have $\text{cl}(S_f) = S_{\text{cl}f}$. If $\inf f$ cannot be attained, then $S_f \ne \emptyset$ implies that f satisfies the Slater condition. Thus, we have the following corollary.

Corollary 2.10. *Suppose that f is a proper convex function in a Banach space. If f satisfies the Slater condition or $\inf f$ cannot be attained, then $\text{cl}(S_f) = S_{\text{cl}f}$.*

Next, we extend the bridge theorems to weak sharp minima. For a proper function f, let $\text{argmin} f = \{x : f(x) \le \inf f\}$. Define $h(x) = f(x) - \inf f$. Since $\inf f = \inf(\text{cl}f)$, $\text{cl}h(x) = \text{cl}f(x) - \inf(\text{cl}f)$. Thus, we have $S_h = \text{argmin} f$ and $S_{\text{cl}h} = \text{argmin}(\text{cl}f)$. Applying Theorems 2.5 and 2.8 to the inequality $h \le 0$, we immediately obtain the following results.

Theorem 2.11. *Suppose that (X, d) is a metric space, f is a proper function, $\alpha \in (0, \infty)$, and $\text{argmin} f \ne \emptyset$. The following statements are equivalent.*
(i) *$f(x) \ge \inf f + \alpha d_{\text{argmin} f}(x)$ for all $x \in X$.*
(ii) *$\text{cl}(\text{argmin} f) = \text{argmin}(\text{cl}f)$ and $\text{cl}f(x) \ge \inf(\text{cl}f) + \alpha d_{\text{argmin}(\text{cl}f)}(x)$ for all $x \in X$.*

Theorem 2.12. *Suppose that (X, d) is a metric space, f is a proper function, and $a \in \text{cl}(\text{argmin} f)$. The following statements are equivalent.*
(i) *There exist $\alpha, \delta \in (0, \infty)$ such that $f(x) \ge \inf f + \alpha d_{\text{argmin} f}(x)$ for all $x \in B(a, \delta)$.*
(ii) *There exist $\alpha, \gamma \in (0, \infty)$ $\text{cl}(\text{argmin} f) \cap B(a, \gamma) = \text{argmin}(\text{cl}f) \cap B(a, \gamma)$ and $\text{cl}f(x) \ge \inf(\text{cl}f) + \alpha d_{\text{argmin}(\text{cl}f)}(x)$ for all $x \in B(a, \gamma)$.*

For a proper convex function in a Banach space, because $f^* = (\text{cl}f)^*$, we have the following result by Theorem 2.11 and [5, Theorem 5.1].

Theorem 2.13. *Suppose that $(X, \|\cdot\|)$ is a Banach space, f is a proper convex function, $\alpha \in (0, \infty)$, and $\text{argmin} f \ne \emptyset$. The following statements are equivalent.*
(i) *$f(x) \ge \inf f + \alpha d_{\text{argmin} f}(x)$ for all $x \in X$.*
(ii) *$\text{cl}(\text{argmin} f) = \text{argmin}(\text{cl}f)$ and $\inf\{\|x^*\|_* : x^* \in \partial(\text{cl}f)(x), x \notin S_{\text{cl}f}\} \ge \alpha$.*
(iii) *$\text{cl}(\text{argmin} f) = \text{argmin}(\text{cl}f)$ and $\partial f^*(x^*) \subseteq \partial f^*(0)$ for all $x^* \in X^*$ satisfying $\|x^*\|_* < \alpha$.*

3 Extension to Systems of Inequalities and Equal Level Set Closure Property

Let $f_1, \dots, f_m$ be proper functions in a metric space X, and $g_1 = \text{cl}f_1, \dots, g_m = \text{cl}f_m$. Consider the system of inequalities

$$f_1(x) \le 0, \dots, f_m(x) \le 0, \tag{3.1}$$

and the related system defined by their closure functions

$$g_1(x) \le 0, \dots, g_m(x) \le 0. \tag{3.2}$$

Let $F(x) = \max\{f_1(x), \dots, f_m(x)\}$ and $G(x) = \max\{g_1(x), \dots, g_m(x)\}$. Let $S_{f_i} = \{x : f_i(x) \le 0\}$, $S_{g_i} = \{x : g_i(x) \le 0\}$, $S_F = \{x : F(x) \le 0\}$, and $S_G = \{x : G(x) \le 0\}$. In order to extend Theorem 2.5 to the system of inequalities (3.1), we need the following topological relation among members of a family of sets.

Definition 3.1. A collection of sets $\{A_i \subseteq X : i = 1, \dots, m\}$ is said to have the closed intersection property (CIP), if $\text{cl}(\bigcap_{i=1}^m A_i) = \bigcap_{i=1}^m \text{cl}(A_i)$ (cf. [3]).

In the next lemma, we characterize $G(x) = \text{cl}F(x)$ and $\text{cl}(S_F) = S_G$.

Lemma 3.2. *Let $f_1, \dots, f_m$ be proper functions in a metric space.*
(i) *$G(x) = \text{cl}F(x)$ for all $x \in X$ if and only if $\{\text{epi}f_i : i = 1, \dots, m\}$ has CIP.*
(ii) *If $f_i \le 0$ has ECP for all $i = 1, \dots, m$, then $\text{cl}(S_F) = S_G$ if and only if $\{S_{f_i} : i = 1, \dots, m\}$ has CIP.*

Proof. (i) $G(x) = \mathrm{cl}F(x)$ for all $x \in X$ if and only if $\mathrm{epi}G = \mathrm{epi}(\mathrm{cl}F)$. By the definition of the closure function,

$$\mathrm{epi}G = \mathrm{epi}(\max\{g_1(x), \dots, g_m(x)\}) = \bigcap_{i=1}^{m} \mathrm{epi}g_i = \bigcap_{i=1}^{m} \mathrm{cl}(\mathrm{epi}f_i),$$

$$\mathrm{epi}(\mathrm{cl}F) = \mathrm{cl}(\mathrm{epi}F) = \mathrm{cl}\big(\mathrm{epi}(\max\{f_1(x), \dots, f_m(x)\})\big) = \mathrm{cl}(\bigcap_{i=1}^{m} \mathrm{epi}f_i).$$

Consequently, $\mathrm{epi}G = \mathrm{epi}(\mathrm{cl}F)$ if and only if $\{\mathrm{epi}f_i : i = 1, \dots, m\}$ has CIP.

(ii) $\mathrm{cl}(S_F) = \mathrm{cl}(\bigcap_{i=1}^m S_{f_i})$. If $f_i \le 0$ has ECP for all $i = 1, \dots, m$, then $S_G = \bigcap_{i=1}^m S_{g_i} = \bigcap_{i=1}^m \mathrm{cl}(S_{f_i})$. Thus, $\mathrm{cl}S_F = S_G$ if and only if $\{S_{f_i} : i = 1, \dots, m\}$ has CIP. □

Now we are ready to extend Theorem 2.5 to the system of inequalities (3.1).

Theorem 3.3. *Suppose that $f_1, \dots, f_m$ are proper functions in a metric space, $\{\mathrm{epi}f_i : i = 1, \dots, m\}$ has CIP, and $\tau \in (0, \infty)$. Then, the following statements are equivalent.*
(i) $d_{S_F}(x) \le \tau F_+(x)$ *for all* $x \in X$.
(ii) $\mathrm{cl}(S_F) = S_G$ *and* $d_{S_G}(x) \le \tau G_+(x)$ *for all* $x \in X$.

Proof. By Theorem 2.5, $d_{S_F}(x) \le \tau F_+(x)$ for all $x \in X$ if and only if $\mathrm{cl}(S_F) = S_{\mathrm{cl}F}$ and $d_{S_{\mathrm{cl}F}}(x) \le \tau(\mathrm{cl}F)_+(x)$ for all $x \in X$. By the assumption that $\{\mathrm{epi}f_i : i = 1, \dots, m\}$ has CIP and Lemma 3.2 (i), $\mathrm{cl}F(x) = G(x)$. □

Theorem 3.4. *Suppose that $f_1, \dots, f_m$ are proper functions in a metric space, $f_i \le 0$ has ECP for all $i = 1, \dots, m$, $\{\mathrm{epi}f_i : i = 1, \dots, m\}$ has CIP, and $\tau \in (0, \infty)$. Then, the following statements are equivalent.*
(i) $d_{S_F}(x) \le \tau F_+(x)$ *for all* $x \in X$.
(ii) $\{S_{f_i} : i = 1, \dots, m\}$ *has CIP and* $d_{S_G}(x) \le \tau G_+(x)$ *for all* $x \in X$.

Proof. By the assumption that $\{\mathrm{epi}f_i : i = 1, \dots, m\}$ has CIP and Lemma 3.2 (i), $\mathrm{cl}F(x) = G(x)$. By the assumption that $f_i \le 0$ has ECP for all $i = 1, \dots, m$ and Lemma 3.2 (ii), $\mathrm{cl}(S_F) = S_G$ if and only if $\{S_{f_i} : i = 1, \dots, m\}$ has CIP. Then, Theroem 3.4 follows directly form Theorem 2.5. □

Remark 3.5. Theorems 3.3 and 3.4 remain valid for an infinite system of inequalities without additional assumptions. Local versions of Theorems 3.3 and 3.4 can be obtained from Theorem 2.8 and Lemma 3.2.

Next, we look at the equal closure condition for all level sets of f. For any $\alpha \in (-\infty, \infty)$, let $S_{f,\alpha} = \{x : f(x) \le \alpha\}$ be the α-level set. Let $h_\alpha(x) = f(x) - \alpha$. Then, $\mathrm{cl}h_\alpha = \mathrm{cl}f - \alpha$. By applying Theorem 2.5 to $h_\alpha \le 0$, we know that $\mathrm{cl}(S_{f,\alpha}) = S_{\mathrm{cl}f,\alpha}$ whenever a global error bound exists for $f(x) \le \alpha$. This observation motivates the definition of the equal level set closure property.

Definition 3.6. A proper function f is said to have the equal level set closure property (ELSC) if $\mathrm{cl}(S_{f,\alpha}) = S_{\mathrm{cl}f,\alpha}$ for all $S_{f,\alpha} \ne \emptyset$, $\alpha \in (-\infty, \infty)$.

The class of functions having ELSC is broader than the lower semicontinuous class. It is the necessary class such that a global error bound exists for all $\{x : f(x) \le \alpha\} \ne \emptyset$. In the next theorem, we show that for a proper convex function in a Banach space, its membership in the ELSC class can be verified at a single level set.

Theorem 3.7. *If f is a proper convex function in a Banach space, then f belongs to the ELSC class if and only if either* $\operatorname{argmin} f = \emptyset$ *or* $\operatorname{cl}(\operatorname{argmin} f) = \operatorname{argmin}(\operatorname{cl} f)$*.*

Proof. Let $\alpha \in (-\infty, \infty)$ and $h_\alpha(x) = f(x) - \alpha$. If $\alpha > \inf f = \inf \operatorname{cl} f$, then $h_\alpha \leq 0$ satisfies the Slater condition. Therefore, $h_\alpha \leq 0$ has a local error bound for all $x \in \operatorname{cl}(S_{h_\alpha})$ (cf. [14]). By Theorem 2.8, for any $a \in \operatorname{cl}(S_{h_\alpha})$, there exists $r \in (0, \infty)$ such that $\operatorname{cl}(S_{h_\alpha}) \cap B(a, r) = S_{\operatorname{cl} h_\alpha} \cap B(a, r)$. By Facts 2.1 and 2.2, $\operatorname{cl}(S_{h_\alpha}) = S_{\operatorname{cl} h\alpha}$. Therefore, $\operatorname{cl}(S_{f,\alpha}) = S_{\operatorname{cl} f,\alpha}$ for all $\alpha > \inf f$. Thus, for a convex inequality in a Banach space, the critical level for ELCS is $\alpha = \inf f$. At this level, $\operatorname{cl}(\operatorname{argmin} f) = \operatorname{argmin}(\operatorname{cl} f)$ when $\operatorname{argmin} f \neq \emptyset$. □

By Theorems 2.13 and 3.7, we know that for a proper convex function f in a Banach space, its membership in the ELSC class is a necessary condition for the existence of a global weak sharp minima.

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